Mean-Variance Optimal Portfolios in the Presence of a Benchmark with Applications to Fraud Detection

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Contributions

▶ **Part 1: Mean-Variance efficient payoffs**

- Optimal payoffs when you only care about mean and variance
- Payoffs with maximal possible Sharpe ratio
- Application to fraud detection

▶ **Part 2: Constrained Mean-Variance efficient payoffs**

- Drawbacks of traditional mean-variance efficient payoffs
- Optimal payoffs in presence of a random benchmark
- Sharpening the maximal possible Sharpe ratios
- Application to improved fraud detection

Financial Market

- The market $(\Omega, \mathcal{F}, P)$ is arbitrage-free.
- There is a risk-free account earning $r > 0$.
- Consider a strategy with payoff X_T at time $T > 0$.
- There exists $\mathbb{Q}$ so that its initial price writes as

$$c(X_T) = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[X_T\right],$$

- Equivalently, there exists a stochastic discount factor ξ_T such that

$$\boxed{c(X_T) = \mathbb{E}_P\left[\xi_T X_T\right].}$$

- Assume ξ_T is continuously distributed.

Mean Variance Optimization

- A Mean-Variance efficient problem:

$$(\mathcal{P}_1) \qquad \begin{array}{ll} \max\limits_{X_T} & \mathbb{E}[X_T] \\ \text{subject to} & \begin{cases} \mathbb{E}[\xi_T X_T] = W_0 \\ \text{var}[X_T] = s^2 \end{cases} \end{array}$$

Proposition (Mean-variance efficient portfolios)

Let $W_0 > 0$ denote the initial wealth and assume the investor aims for a strategy that maximizes the expected return for a given variance s^2 for $s \geqslant 0$. The a.s. unique solution to $(\mathcal{P}_1)$ writes as

$$X_T^\star = a - b\xi_T,$$

where $a = \left(W_0 + b\mathbb{E}[\xi_T^2]\right) e^{rT} \geqslant 0$, $b = \frac{s}{\sqrt{var(\xi_T)}} \geqslant 0$.

Proof

Choose a and $b \geqslant 0$ such that $X_T^\star = a - b\xi_T$ satisfies the constraints $\text{var}(X_T^\star) = s^2$ and $c(X_T^\star) = W_0$.

Observe that $\text{corr}(X_T^\star, \xi_T) = -1$ and $X_T^\star$ is thus the unique payoff that is perfectly negatively correlated with ξ_T while satisfying the variance and cost constraints.

Consider any other strategy X_T which also verifies these constraints (but is not negatively linear in ξ_T). We find that

$$\text{corr}(X_T, \xi_T) = \frac{\mathbb{E}[\xi_T X_T] - \mathbb{E}[\xi_T]\mathbb{E}[X_T]}{\sqrt{\text{var}(\xi_T)}\sqrt{\text{var}(X_T)}} > -1 = \text{corr}(X_T^\star, \xi_T).$$

Since $\text{var}(X_T) = s^2 = \text{var}(X_T^\star)$ and $\mathbb{E}[\xi_T X_T] = W_0 = \mathbb{E}[\xi_T X_T^\star]$ it follows that

$$\mathbb{E}[\xi_T]\mathbb{E}[X_T] < \mathbb{E}[\xi_T]\mathbb{E}[X_T^\star],$$

which shows that $X_T^\star$ maximizes the expectation and thus solves Problem $(\mathcal{P}_1)$.

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Maximum Sharpe Ratio

- The Sharpe Ratio (SR) of a payoff X_T (terminal wealth at T when investing W_0 at $t = 0$) is defined as

$$SR(X_T) = \frac{\mathbb{E}[X_T] - W_0 e^{rT}}{\text{std}(X_T)},$$

- All mean-variance efficient portfolios $X_T^\star$ have the same maximal Sharpe Ratio ($SR^\star$) given by

$$SR^\star := SR(X_T^\star) = e^{rT}\text{std}(\xi_T),$$

$\Rightarrow$ For all portfolios X_T we have

$$SR(X_T) \leqslant e^{rT}\text{std}(\xi_T),$$

- This can be used to show Madoff's investment strategy was a fraud (Bernard & Boyle (2007)).
- Estimation of the upper bound $e^{rT}\text{std}(\xi_T)$?

Example in a Black-Scholes market

- There is a risk-free rate $r > 0$ and a risky asset with price process,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

where W_t is a standard Brownian motion, μ is the drift and σ is the volatility.

- The state-price density ξ_T is given as

$$\xi_T = e^{-rT} e^{-\theta W_T - \frac{1}{2}\theta^2 T} = \alpha S_T^{-\beta},$$

for known coefficients $\alpha, \beta > 0$ (assume $\mu > r$ and $\theta = \frac{\mu - r}{\sigma}$).

- The maximal Sharpe ratio is given by

$$SR^\star = \sqrt{e^{\theta^2 T} - 1}.$$

see Goetzmann et al. (2007) for another proof.

Improving Fraud Detection by Adding Constraints

- Detect fraud based on mean and variance only
- Ignored so far additional information available in the market.
- How to take into account the dependence features between the investment strategy and the financial market?
- Include correlations of the fund with market indices to refine fraud detection.

Ex: the so-called "market-neutral" strategy is typically designed to have very low correlation with market indices $\Rightarrow$ it reduces the maximum possible Sharpe ratio!

Improving Investment by Adding Constraints

- Optimal strategies $X_T^* = a - b\xi_T$ give their lowest outcomes when ξ_T is high. Bounded gains but unlimited losses!
- Highest state-prices $\xi_T(\omega)$ correspond to states ω of bad economic conditions as these are more expensive to insure:
  - E.g. in a Black-Scholes market: $\xi_T = \alpha S_T^{-\beta}$, $\alpha, \beta > 0$.
  - Also, $\mathbb{E}[X_T^* | \xi_T > c] < \mathbb{E}[Y_T | \xi_T > c]$, for any other strategy Y_T with the same distribution as X_T^* showing that X_T^* does not provide protection against crisis situations (event "$\xi_T > c$").
  - in a Black-Scholes market: $X_T^* = -\infty$ when $S_T = 0$.
- To cope with this observation: we impose the strategy to have some desired dependence with ξ_T, or more generally with a benchmark B_T.

Proposition (Optimal portfolio with a correlation constraint)

Let B_T be a benchmark, linearly independent from ξ_T with $0 < var(B_T) < +\infty$. Let $|\rho| < 1$ and $s > 0$. A solution to the following mean-variance optimization problem

$$(\mathcal{P}_2) \qquad \max_{\begin{cases} var(X_T) = s^2 \\ c(X_T) = W_{0,} \\ corr(X_T, B_T) = \rho \end{cases}} \mathbb{E}[X_T] \tag{1}$$

is given by $X_T^\star = a - b(\xi_T - cB_T)$, where a, b and c are uniquely determined by the set of equations

$$\begin{aligned} \rho &= corr(cB_T - \xi_T, B_T) \\ s &= b\sqrt{var(\xi_T - cB_T)} \\ W_0 &= ae^{-rT} - b(E[\xi_T^2] - cE[\xi_T B_T]). \end{aligned}$$

Proof

Observe that $f(c) := \text{corr}(cB_T - \xi_T, B_T)$ verifies $\lim_{c\to-\infty} f(c) = -1$, $\lim_{c\to+\infty} f(c) = 1$ and $f'(c) > 0$ so that $\rho = f(c)$ has a unique solution. Take $X_T^\star = a - b(\xi_T - cB_T)$ linear in $\xi_T - cB_T$ and satisfying all constraints and $b > 0$.

Consider any other X_T that satisfies the constraints and which is non-linear in $\xi_T - cB_T$, then

$$\begin{aligned}\text{corr}(X_T, \xi_T - cB_T) &= \frac{\mathbb{E}[X_T(\xi_T - cB_T)] - \mathbb{E}[\xi_T - cB_T]\mathbb{E}[X_T]}{\text{std}(\xi_T - cB_T)\text{std}(X_T)} \\ &> -1 = \text{corr}(X_T^\star, \xi_T - cB_T)\end{aligned}$$

Since both X_T and $X_T^\star$ satisfy the constraints we have that $\text{std}(X_T) = \text{std}(X_T^\star)$, $\mathbb{E}[X_T\xi_T] = \mathbb{E}[X_T^\star\xi_T]$ and $\text{cov}(X_T, B_T) = \text{cov}(X_T^\star, B_T)$. Hence the inequality holds true if and only if $\mathbb{E}[X_T^\star] > \mathbb{E}[X_T]$. □

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$S_T^\star$: Growth Optimal Portfolio (GOP)

- The **Growth Optimal Portfolio (GOP)** maximizes expected logarithmic utility from terminal wealth.
- It has the property that it **almost surely accumulates more wealth than any other strictly positive portfolios after a sufficiently long time**.
- Under general assumptions on the market, the GOP is a diversified portfolio (proxy: a world stock index).
- The GOP can be used as numéraire to price under P, so that $\xi_T = \frac{1}{S_T^\star}$

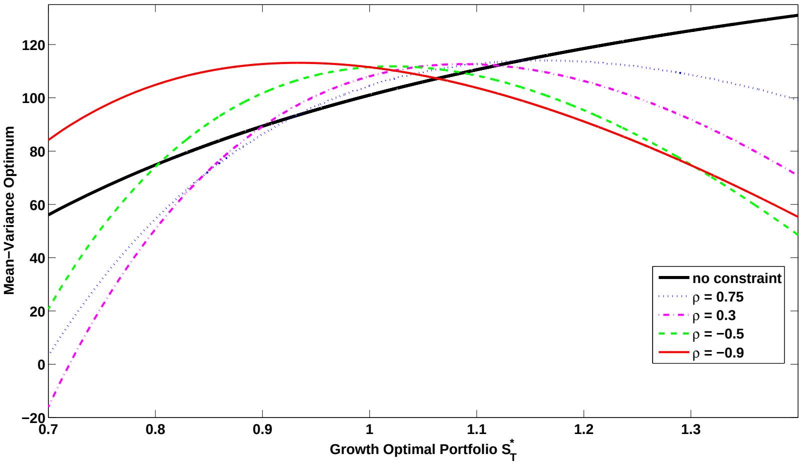
$$c(X_T) = E_P\left[\xi_T X_T\right] = E_P\left[\frac{X_T}{S_T^\star}\right]$$

  where $S_0^\star = 1$.
- Details in Platen & Heath (2006).

Example when $B_T = S_T^\star$

An optimal solution is of the form $X_T^\star = a - b(\xi_T - cS_T^\star)$, where c is computed from the equation $\rho = \text{corr}(cS_T^\star - \xi_T, S_T^\star)$, b is derived from $b = \frac{s}{\sqrt{\text{var}(\xi_T - cS_T^\star)}}$ and
$a = W_0 e^{rT} + b\left(e^{-2rT+\theta^2 T} - c\right)e^{rT}$.

Optimal payoffs as a function of the GOP for different values of the correlation ρ with the benchmark $S_T^\star$ using the following parameters: $W_0 = 100$, $r = 0.05$, $\mu = 0.07$, $\sigma = 0.2$, $T = 1$, $\theta = (\mu - r)/\sigma$, $S_0 = 100$, $s = 10$.



Mean−Variance Optimum
Growth Optimal Portfolio S_T^*
no constraint
ρ = 0.75
ρ = 0.3
ρ = −0.5
ρ = −0.9

Fraud Detection

Proposition (Constrained Maximal Sharpe Ratio)

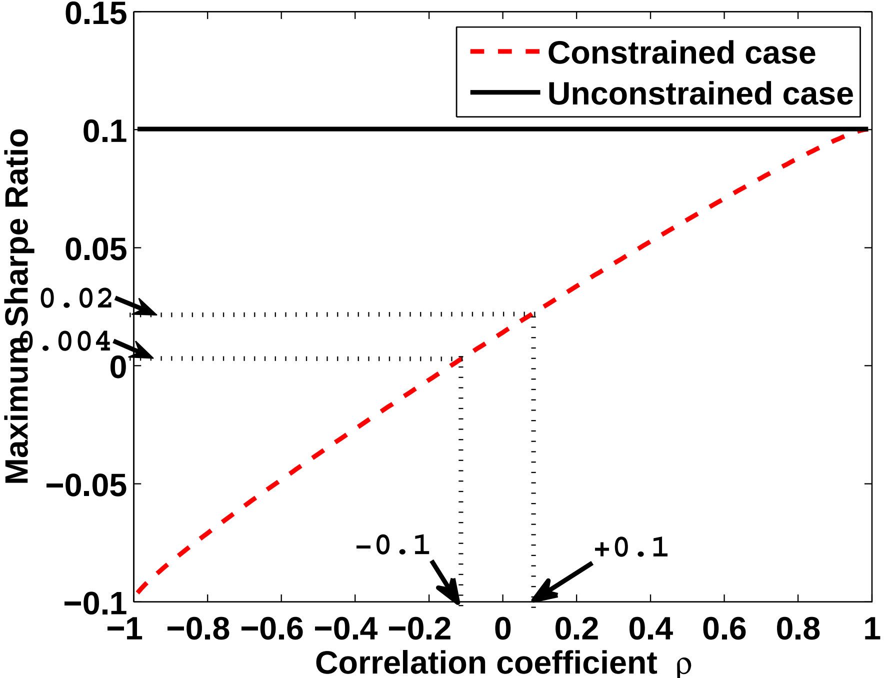
All mean-variance efficient portfolios $X_T^\star$ which satisfy the additional constraint $corr(X_T^\star, B_T) = \rho$ with a benchmark asset B_T (that is not linearly dependent to ξ_T) have the same maximal Sharpe ratio $SR_\rho^\star$ given by

$$SR_\rho^\star = e^{rT}\frac{cov(\xi_T, \xi_T - cB_T)}{std(\xi_T - cB_T)} \leqslant SR^\star = e^{rT} std(\xi_T). \qquad (2)$$

where $SR^\star$ is the unconstrained Sharpe ratio.

Illustration in the Black-Scholes model

Maximum Sharpe ratio $SR^{\star}_{\rho}$ for different values of the correlation ρ when the benchmark is $B_T = S^{\star}_T$. We use the following parameters: $W_0 = 100$, $r = 0.05$, $\mu = 0.07$, $\sigma = 0.2$, $T = 1$, $S_0 = 100$.



Constrained case
Unconstrained case
Maximum Sharpe Ratio
0.15
0.1
0.05
0.02
0.004
0
−0.05
−0.1
−0.1
+0.1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
Correlation coefficient ρ

M-V Optimization with a Benchmark

- Dependence (interaction) between X_T and B_T cannot be fully reflected by correlation.
- A useful device to do so is the copula. Sklar's theorem shows that the joint distribution of (B_T, X_T) can be decomposed as

$$P(B_T \leqslant y, X_T \leqslant x) = C(F_{B_T}(y), F_{X_T}(x)),$$

where C is the joint distribution (also called the copula) for a pair of uniform random variables over $(0,1)$. Hence, the copula C fully describes the interaction between the strategy's payoff X_T and the benchmark B_T.

- Constrained Mean-Variance efficient problem:

$$(\mathcal{P}_3) \qquad \max_{X_T} \ \mathbb{E}[X_T] \quad \text{subject to} \quad \begin{cases} \mathbb{E}[\xi_T X_T] = W_0 \\ \text{var}(X_T) = s^2 \\ C := \text{Copula}(X_T, B_T) \end{cases}$$

Proposition (Constrained Mean-Variance Efficiency)

Let $s > 0$. Assume that the benchmark B_T has a joint density with ξ_T. Define $\mathcal{A}$ as $\mathcal{A} = \left(c_{F_{B_T}(B_T)}\right)^{-1}\left[j_{F_{B_T}(B_T)}(1 - F_{\xi_T}(\xi_T))\right]$, where the functions $j_u(v)$ and $c_u(v)$ are defined as the first partial derivative for $(u, v) \to J(u, v)$ and $(u, v) \to C(u, v)$ respectively, and where J denotes the copula for the random pair (B_T, ξ_T). If $\mathbb{E}[\xi_T|\mathcal{A}]$ is decreasing in $\mathcal{A}$, then the solution to the problem

$$\max \quad \mathbb{E}[X_T] \qquad (3)$$
$$\begin{cases} var(X_T) = s^2 \\ c(X_T) = W_0 \\ C : \text{copula between } X_T \text{ and } B_T \end{cases}$$

is uniquely given as $X_T^\star = a - b\mathbb{E}[\xi_T|\mathcal{A}]$ where a, b are non-negative and can be computed explicitly.

Case $B_T = \xi_T$ can be solved similarly.

- All portfolios with copula C with B_T must now have a Sharpe Ratio bounded by

$$e^{rT}\text{std}[\mathbb{E}[\xi_T|\mathcal{A}]],$$
$$\left(\leqslant e^{rT}\text{std}[\xi_T]\right).$$

- This is useful to develop improved fraud detection schemes.

Proposition (Case $B_T = S_t^\star$)

Let W_0 denote the initial wealth and let $B_T = S_t^\star$ $(0 < t < T)$ be the benchmark. Assume that $\rho \geqslant -\sqrt{1-\frac{t}{T}}$. Then, the solution to $(\mathcal{P}_3)$ when the copula C is the Gaussian copula with correlation ρ, C_ρ^{Gauss} is given by $X_T^\star$,

$$X_T^\star = a - bG_T^c. \tag{4}$$

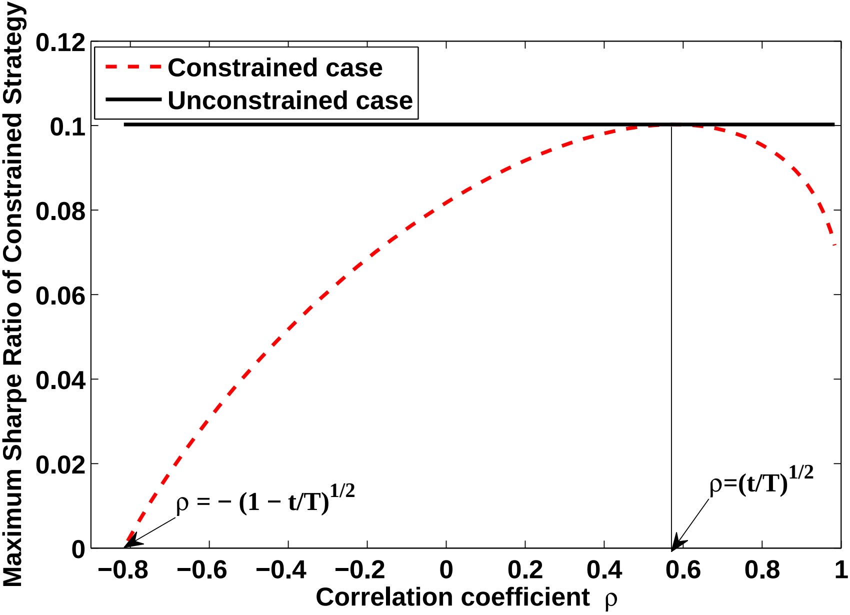
Here G_T is a weighted average of the benchmark and the GOP. It is given as $G_T = (S_t^\star)^\alpha S_T^\star$ with α,

$$\alpha = \rho\sqrt{\frac{T-t}{t}\frac{1}{1-\rho^2}} - 1.$$

Furthermore
$a = W_0 e^{rT} + be^{rT}\mathbb{E}[\xi_T G_T^c]$, $b = \frac{s}{\sqrt{var(G_T^c)}}$, $c = -\frac{\alpha t + T}{(\alpha+1)^2 t + (T-t)}$.

Illustration

- Maximum Sharpe ratio $SR^{\star}_{\rho,G}$ for different values of the correlation ρ when the benchmark is $B_T = S^{\star}_t$. We use the following parameters: $t = 1/3$, $\sqrt{t/T} = 0.577$, $-\sqrt{1-t/T} = -0.816$, $W_0 = 100$, $r = 0.05$, $\mu = 0.07$, $\sigma = 0.2$, $T = 1$, $S_0 = 100$.
- Observe that the constrained case reduces to the unconstrained maximum Sharpe ratio when the correlation in the Gaussian copula is $\rho = \sqrt{t/T}$. The reason is that the copula between the unconstrained optimum and $S^{\star}_t$ is the Gaussian copula with correlation $\rho = \sqrt{t/T}$. The constraint is thus redundant in that case.



Maximum Sharpe Ratio of Constrained Strategy
0.12
0.1
0.08
0.06
0.04
0.02
0
Constrained case
Unconstrained case
ρ = − (1 − t/T)^1/2
ρ=(t/T)^1/2
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
Correlation coefficient ρ

Conclusions

- Mean-variance efficient portfolios when there are no trading constraints
- Mean-variance efficiency with a stochastic benchmark (linked to the market) as a reference portfolio (given correlation or copula with a stochastic benchmark).
- Improved upper bounds on Sharpe ratios useful for example for fraud detection. For example it is shown that under some conditions it is not possible for investment funds to display negative correlation with the financial market and to have a positive Sharpe ratio.
- Related problems can be solved: case of multiple benchmarks, partial hedging problem...

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