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Non-Homogeneous Discrete Time Alternating Renewal Processes For Health Insurance Evolution and Evaluation

By

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OUTLINE

Non-homogeneity;
Homogeneous phenomena;
Homogeneity vs non-homogeneity
Running time non homogeneity;
Age & seniority non-homogeneity;
Relevance of non-homogeneity in health and temporary disability insurance;
Homogeneous renewal and alternating renewal processes;
Application of homogeneous alternating renewal processes in health insurance;
Definition of non-homogeneous alternating renewal processes;
Application of non-homogeneous alternating renewal processes in health and temporary disability insurance
Some conclusions
Future lines of research

HOMOGENEITY VS NON-HOMOGENEITY I

Homogeneous models are simpler (only 1 Time Variable) Non Homogeneous have 2-time variables
The results given by a homogeneous model for a horizon time of n years are a linear function of n
In a non-homogeneous setting the results that should be obtained are a quadratic function of n
Non-homogeneous environment is closer to the reality but there are more applications of homogeneous models
There are some phenomena that can be modelled by homogeneous models (earthquakes, eruptions)
If the non-homogeneous variable is the running time, then the forecasting of the future events is function of the past data and this fact can be considered like biting its tail.
In insurance setting can be considered three times variables, running time, age and seniority.
In insurance the main time variable can be considered the age of the insured. This is evident in life and in pension insurance contracts but also in non-life insurance the age of the insured assumes a great relevance.
A great importance in pension schemes is given by the seniority. In Age and Seniority the non-homogeneity can be applied without the problem of the running time because, for example, it is possible considering that a person of age k will have the same behavior of a person of the same age ten years later. The same could hold for two workers of the same seniority in different time, i.e. the past data could be used for the forecasting of future. Instead, if the non-homogeneity is on running time the most phenomena changes and the forecasting of the non-homogeneous models becomes more difficult.

HOMOGENEITY VS NON-HOMOGENEITY II

In the mortality case, that is a typical variable that depends on running time, the seminal paper (Lee & Carter (1992)) gave an extrapolation model for the mortality forecasting. We think that the only way for the forecasting of the running time non-homogeneous phenomena is the construction of extrapolation models. The Lee Carter model is a parametric model. It is clear that extrapolation implies the use of parametric models but we would use a different approach. More precisely, once that it will be chosen a probability distribution (p.d.) that fits well the collected data, we would construct a sequence of parameters of the chosen p.d. From this sequence we will extrapolate the parameters of future distributions.

Summarizing:

Non homogeneous setting, with age as time variable and with sufficient data, it will be possible the construction of non-parametric models. The same holds with seniority as time variable;

Non-homogeneous environment with running time variable needs many data because of construction of the sequence of parameters. If there are few data it could be better using a homogeneous model;

Homogeneous environment for the construction of non-parametric model asks less data because it gives less results;

If data are not enough also for a non parametric homogeneous model it could be construct a homogeneous parametric model; (anyway the data should be always enough for the evaluation of parameters)

NON-HOMOGENEITY IN HEALTH AND TEMPORARY DISABILITY INSURANCE

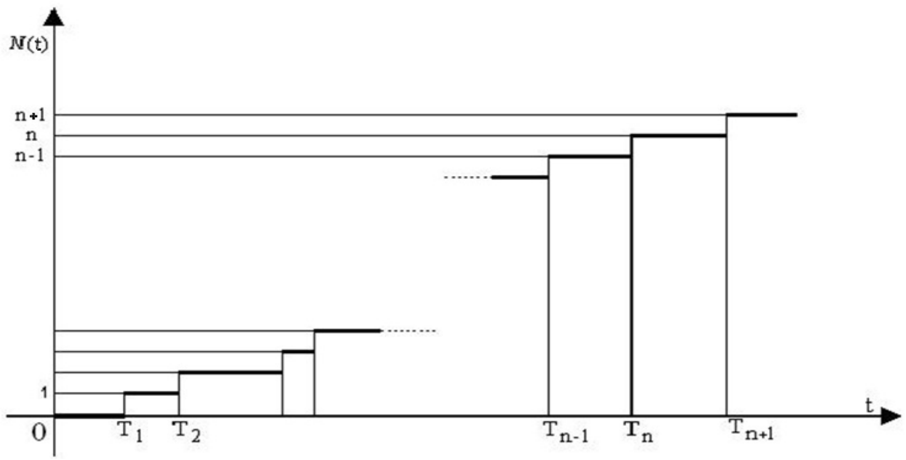
The contracts for health and temporary disability insurance have two possible states health or ill (disable). When the insured is healthy then he will pay the premium to the insurance company. If he is ill then he will receive the claim payments by the insurance company. The kind of the payment that he will receive will be function of contract. Certainly, he will receive the reimbursement of the expenses that he will pay because of the illness. Furthermore, he could receive for each day or week or month a sum of money.

In our opinion, the most important time variable for the health and temporary insurance is the age. Indeed, the healing time and also the probability to be ill changes in function of the age. Regarding the probability of a temporary disability the most important influence of age is the recovering time.

RENEWAL PROCESSES

Renewal processes work in this way. We have a phenomenon that will be verified but we do not know when. As soon as the studied phenomenon happens the system is suddenly renewed and it restarts

Homogeneous case: the system restarts with the same initial characteristics
Non-homogeneous case: the system takes into account the time in which it restarts
In homogeneous case the duration of the renewal time is relevant
In non-homogeneous case the model takes into account the starting and ending time of the renewal.
It results that simple actuarial models can be well simulated by this kind of processes.
Only recently the non-homogeneous renewal processes were defined in a general way (Gismondi et al. (2015))



HOMOGENEOUS & NON-HOMOGENEOUS RENEWAL PROCESSES

$(X_n, \quad n \geq 1)$ n.n.i.i.d. r.v. defined on (Ω, F, P) $T_0 = 0, T_n = X_1 + \cdots + X_n, n \geq 1,$ Renewal process

$X_n, \quad n \geq 1$ Interarrival times $T_n, \quad n \geq 1$ Renewal Times

$(N(t), \quad t \geq 0)$ Renewal counting process $N(t) > n-1 \Leftrightarrow T_n \leq t, \qquad n \in \mathbb{N}$

H. renewal function $H(t) = \mathrm{E}[N(t)]$ N.H. renewal function $H(s,t) = \mathrm{E}[N(t) - N(s)]$

Homogeneous D.F. Non-Homogeneous D.F.

$F(t), \quad F(0) < 1, \quad F(\infty) = 1$ $F(s,t) = P(X_n \leq t - s | T_{n-1} = s), 0 < s \leq t; \; F(s,s) > 1, F(s,+\infty) = 1$

Homogeneity particular as case of non-homogeneity: $F(s,t) = F(s+h, t+h); \; \forall 0 \leq s \leq t$ and $h: -s \leq h.$

$$H(k) = F(k) + \sum_{\tau=1}^{k} H(k-\tau) v(\tau), \; . \quad v(\tau) = \begin{cases} F(\tau) = 0 & \tau = 0 \\ F(\tau) - F(\tau - 1) & \tau > 0 \end{cases}$$ Homogeneous evolution equation

$$H(u,k) = F(u,k) + \sum_{\tau=u+1}^{k} H(\tau, k) v(u,\tau). \quad v(u,\tau) = \begin{cases} F(u,\tau) = 0 & \tau = u, \\ F(u,\tau) - F(u,\tau-1) & \tau > u, \end{cases}$$ Non-homogeneous evolution equation

In non-homogeneous case $(X_n, \quad n \geq 1)$ Are n.n. conditional independent r.v. $\mathrm{P}(R \cap B / Y) = \mathrm{P}(R/Y)\mathrm{P}(B/Y)$

HOMOGENEOUS ALTERNATING PROCESSES I

Renewals can start after a non-negligible random time

Taking into account this phenomenon, defining a renewal process in which the renewal time after the failure is assumed being an integer non-negative random variable

$\{Y_1, Y_2, \ldots\}$ and $\{Z_1, Z_2, \ldots\}$ two independent sequences of (i.i.d.n.n.) r.v.

$\mathbf{X} = \{(Y_1, Z_1), (Y_2, Z_2), \ldots, (Y_k, Z_k), \ldots\},\ Y_i, Z_i \in \mathbb{R}^+,\ i \in \mathbb{N},$ Y_i denotes the i^{th} working period in a Up state

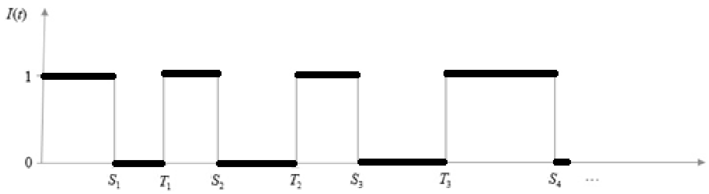
In insurance the Up state corresponds to the healthy Z_i denotes the i^{th} non-working period in a Down state.

$S_1 = Y_1,\ S_n = \sum_{i=1}^{n-1}(Y_i + Z_i) + Y_n, n = 2,3,\ldots;\ S_n \in \mathbb{N}$ $T_n = \sum_{i=1}^{n}(Y_i + Z_i), n = 1,2,\ldots;\ T_n \in \mathbb{N}$ $S_n = T_{n-1} + Y_n;\ T_n = S_n + Z_n;$

$$E[T_n] = nE[Y] + nE[Z];\ E[S_n] = nE[Y] + (n-1)E[Z].$$

indicator variable $I(t) = \begin{cases} 0 & \text{if} \quad t \in [S_n, T_n) \\ 1 & \text{if} \quad t \in ((0, S_1) \vee [T_n, S_{n+1})) \end{cases}$

Trajectory of $I(t)$



HOMOGENEOUS ALTERNATING PROCESSES II

Y_i and Z_i are n.n.i.i.d. r.v. $F_Y(Y)$ & $F_Z(Z)$ are c.d.f. of Y_i & Z_i; $N_Y(t)$ & $N_Z(t)$ are the random numbers of failures and renewals that happened in $(0, t]$.

$$F_{YZ} = F_Y * F_Z$$

$$H_Y(t) = \mathrm{E}(N_Y(t)) \;\&\; H_Z(t) = (N_Z(t))$$

represent the renewal functions of the discrete time alternating renewal process

$$\mathrm{E}(N_Z(t)) = \sum_{n=1}^{\infty} F_{YZ}^{(n)}(t) = H_Z(t) = F_{YZ}(t) + H_Z(t); \quad \mathrm{E}(N_Y(t)) = H_Y(t) = F_Y(t) + (F_{YZ}) * H_Y(t)), \; t = 1, \ldots, T$$

(see Beichelt 2006)

THE DATA

We did not have data

For the application we mixed data from public Italian disability (also temporary) insurance and Spanish private health insurance data

We obtained the Y and Z D.F.

Alternating renewal D.F.		
year	F_S	F_T
3	0.160069	0.077669
6	0.398304	0.238632
9	0.584982	0.411431
12	0.707825	0.553571
15	0.794232	0.664585
18	0.856648	0.751378
21	0.900255	0.817626
24	0.930225	0.866798
27	0.950925	0.902815
30	0.965014	0.928876
33	0.974507	0.947488
36	0.981278	0.960931
39	0.986685	0.971115
42	0.99247	0.980039
45	1	1

The time interval was 3 years because of data

ALTERNATING PREMIUMS & CLAIM AMOUNTS

Alternating Comp. process		
year	C_1	C_2
3	215.7945	-51.17219
6	560.2663	-268.4785
9	1886.558	-866.2952
12	3244.654	-1779.064
15	6172.119	-3767.592
18	12833.92	-6097.254
21	20251.07	-8593.849
24	26038.33	-12920.41
27	35209.08	-17820.79
30	47948.73	-23014.83
33	57017.84	-29670.49
36	62633.62	-37631.34
39	67490.75	-49190.22
42	75359.8	-59300.49

NON-HOMOGENEOUS ALTERNATING PROCESSES I

$\{Y_1, Y_2, \ldots\}$ and $\{Z_1, Z_2, \ldots\}$ two sets of r.v. that are supposed to be two independent sequences of conditionally independent not identically defined non-negative r.v

$\mathbf{X} = \{(Y_1, Z_1), (Y_2, Z_2), \ldots, (Y_k, Z_k), \ldots\}$ $Y_i, Z_i \subseteq \mathbb{R}^+$, $i \in \mathbb{N}$ non-homogeneous alternating renewal process.

$$S(s, s+1) = Y_{s+1},\ S(s, s+n) = \sum_{i=1}^{n-1} (Y_{s+i} + Z_{s+i}) + Y_{s+n}, s \in \mathbb{N}, n = 2, 3, \ldots;\ S(s, s+n) \in \mathbb{N}$$

$$T(s, s+n) = \sum_{i=1}^{n} (Y_{s+i} + Z_{s+i}), s \in \mathbb{N}, n = 1, 2, \ldots;\ T(s, s+n) \in \mathbb{N}$$

$S(s, s+n)$ Times of the n^{th} failure given the starting time s

$T(s, s+n)$ Times of the n^{th} renewal given the starting time s

It is well known that given two independent and not identically defined r.v. it results $P(Y_i + Z_i) = P(Z_i + Y_i)$.

$\forall \left(Y_i, Z_i\right) \in \mathbf{X}$ it results that $F_{Y_i Z_i} = F_{Y_i} * F_{Z_i} = F_{Z_i} * F_{Y_i} = F_{Z_i Y_i}$.

NON-HOMOGENEOUS ALTERNATING PROCESSES II

$$T(s,s_n)=S(s,s_n)+Z_{s_n} \ \ \& \ S(s,s_n)=T(s,s_{n-1})+Y_{s_n}.$$

$$E\left[S(s,s_n)\right]=E\left[Y_s+\sum_{i=1}^{n-1}(Z_{s_i}+Y_{s_i})\right]=\sum_{i=1}^{n-1}E\left[Z_{s_i}\right]+\sum_{i=0}^{n}E\left[Y_{s_i}\right]$$

$$E\left[T(s,s_n)\right]=E\left[\sum_{i=1}^{n}(Y_{s_i}+Z_{s_i})\right]=\sum_{i=1}^{n}E\left[Y_{s_i}\right]+\sum_{i=0}^{n}E\left[Z_{s_i}\right]$$

NON-HOMOGENEOUS ALTERNATING PROCESSES III

$$T(s,s_n) = S(s,s_n) + Z_{s_n} \ \& \ S(s,s_n) = T(s,s_{n-1}) + Y_{s_n}.$$

$$F_{S(s,s_n)}(s,t) = P(S(s,s_n) \leq t \mid F_Y(s,s) = 0);$$

$$F_{T(s,s_n)}(s,t) = P(Z \leq t \mid F_Z(s,s) = 0)$$

$N_f(s,t)$ & $N_r(s,t)$ are the random numbers of failures and renewals that happened in $(s,t]$ respectively.

$$i - \mathrm{E}\left[N_r(t) - N_r(s)\right] = H_r(s,t) = \sum_{n=1}^{\infty} n\left(F_{YZ}^{(n)}(s,t) - F_{YZ}^{(n+1)}(s,t)\right) = \sum_{n=1}^{\infty} F_{YZ}^{(n)}(s,t) = F_{YZ}(s,t) + H_r * F_{YZ}(s,t);$$

$$ii - \mathrm{E}\left[N_f(t) - N_f(s)\right] = H_f(s,t) = \sum_{n=1}^{\infty} F_Y * \left(F_Y * F_Z\right)^{(n-1)}(s,t) = F_Y(s,t) + \sum_{n=2}^{\infty} F_Y * \left(F_Y * F_Z\right)^{(n-1)}(s,t)$$

$$= F_Y(s,t) + \sum_{n=1}^{\infty} F_Y * \left(F_Y * F_Z\right)^{(n)}(s,t)$$

For the proof of *i* see Gismondi et al. (2015)

NON HOMOGENEOUS D.F. OF HAVING CLAIMS

10	0	1	2	3	4	5	6	7	8	9	10
0	0.020312	0.269774	0.298637	0.4398	0.494343	0.612426	0.664135	0.820701	0.870884	0.97806	1
1	0	0.03507	0.270311	0.31539	0.497784	0.528349	0.691497	0.72894	0.926087	0.956382	1
2	0	0	0.033828	0.2877	0.336119	0.540305	0.589614	0.781902	0.818078	0.958734	1
3	0	0	0	0.016225	0.284712	0.331977	0.561905	0.592657	0.881057	0.97037	1
4	0	0	0	0	0.158225	0.321213	0.431488	0.779834	0.797539	0.985294	1
5	0	0	0	0	0	0.177132	0.461443	0.552113	0.838577	0.966408	1
6	0	0	0	0	0	0	0.207775	0.527553	0.693146	0.991319	1
7	0	0	0	0	0	0	0	0.07731	0.69908	0.999454	1
8	0	0	0	0	0	0	0	0	0.2990452	0.953227	1
9	0	0	0	0	0	0	0	0	0	0.585349	1
10	0	0	0	0	0	0	0	0	0	0	0

Probability of having at least 1 claim from time s to time n. The first s=0

THE AGGREGATE CLAIM AMOUNTS

10	0	1	2	3	4	5	6	7	8	9	10
0	-960.586	-1870.77	-2689.83	-3427.97	-4090.92	-4664.34	-5153.21	-5548.55	-5851.55	-6052.71	-6527.22
1	0	-910.181	-1729.25	-2467.38	-3130.33	-3703.75	-4192.62	-4587.97	-4890.96	-5092.12	-5438.17
2	0	0	-819.065	-1557.2	-2220.15	-2793.57	-3282.44	-3677.79	-3980.78	-4181.94	-4466.34
3	0	0	0	-738.137	-1401.09	-1974.51	-2463.38	-2858.72	-3161.72	-3362.87	-3609.38
4	0	0	0	0	-662.95	-1236.37	-1725.24	-2120.59	-2423.58	-2624.74	-2844.45
5	0	0	0	0	0	-573.42	-1062.29	-1457.64	-1760.63	-1961.79	-2109.16
6	0	0	0	0	0	0	-488.871	-884.216	-1187.21	-1388.37	-1488.35
7	0	0	0	0	0	0	0	-395.345	-698.339	-899.497	-963.449
8	0	0	0	0	0	0	0	0	-302.994	-504.152	-541.668
9	0	0	0	0	0	0	0	0	0	-201.158	-214.38
10	0	0	0	0	0	0	0	0	0	0	0

Aggregate claim amounts paid from time s to time n; the first s=0. All is discounted at time 0

NON-HOMOGENEOUS D.F. OF UP

	0	1	2	3	4	5	6	7	8	9	10
0	0	0.021555	0.091923	0.112368	0.361458	0.423425	0.642944	0.697922	0.886828	0.951288	1
1	0	0	0.030892	0.263912	0.290526	0.398014	0.446405	0.613022	0.691036	0.944817	1
2	0	0	0	0.041784	0.281046	0.312444	0.562247	0.609973	0.792631	0.891243	1
3	0	0	0	0	0.05408	0.244545	0.264825	0.466583	0.509319	0.720325	1
4	0	0	0	0	0	0.105032	0.476739	0.567076	0.828616	0.897488	1
5	0	0	0	0	0	0	0.150051	0.471383	0.580857	0.89425	1
6	0	0	0	0	0	0	0	0.203149	0.435566	0.5888	1
7	0	0	0	0	0	0	0	0	0.337001	0.774156	1
8	0	0	0	0	0	0	0	0	0	0.802503	1
9	0	0	0	0	0	0	0	0	0	0	1
10	0	0	0	0	0	0	0	0	0	0	0

Non-homogeneous Distribution Function to be Up (healthy)

MEAN PREMIUMS DISCOUNTED AT TIME 0

	0	1	2	3	4	5	6	7	8	9	10
0	0	1056.446	2057.433	2958.147	3770.65	4500.099	5130.718	5668.304	6103.098	6436.033	6657.766
1	0	0	1001.986	1902.701	2714.204	3443.653	4074.272	4611.858	5046.652	5380.587	5601.32
2	0	0	0	900.7142	1712.218	2442.666	3072.286	3610.871	4045.666	4378.601	4600.334
3	0	0	0	0	811.5036	1541.952	2171.572	2709.157	3144.951	3477.886	3699.619
4	0	0	0	0	0	729.4485	1360.068	1897.653	2332.448	2665.383	2887.116
5	0	0	0	0	0	0	630.6195	1168.205	1603.999	1936.934	2157.667
6	0	0	0	0	0	0	0	537.5855	972.3798	1305.315	1527.048
7	0	0	0	0	0	0	0	0	434.7943	768.7295	989.4624
8	0	0	0	0	0	0	0	0	0	333.9352	554.6681
9	0	0	0	0	0	0	0	0	0	0	221.733
10	0	0	0	0	0	0	0	0	0	0	0

Mean premiums cashed from s to t and discounted at time 0

CONCLUSIONS

-Alternating renewal stochastic processes is a tool that is particularly efficient in the study of health and temporary disability insurance contracts
-The non homogeneous environment gives the possibility to take into account the age of insured people that is a fundamental issue in health and temporary disability setting
- The main difficulty that we encountered in the work was given by the difficulty of getting real data

FUTURE WORK ON THIS TOPIC

Acquisition of real data that is the most important issue that we should do
Definition of continuous time alternating renewal process
Numerical solution of continuous alternating renewal process
Relations between the continuous and discrete time alternating renewal processes
Definition of discrete time alternating compound renewal processes (they are a class of stochastic processes)
Definition of continuous time alternating compound renewal processes and their numerical solutions
Relations among the continuous and discrete time alternating compound renewal processes
Applications of the alternating compound renewal processes in Actuarial problems

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I Thank you for the patience

CIAO