The Herd Behavior Index

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Introduction

Herd behavior in stock markets
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► Our goal: Define an index which:
  ► reflects market’s perception of degree of co-movement of stock prices,
  ► is forward looking,
  ► is based on observed option data,
  ► is model-independent.

► We will call this index the **Herd Behavior Index (HIX)**.
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The financial market

- The usual set up:

\[(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\]

- Current time is denoted by 0. Time span is \(T\) years.

- The stock market:
  - \(n\) stocks.
  - \(X_i(T) \equiv X_i = \text{price of (dividend paying) stock } i \text{ at time } T\).

- Assumptions:
  - The market is arbitrage-free.
  - The time - 0 price of any traded contingent claim with pay-off \(A(T)\) at time \(T\) is given by

\[e^{-rT} \mathbb{E}[A(T)]\]

- \(r\) = risk-free interest rate (deterministic).
- Expectation is taken w.r.t. the (unknown) measure \(Q\).
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The financial market

Stock options

- **European options on stock** $i$:
  - Maturity $T$.
  - Arbitrage-free prices:
    \[
    C_i[K] = e^{-rT} \mathbb{E}[(X_i - K)_+] \\
    P_i[K] = e^{-rT} \mathbb{E}[(K - X_i)_+] 
    \]

- **Traded strikes for stock** $i$:
  \[
  0 = K_{i,0} < K_{i,1} < \ldots < K_{i,m_i} < K_{i,m_i+1} = F_{X_i}^{-1}(1) < \infty
  \]

- **Option prices vs. risk-neutral distributions**:
  - $F_{X_i} = \text{risk-neutral cdf of } X_i$:
    \[
    F_{X_i}(x) = 1 + e^{rT} C_i'[x+] = e^{rT} P_i'[x+]
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The financial market

Stock options

Options on Walt Disney:

Walt Disney Co: 23/01/2012
Time to maturity: 25 days

Call Prices
Put Prices
The financial market
The stock market index and its options

- **The stock market index:**
  - $S(T) \equiv S = \text{value of stock market index at time } T$:
    $$S = w_1 X_1 + \cdots + w_n X_n$$
  - $w_i = \text{positive weight factors.}$

- **European options on the index:**
  - Maturity $T$.
  - Arbitrage-free prices:
    $$C[K] = e^{-rT} \mathbb{E}[(S - K)_+]$$
    $$P[K] = e^{-rT} \mathbb{E}[(K - S)_+]$$
  - Traded strikes:
    $$K_{-1} < K_{-1+1} < \cdots < K_{-1} < K_0 \leq \mathbb{E}[S] < K_1 < \cdots < K_{h-1} < K_h$$
The financial market
The stock market index and its options

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The stock market index and its options

- European options on the DJ index:

Dow Jones: 21/03/2012
Time to maturity = 30 days

- Call Prices
- Put Prices
Risk neutral stock price distributions

- Approximate the unknown option curve $C_i[K]$ of each stock $i$ (dashed line) by the piecewise linear curve $\overline{C}_i[K]$ connecting the traded strikes (solid line).
Risk neutral stock price distributions

- The approximation $\bar{F}_{X_i}$ of $F_{X_i}$ follows from:

$$\bar{F}_{X_i}(x) = 1 + e^{rT} C'_i[x+]$$
Contingent claims on the stock market index

- \( S = \) value of stock market index at time \( T \).
- Consider the contingent claim with pay-off \( f(S) \) at \( T \).
- For any \( a \geq 0 \), this pay-off can be expressed as:

\[
\begin{align*}
    f(S) &= f(a) + f'(a) [(S - a)_+ - (a - S)_+] \\
         &+ \int_0^a f''(K) (K - S)_+ dK + \int_a^{+\infty} f''(K) (S - K)_+ dK
\end{align*}
\]

- Model-free static replication of \( f(S) \):
  - Buy \( f(a) \) bonds, each paying an amount 1 at maturity.
  - Buy \( f'(a) \) calls \( C[a] \).
  - Sell \( f'(a) \) puts \( P[a] \).
  - Buy \( f''(K) \) \( dK \) puts \( P[K] \), for \( K < a \).
  - Buy \( f''(K) \) \( dK \) calls \( C[K] \), for \( K > a \).

---

\(^2\) Carr & Madan (2001).
Contingent claims on the stock market index

- $S = \text{value of stock market index at time } T$.
- Consider the contingent claim with pay-off $f(S)$ at $T$.
- For any $a \geq 0$, this pay-off can be expressed as\(^2\):

\[
f(S) = f(a) + f'(a) \left[(S - a)_+ - (a - S)_+\right] \\
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  - Sell $f'(a)$ puts $P[a]$.
  - Buy $f''(K) \, dK$ puts $P[K]$, for $K < a$.
  - Buy $f''(K) \, dK$ calls $C[K]$, for $K > a$.

Contingent claims on the stock market index

- $S =$ value of stock market index at time $T$.
- Consider the contingent claim with pay-off $f(S)$ at $T$.
- For any $a \geq 0$, this pay-off can be expressed as:

$$f(S) = f(a) + f'(a) [(S - a)_+ - (a - S)_+]$$

$$+ \int_0^a f''(K) (K - S)_+ dK + \int_a^{+\infty} f''(K) (S - K)_+ dK$$

- Model-free static replication of $f(S)$:
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---

Contingent claims on the stock market index

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- Consider the **contingent claim** with pay-off \( f(S) \) at \( T \).
- For any \( a \geq 0 \), this pay-off can be expressed as\(^2\):

\[
    f(S) = f(a) + f'(a) \left[ (S - a)_+ - (a - S)_+ \right] \\
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\(^2\)Carr & Madan (2001).
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Forward contracts

- Consider the swap contract, underwritten at time 0, with pay-offs at time $T$ given by

$$P = \mathbb{E}[f(S)]$$

- The time-0 price of the forward contract is 0:

$$P = \mathbb{E}[f(S)]$$

- The time-0 forward price in terms of option prices:

$$\mathbb{E}[f(S)] = f(a) + e^{rT}f'(a)(C[a] - P[a])$$
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Forward contracts

- **Traded strikes**: \( K_{-l}, K_{-l+1}, \ldots, K_{-1}, K_0, K_1, \ldots, K_{h-1}, K_h \).
- **Approximation for \( \mathbb{E} [f(S)] \)**: (composite trapezoidal rule)

\[
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\]
Perfect herd behavior

- Perfect herd behavior = comonotonicity:
Perfect herd behavior

- Perfect herd behavior = comonotonicity\(^3\):
  \[
  (X_1, \ldots, X_n) \overset{d}{=} \left( F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U) \right)
  \]

  - \(U\) is a uniform \((0, 1)\) r.v.

- Comonotonicity gap\(^4\):
  - Distance between observed market situation and comonotonic situation.
  - Problem: how to capture this gap in a single number?

---

\(^3\)D, Denuit, Goovaerts, Kaas & Vyncke (2002a,b)

\(^4\)Laurence (2008).
Perfect herd behavior

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³D, Denuit, Goovaerts, Kaas & Vyncke (2002a,b)
⁴Laurence (2008).
The comonotonic stock market index

- The stock market index:

\[ S = \sum_{i=1}^{n} w_i X_i \]

- The comonotonic stock market index:

\[ S^c = \sum_{i=1}^{n} w_i F_{X_i}^{-1} (U) \]

- The (approximated) comonotonic stock market index\(^5\):

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Comonotonic index option prices

- Comonotonic index call option prices:
  - Definition:
    \[ \overline{C}^c[K] = e^{-rT} E[(\overline{S}^c - K)_+] \]
  - \( \overline{C}^c[K] \) can be determined from observed stock option prices\(^6\):
    \[ \overline{C}^c[K] = \sum_{i=1}^{n} w_i \overline{C}_i[K_i^*] \]
    for appropriately chosen strikes \( K_i^* \).

- Comonotonic index put option prices: \( \overline{P}^c[K] \), similar\(^7\).

---


\(^7\)Linders, D., Hounnon, Vanmaele (2012).
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Observed and comonotonic DJ - index option prices

Index Option Prices: June 23, 2000

- DJ Calls
- DJ Puts
- DJ Comonotonic Calls
- DJ Comonotonic Puts

- $S(0) = 103.76$, $T = 30$ days
- $HIX[T] = 0.28579$, $CIX[T] = 0.28375$
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Index Option Prices: June 23, 2000

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Observed and comonotonic DJ - index option prices

$S(0) = 126, \ T = 30 \text{ days}$

$HIX[T] = 0.36946, \ CIX[T] = 0.36528$
Observed and comonotonic DJ - index option prices

\[ S(0) = 126, \ T = 30 \text{ days} \]

\[ \text{HIX}[T] = 0.36946, \ \text{CIX}[T] = 0.36528 \]
$S(0) = 86.91, \quad T = 30 \text{ days}$

$\text{HIX}[T] = 0.74353, \quad \text{CIX}[T] = 0.72836$
- $S(0) = 86.91$, $T = 30$ days
- $HIX[T] = 0.74353$, $CIX[T] = 0.72836$
Comonotonic forward prices

Expected value of functions of the comonotonic index price:

\[
\mathbb{E} \left[ f \left( \bar{S}^c \right) \right] = f \left( \mathbb{E} \left[ S \right] \right) + e^{rT} \int_{0}^{\mathbb{E}[S]} f''(K) \, \bar{P}^c \left[ K \right] \, dK \\
+ e^{rT} \int_{\mathbb{E}[S]}^{+\infty} f''(K) \, \bar{C}^c \left[ K \right] \, dK
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Approximation:

\[
\mathbb{E} \left[ f \left( \bar{S}^c \right) \right] \approx f \left( \mathbb{E} \left[ S \right] \right) + e^{rT} \sum_{i=-l}^{h} f''(K_i) \, \Delta K_i \, \bar{Q}^c \left[ K_i \right] \\
- \frac{f''(K_0)}{2} \left( \mathbb{E} \left[ S \right] - K_0 \right)^2
\]

- \( \Delta K_i \) defined as above.
- \( \bar{Q}^c \left[ K_i \right] \) defined by

\[
\bar{Q}^c \left[ K_i \right] = \begin{cases} 
\bar{P}^c \left[ K_i \right], & \text{if } K_i < K_0, \\
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Comonotonic forward prices

- Expected value of functions of the comonotonic index price:

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\mathbb{E} \left[ f \left( S^c \right) \right] = f(\mathbb{E}[S]) + e^{rT} \int_{0}^{\mathbb{E}[S]} f''(K) \bar{P}^c[K] \, dK \\
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\]
The implied degree of herd behavior

- Measuring the degree of herd behavior: \( f \) convex

\[
\text{Degree of Herd Behavior} = \frac{\mathbb{E}[f(S)]}{\mathbb{E}[f(S^c)]}
\]

- Definition of the Herd Behavior Index\(^8\):

\[
\text{HIX}_f[T] = \frac{\text{approximation for } \mathbb{E}[f(S)]}{\text{approximation for } \mathbb{E}[f(S^c)]}
\]

- **Nominator:**
  - Captures real market situation.
  - Follows from observed index option prices.

- **Denominator:**
  - Captures comonotonic market situation.
  - Follows from observed stock option prices.

---

\(^8\) Linders, D., Kukush (2012).
The implied degree of herd behavior

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- **Nominator:**
  - Captures real market situation.
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\(^8\)Linders, D., Kukush (2012).
The implied degree of herd behavior

- **Measuring the degree of herd behavior:** \((f\text{ convex})\)

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  \text{Degree of Herd Behavior} = \frac{\mathbb{E}[f(S)]}{\mathbb{E}[f(S^c)]}
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- **Definition of the Herd Behavior Index\(^8\):**

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The implied degree of herd behavior

Example

► The function \( f \):

\[
f (s) = (s - \mathbb{E}[S])^2
\]

► Measuring the degree of herd behavior:

Degree of Herd Behavior = \( \frac{\text{Var} [S]}{\text{Var} [S^c]} \)

► Definition of the HIX:

\[
\text{HIX} [T] = \frac{\text{approximation for Var} [S]}{\text{approximation for Var} [\bar{S}^c]}
\]

\[
= \frac{2e^{rT} \sum_{i=-I}^{h} \Delta K_i \ Q[K_i] - (\mathbb{E}[S] - K_0)^2}{2e^{rT} \sum_{i=-I}^{h} \Delta K_i \ \bar{Q}^c [K_i] - (\mathbb{E}[S] - K_0)^2}
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Implementation considerations

- Suppose we want to calculate daily values of DJ - HIX\([T]\) for \(T = 30\) calendar days.
- In general, options with maturity \(T\) are not available:
  - Consider near term \(T_1\) and next term \(T_2\) options.
  - HIX\([T]\) is a weighted average:

\[
HIX\,[T] = HIX\,[T_1] \times \left(\frac{T_2 - T}{T_2 - T_1}\right) + HIX\,[T_2] \times \left(\frac{T - T_1}{T_2 - T_1}\right)
\]

- Roll to 2\(^{nd}\) and 3\(^{th}\) expiry dates if \(T_1\) is less than 7 days.
- European index options and American stock options:
  - Replace (non-observed) \(C_i\,[K_{i,j}]\) and \(P_i\,[K_{i,j}]\) by corresponding observed American option prices.
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The DJ-HIX

Historical herd behavior over time

- Values of DJ-HIX for $T = 30$ calendar days.
The DJ-HIX

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- Values of **DJ-HIX** for \( T = 30 \) calendar days.
The DJ-HIX
Recent herd behavior over time


- Recent herd behavior: www.kuleuven.be/insurance.
The DJ-HIX

Recent herd behavior over time

- **Time period:** October 24, 2011 - September 27, 2012.

![Dow Jones Industrial Average](image1)

![Market implied degree of herd behavior](image2)

- **Recent herd behavior:** [www.kuleuven.be/insurance](http://www.kuleuven.be/insurance).
Implied herd behavior and the VIX methodology

- **Variance swap contract** (set up at 0, pay-offs at $T$):

  $\text{Fixed leg}$

  $\text{Floating leg}$

  \[ SR[T] \]

  \[ RV[T] \]

- **The Realized Variance**:

  \[ RV[T] = \frac{1}{T} \sum_{j=1}^{365} \left( \ln S \left( \frac{j}{365} \right) - \ln S \left( \frac{j-1}{365} \right) \right)^2 \]

- **The Swap Rate**:

  \[ SR[T] = \mathbb{E}[RV[T]] \]
Implied herd behavior and the VIX methodology

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Implied herd behavior and the VIX methodology

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  $$SR[T] = \mathbb{E}[RV[T]]$$
Implied herd behavior and the VIX methodology

- **Measuring the degree of herd behavior:**

  \[
  \text{Degree of Herd Behavior} = \frac{SR[T]}{SR^c[T]}
  \]

- **The Comonotonicity Index (CIX)\(^9\):**

  \[
  \text{CIX}[T] = \frac{\text{approximation of } SR[T]}{\text{approximation of } SR^c[T]}
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DJ-HIX vs. DJ-CIX

- Values of **DJ-HIX** and **DJ-CIX** for $T = 30$ calendar days.
DJ-HIX vs. DJ-CIX

- Values of **DJ-HIX** and **DJ-CIX** for $T = 30$ calendar days.
- **Time period:** January, 2006 - October, 2009.
DJ-HIX vs. DJ-Volatility Index vs. DJ-Implied Correlation
Conclusions

- The HIX:
  - reflects market’s perception of future co-movement behavior of stock prices,
  - is forward looking,
  - is based on observed option data,
  - HIX is model-independent.

- High levels of the HIX are a sign of a bearish market.

- The HIX may give information about the degree of diversification that is possible by investing in a portfolio of stocks.

- Trading the comonotonicity gap.
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References

References II


