Stochastic Interpolation for Stochastic Asset Models

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Assume that you have a long term stochastic model, like the Wilkie model, simulated at annual intervals.

You would like to have simulated values at shorter intervals, say monthly.

You could alter the model to a monthly one; this may work; but you may need to keep the long term features.

Or you could use stochastic interpolation.

Stochastic bridges:

Brownian bridge

based on Brownian motion

or random walk model

Ornstein-Uhlenbeck bridge

based on Ornstein-Uhlenbeck process

or AR(1) model

Notation:

Long period called a "year"

Short period a "month"

M "months" per "year"

Often M = 12

Years 0, 1, … t, t+1, …

Months in year t to t+1:

t.0, t.1, … t.k, … t.M

t.0 = t

t.M = t+1

Series X_0, X_1, …, X_t, X_{t+1}, …

Brownian bridge:

Assume random walk, so:

$$X_{t+1} = X_t + \mu_y + \sigma_y . Z_{t+1}$$

$E[Z] = 0$ Var$[Z] = 1$

Could be unit normal, or any other standardised distribution.

Put:

$$\mu_m = \mu_y/M$$

$$\sigma_m^2 = \sigma_y^2/M$$

$$Y_{t.0} = Y_t$$

Simulate for k = 0 to M − 1

$$Y_{t.k+1} = Y_{t.k} + \mu_m + \sigma_m.Z_k$$

ending with $Y_{t.M}$

$Y_{t.M}$ will (almost surely) not equal X_{t+1}

so put $d = (X_{t+1} - Y_{t.M})/12$

Adjust Y series to give

$X_{t.k} = Y_{t.k} + k \times d$

so $X_{t.M} = Y_{t.M} + 12d = X_{t+1}$

In fact you can put $\mu_m = 0$

Several things to check:

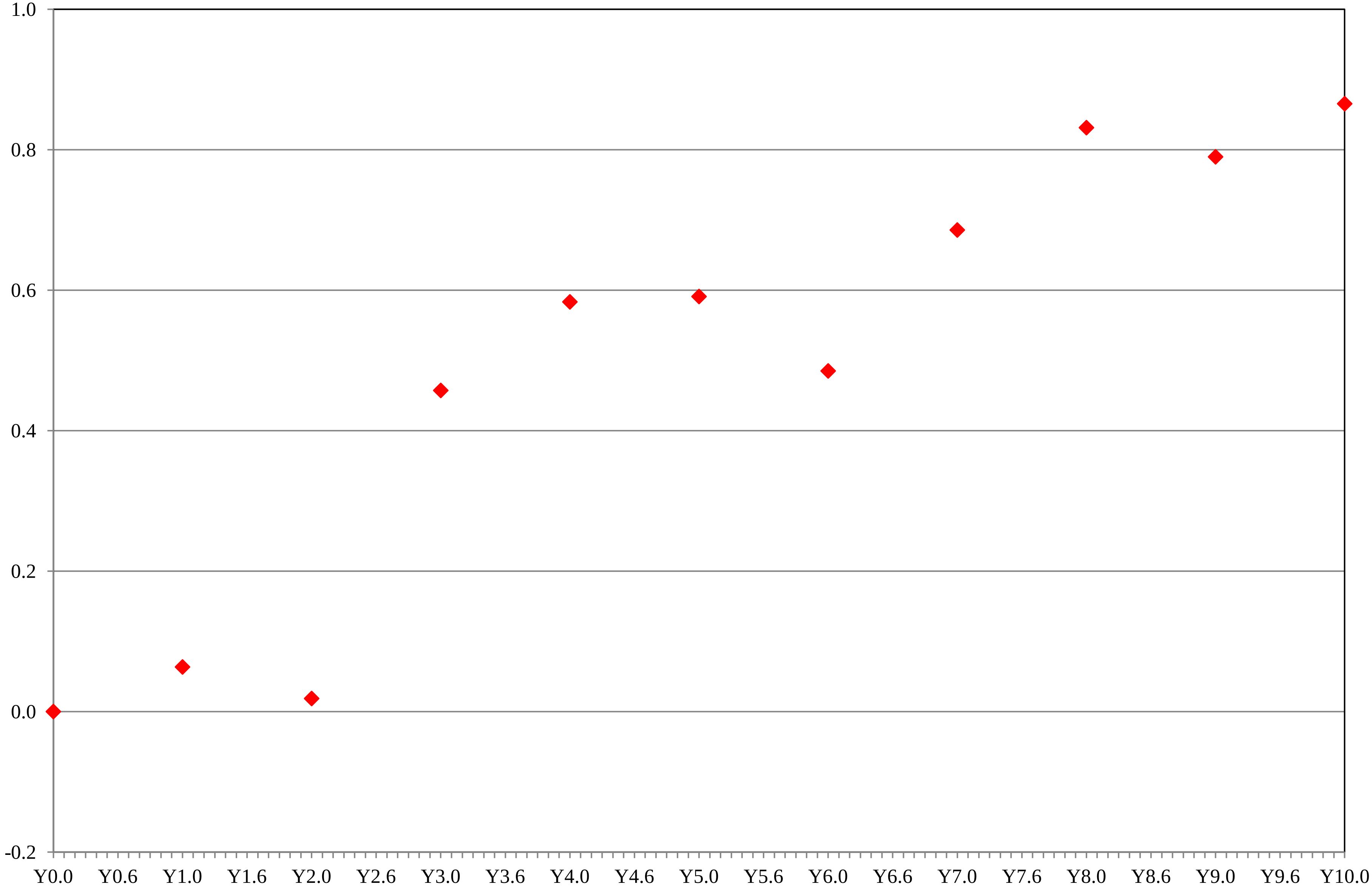
Simulate series, or use "skeleton",

i.e. non-stochastic, with $\sigma = 0$

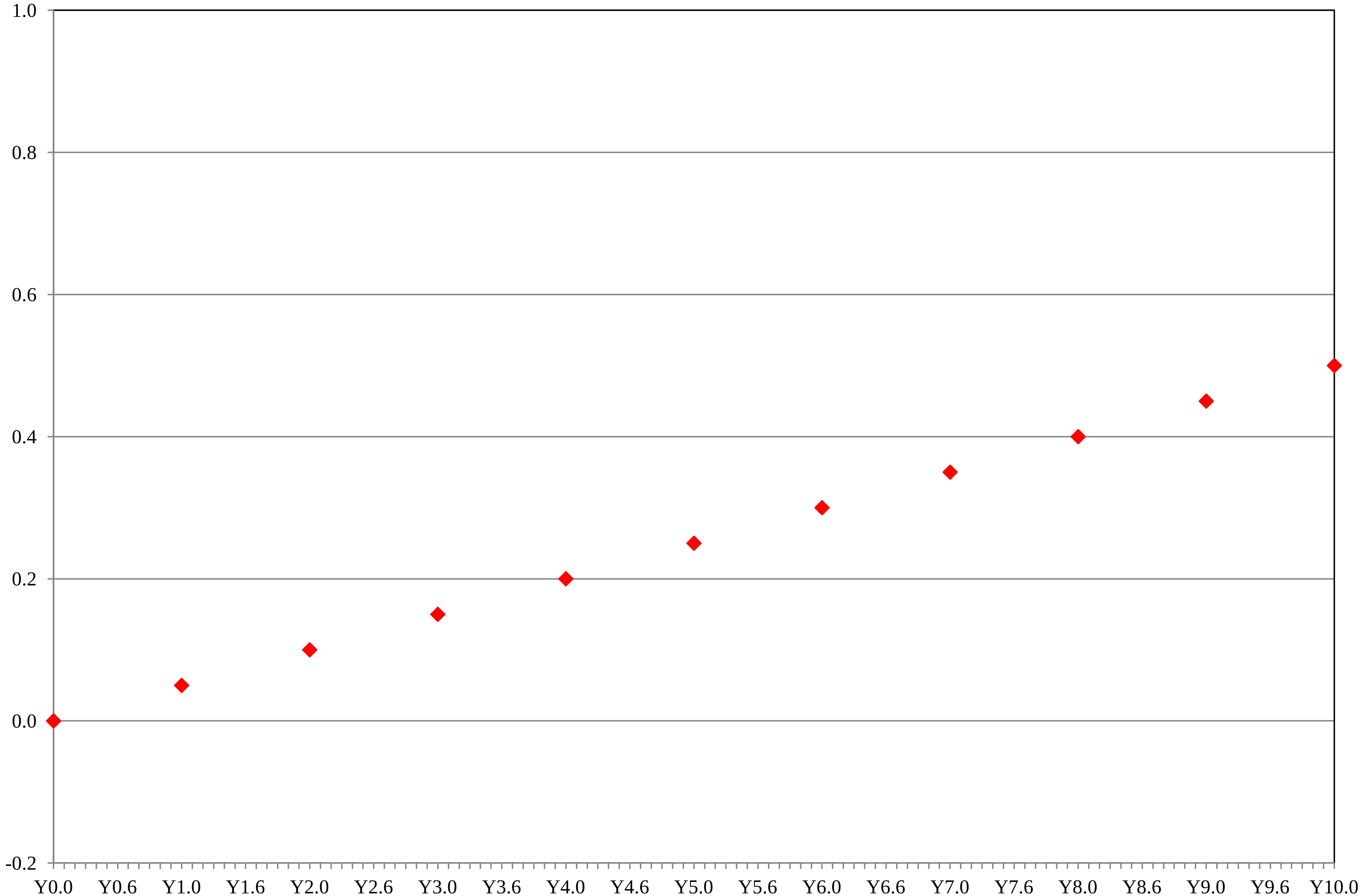
Four trials:

	Yearly	Monthly
1	Skeleton	Skeleton
2	Skeleton	Stochastic
3	Stochastic	Skeleton
4	Stochastic	Stochastic

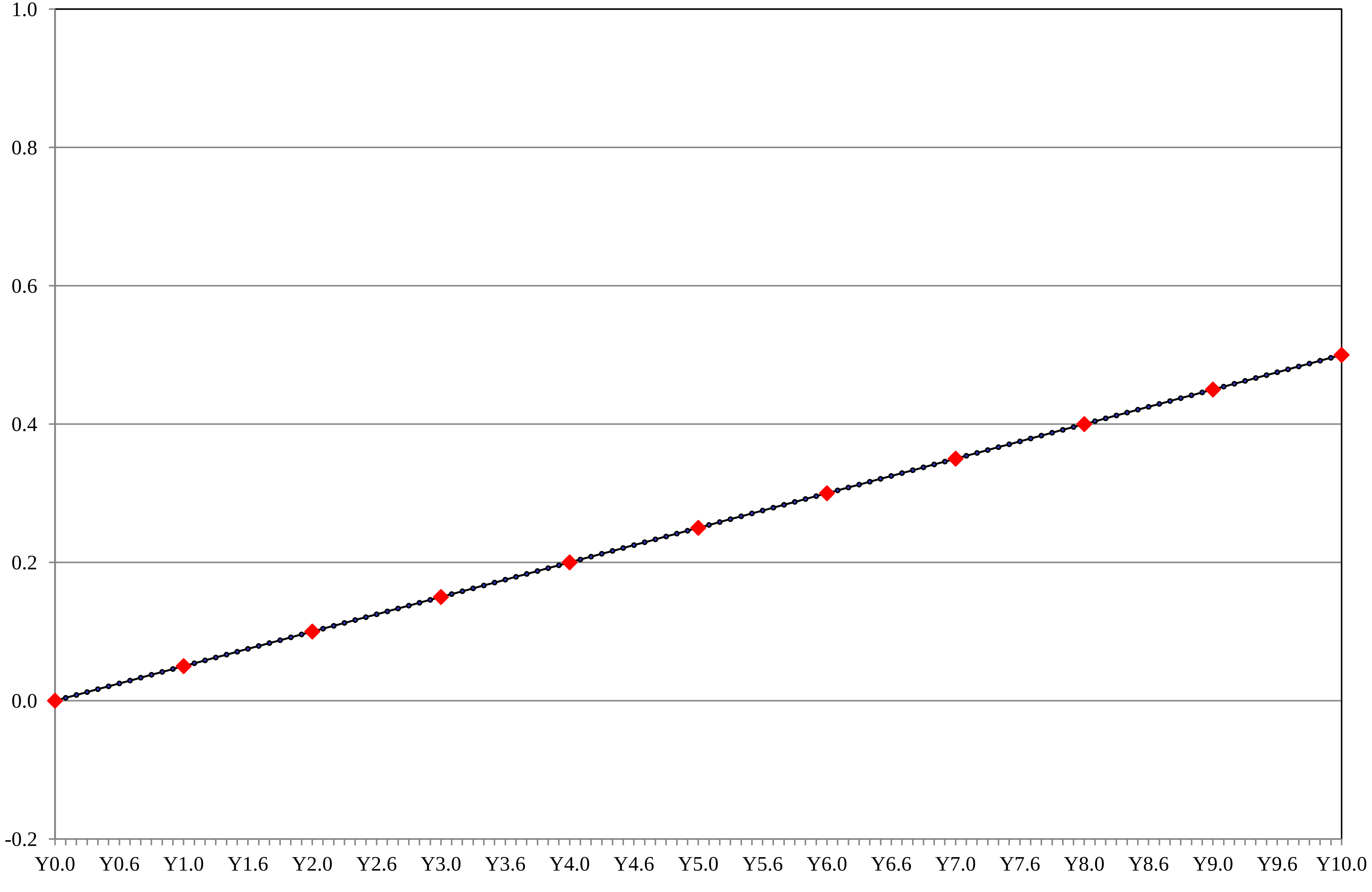
Random walk, stochastic, yearly



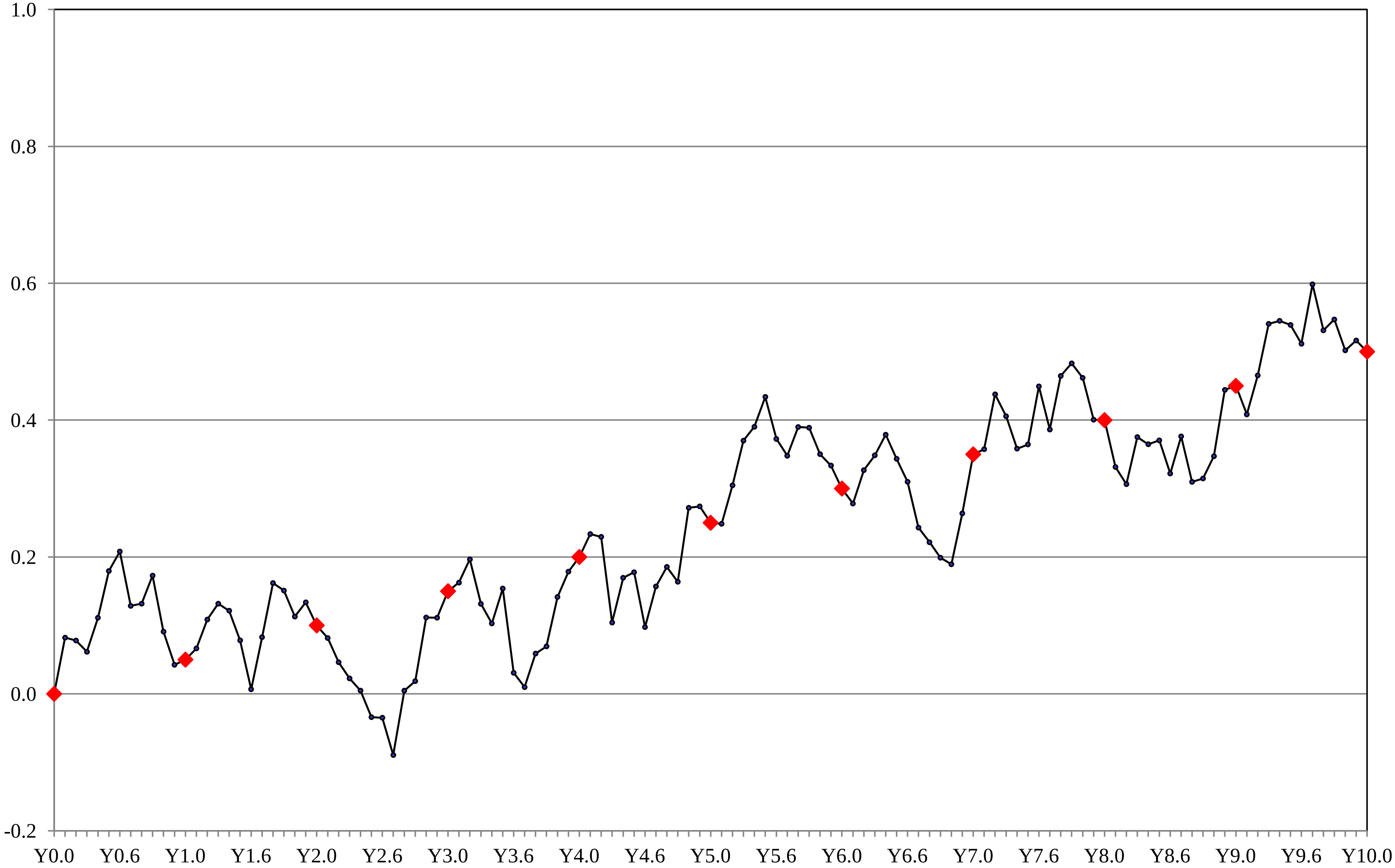
Random walk, skeleton, yearly



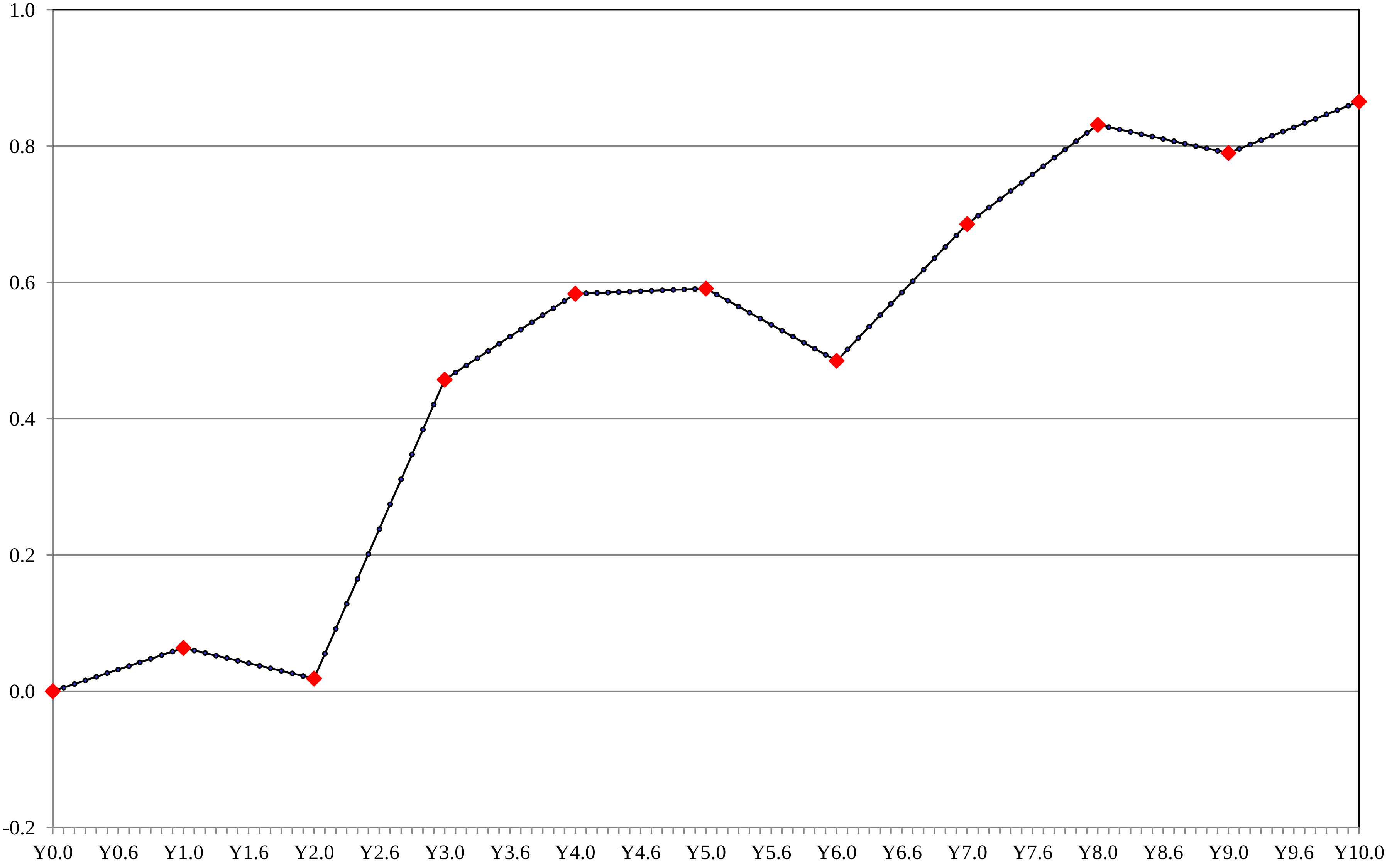
Yearly skeleton, monthly skeleton



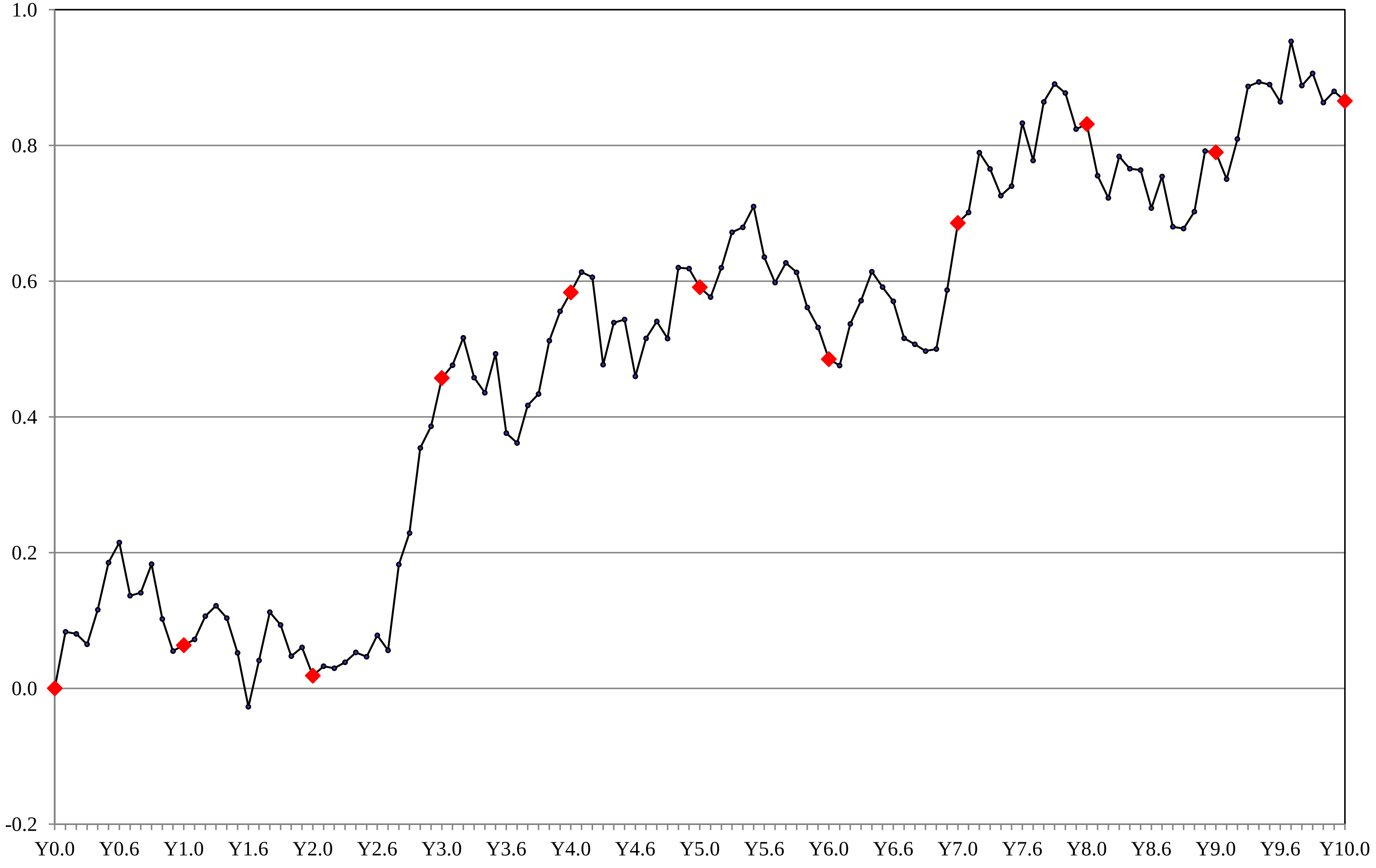
Yearly skeleton, monthly stochastic



Yearly stochastic, monthly skeleton



Yearly stochastic, monthly stochastic



Another series $X_0, X_1, \ldots, X_t, X_{t+1}, \ldots$

Ornstein-Uhlenbeck (or AR(1)) bridge:

AR(1) model:

$$X_{t+1} = \mu_y + \alpha_y.(X_t - \mu_y) + \sigma_y.Z_{t+1}$$

So autoregressive towards mean μ_y

Put:

$$\mu_m = \mu_y$$

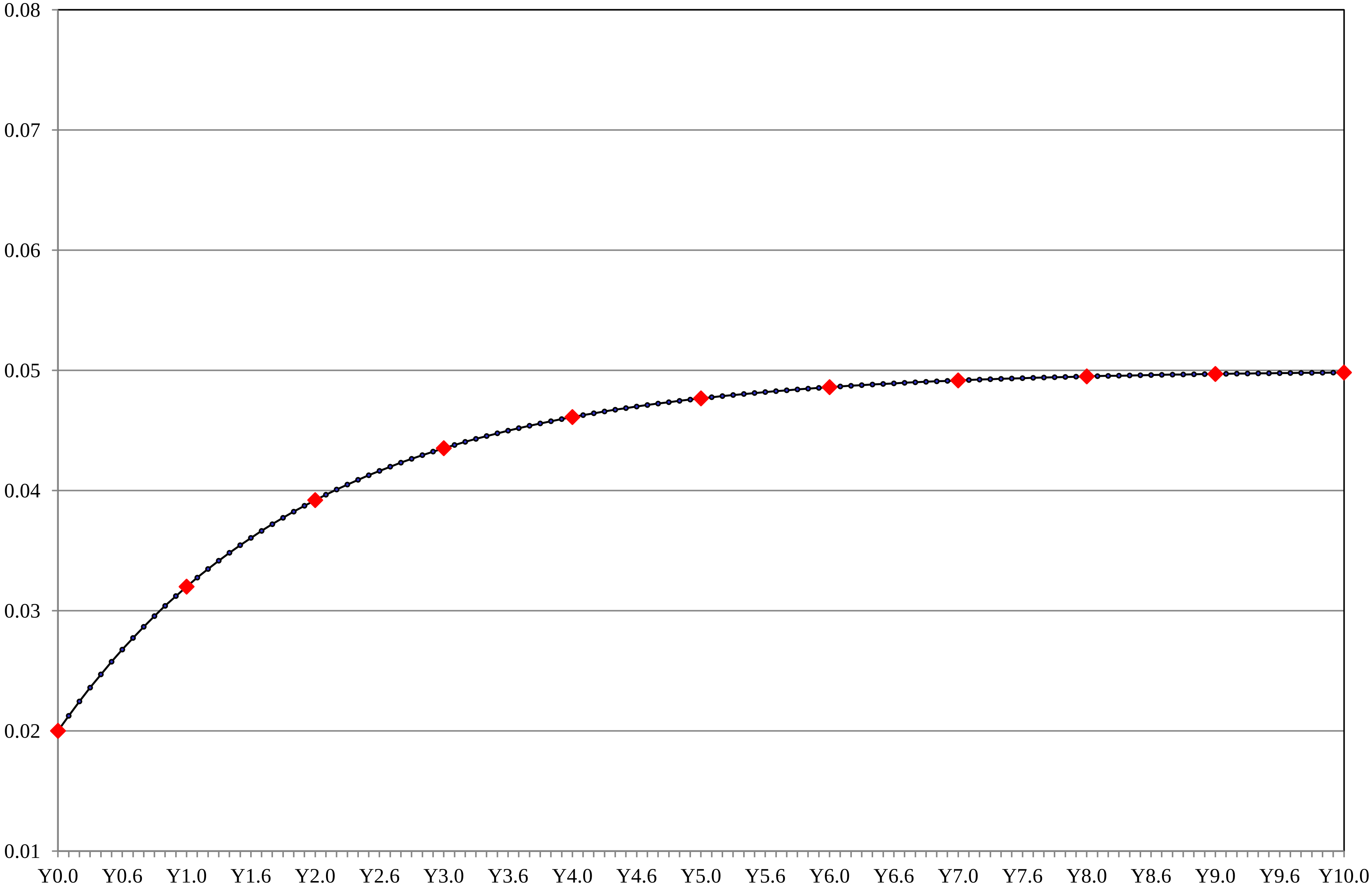
$$\alpha_m = \alpha_y^{1/M}$$

$$\sigma_m^2 = \sigma_y^2.(1 - \alpha_m^2)/(1 - \alpha_y^2)$$

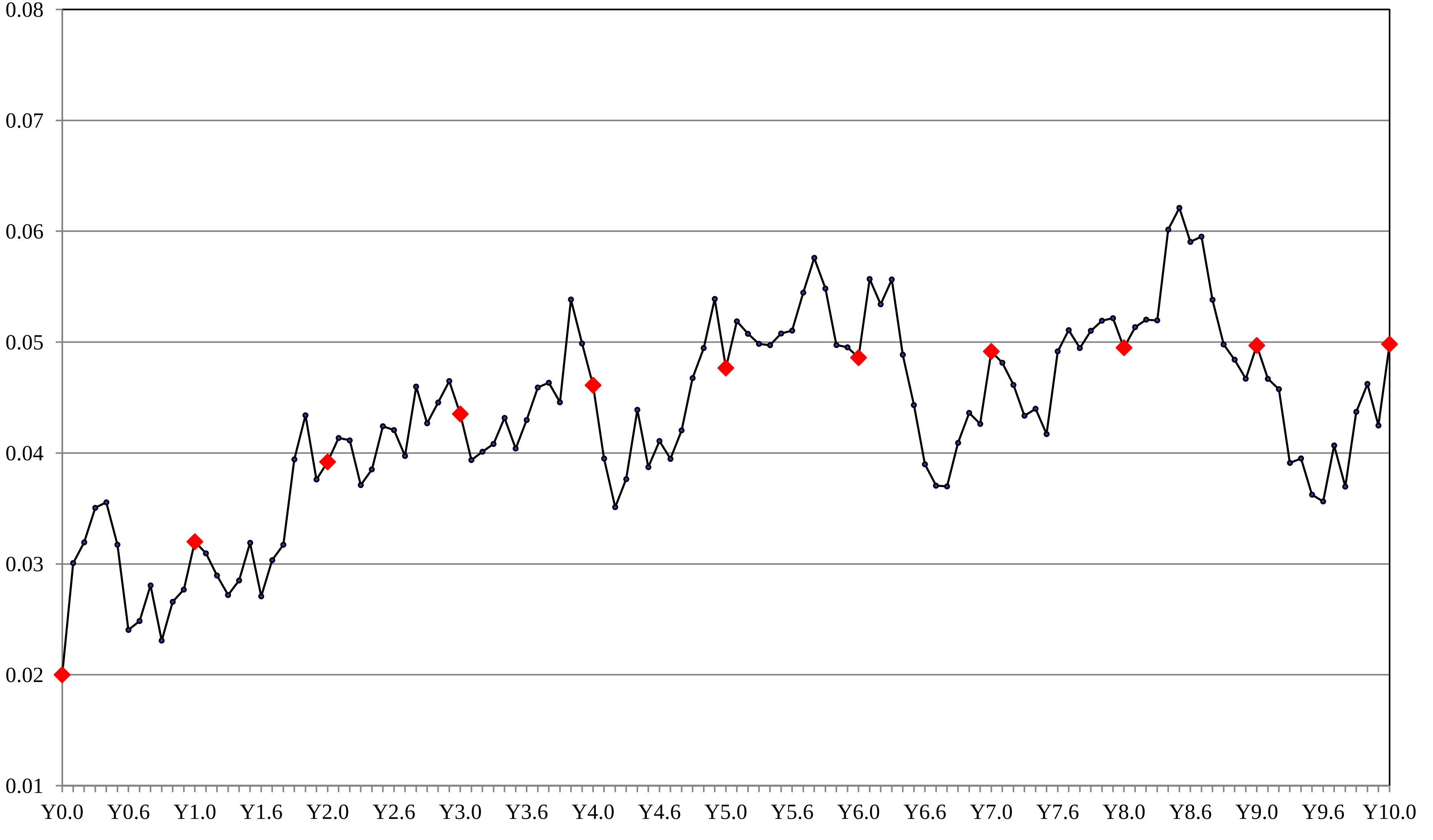
$$Y_{t.0} = Y_t$$

Simulate residuals Z_k and then with some algebra (not shown) we get a series that ends with $X_{t,M} = X_{t+1}$

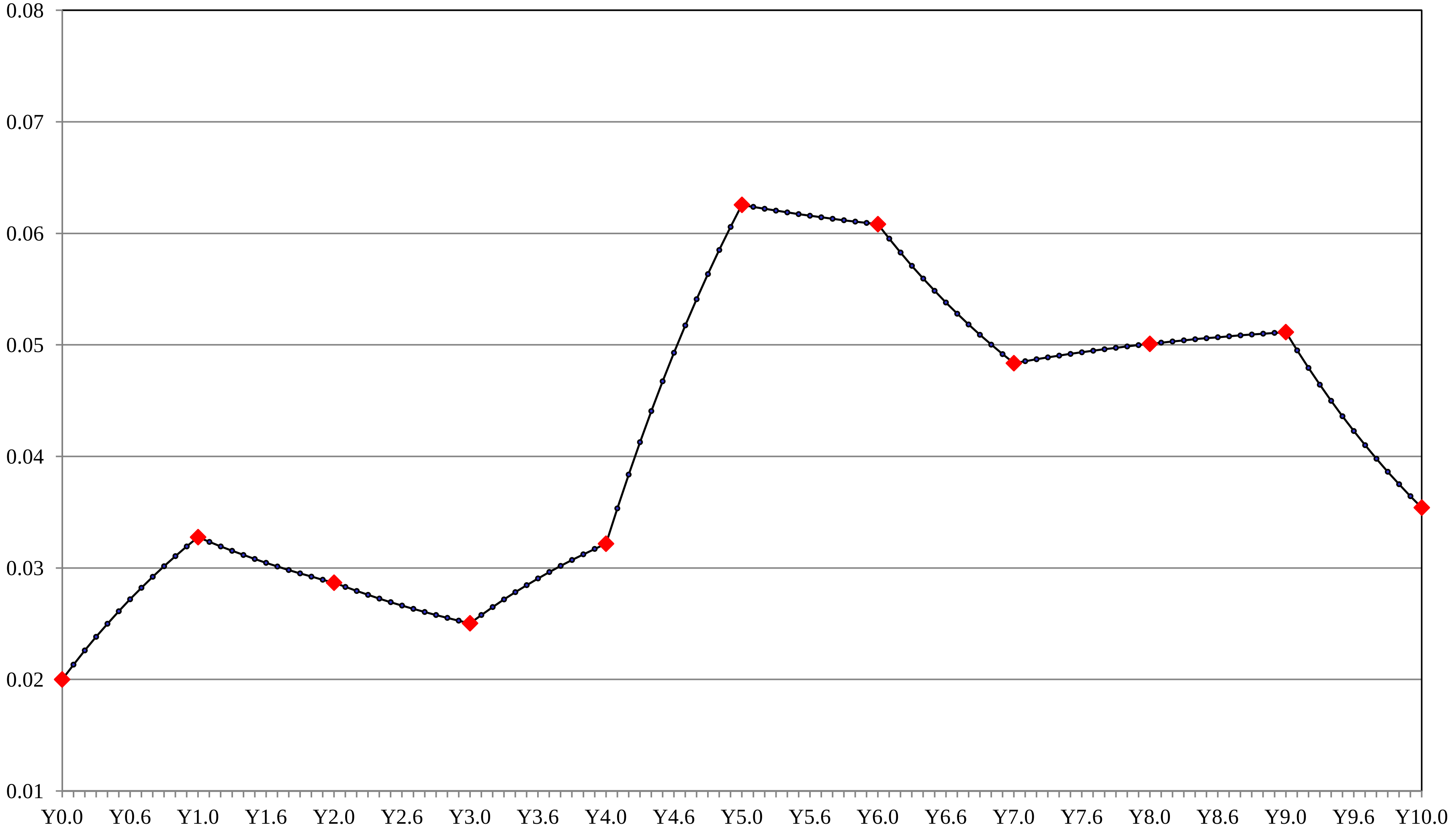
Yearly skeleton, monthly skeleton



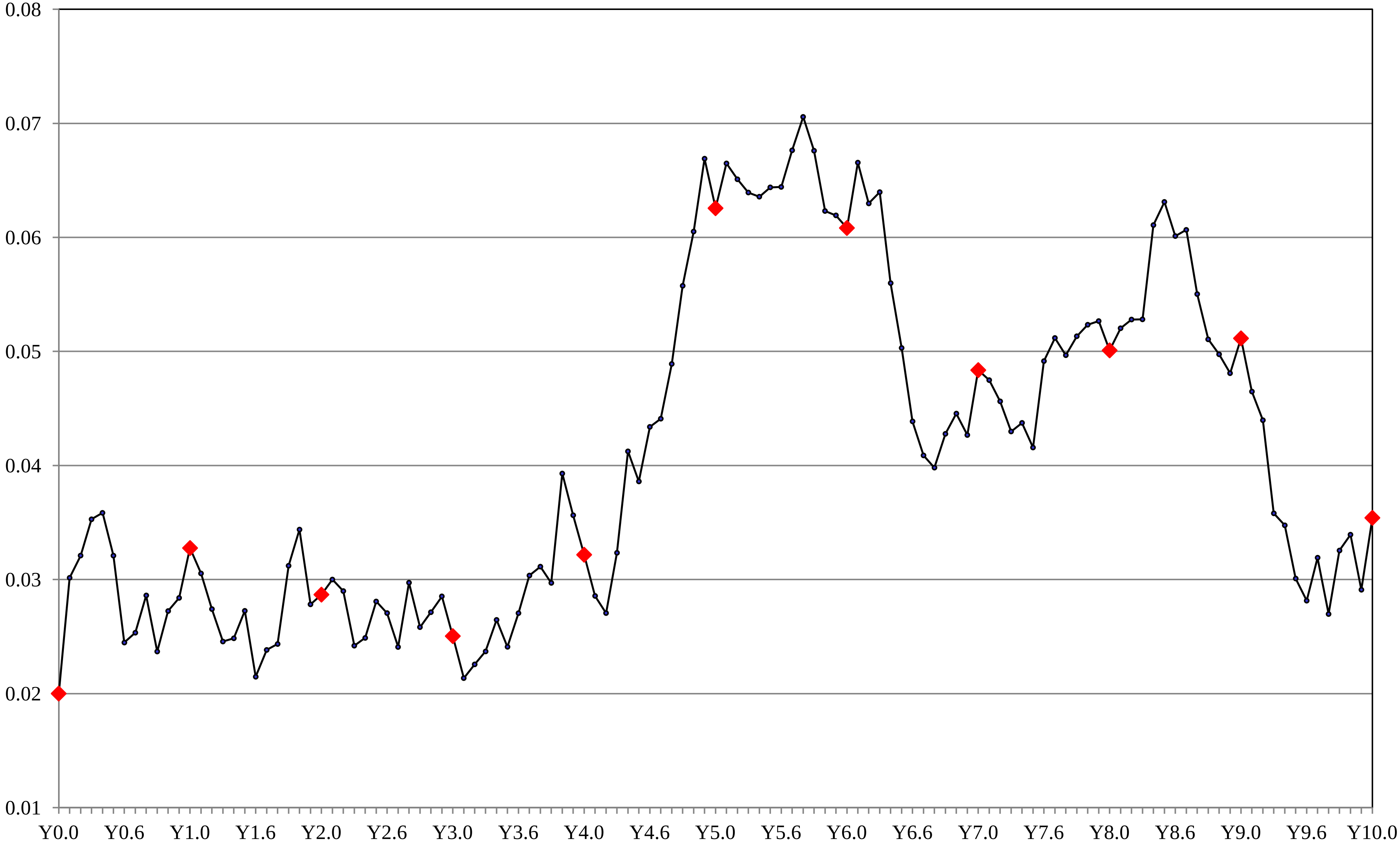
Yearly skeleton, monthly stochastic



Yearly stochastic, monthly skeleton



Yearly stochastic, monthly stochastic



So far we could just have used the monthly equivalent of the yearly model, and simulated all the monthly values directly.

But it is not always simple.

Wilkie model for Consumer Prices and inflation:

Q(t) is CPI at time t

QL(t) = lnQ(t)

I(t) = QL(t) – QL(t–1)

AR(1) model for I(t):

I(t) = QMU + QA.(I(t–1) – QMU)

+ QSD.QZ(t)

Then QL(t) = QL(t–1) + I(t)

First:

- **Simulate I(t) yearly with AR(1) model**
- **Calculate QL(t) from I(t)**

Then:

- **OU Stochastic Bridge Monthly for I**
- **We have I(t.1) to I(t.M–1) for all t**

We need to deal with the first year.

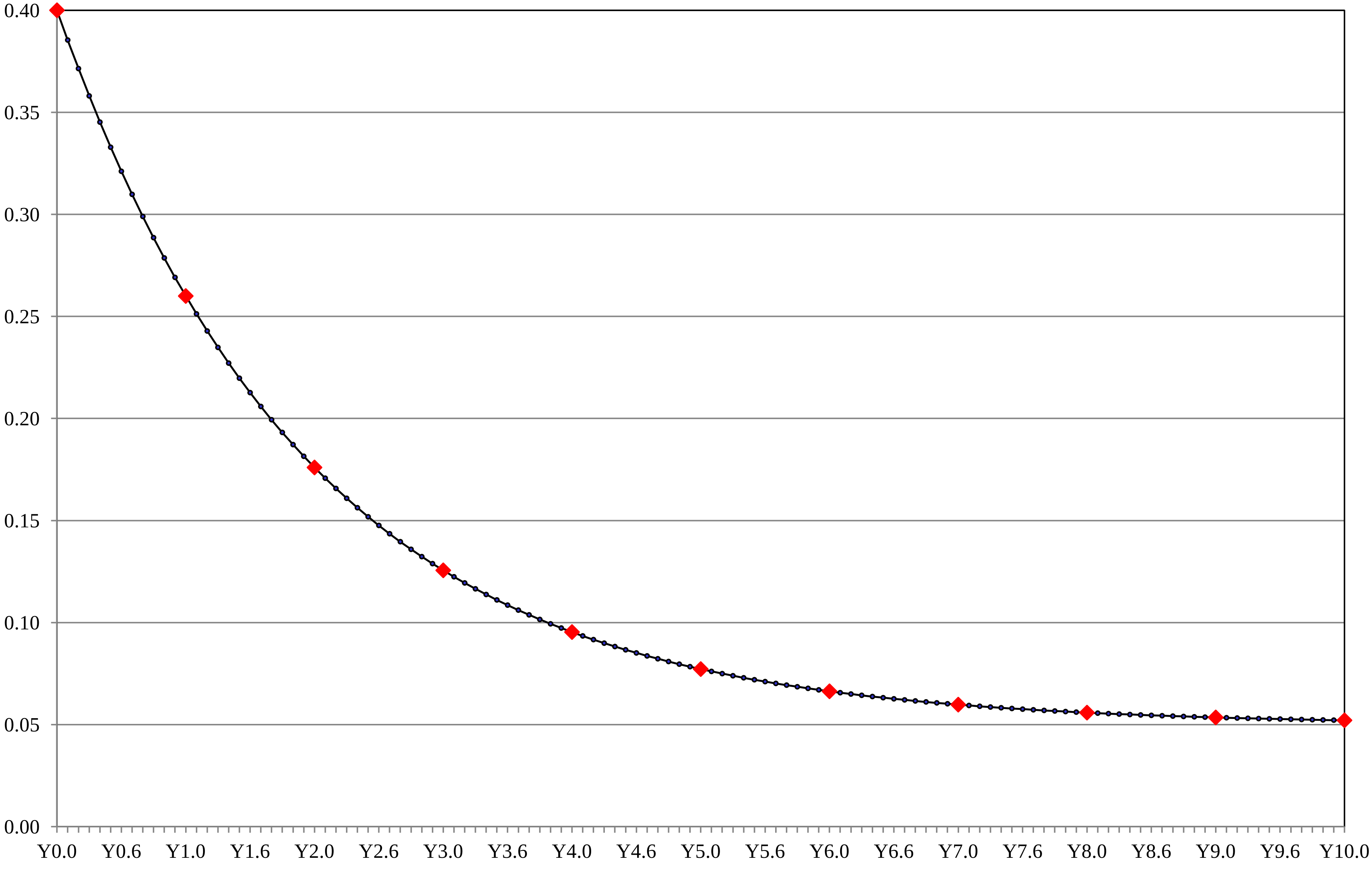
We have QL(–1) = QL(0) – I(0)

But we need QL(–1.1) to QL(–1.M–1)

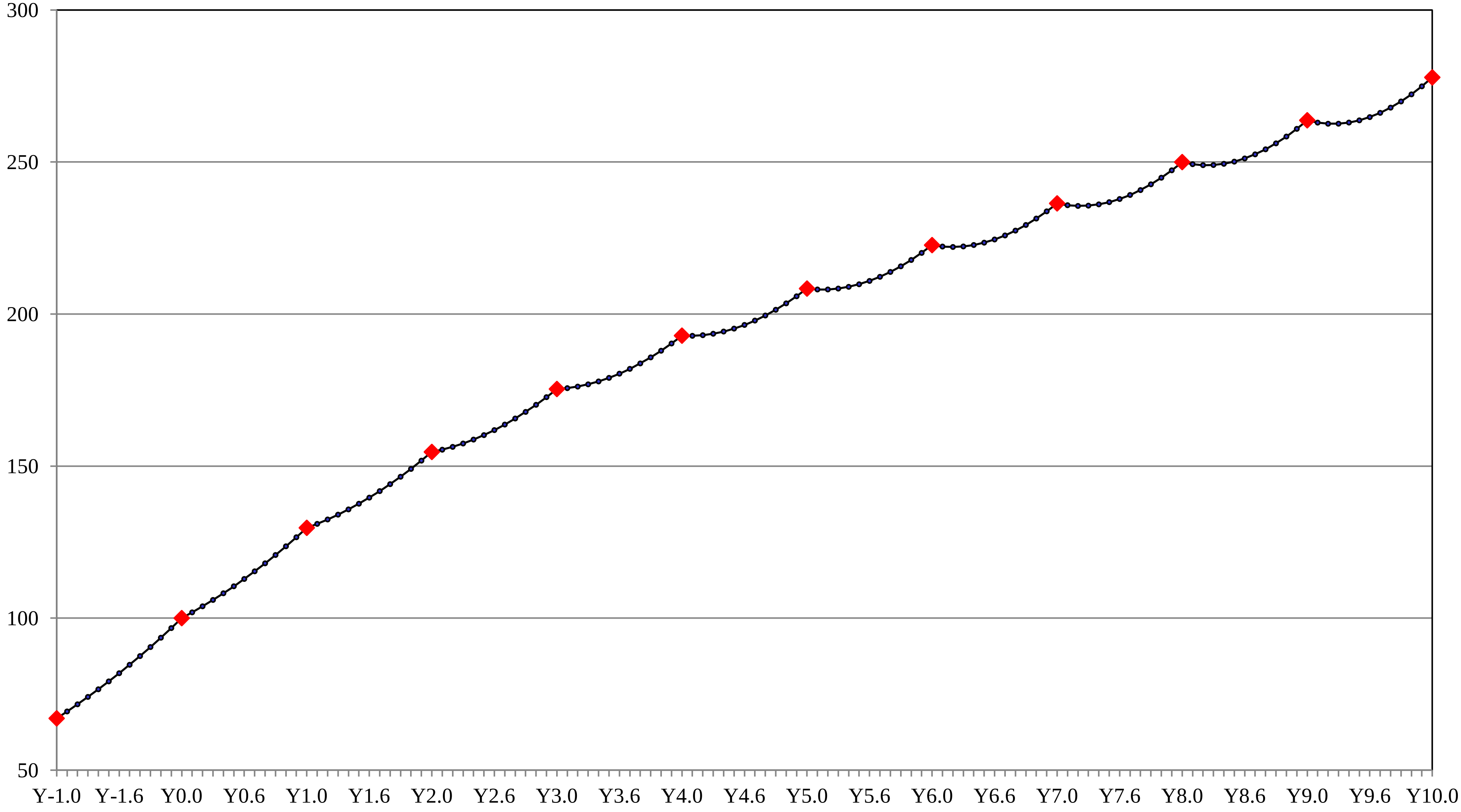
We could use:

- **Last year's actual, but problems**
- **Brownian Bridge**

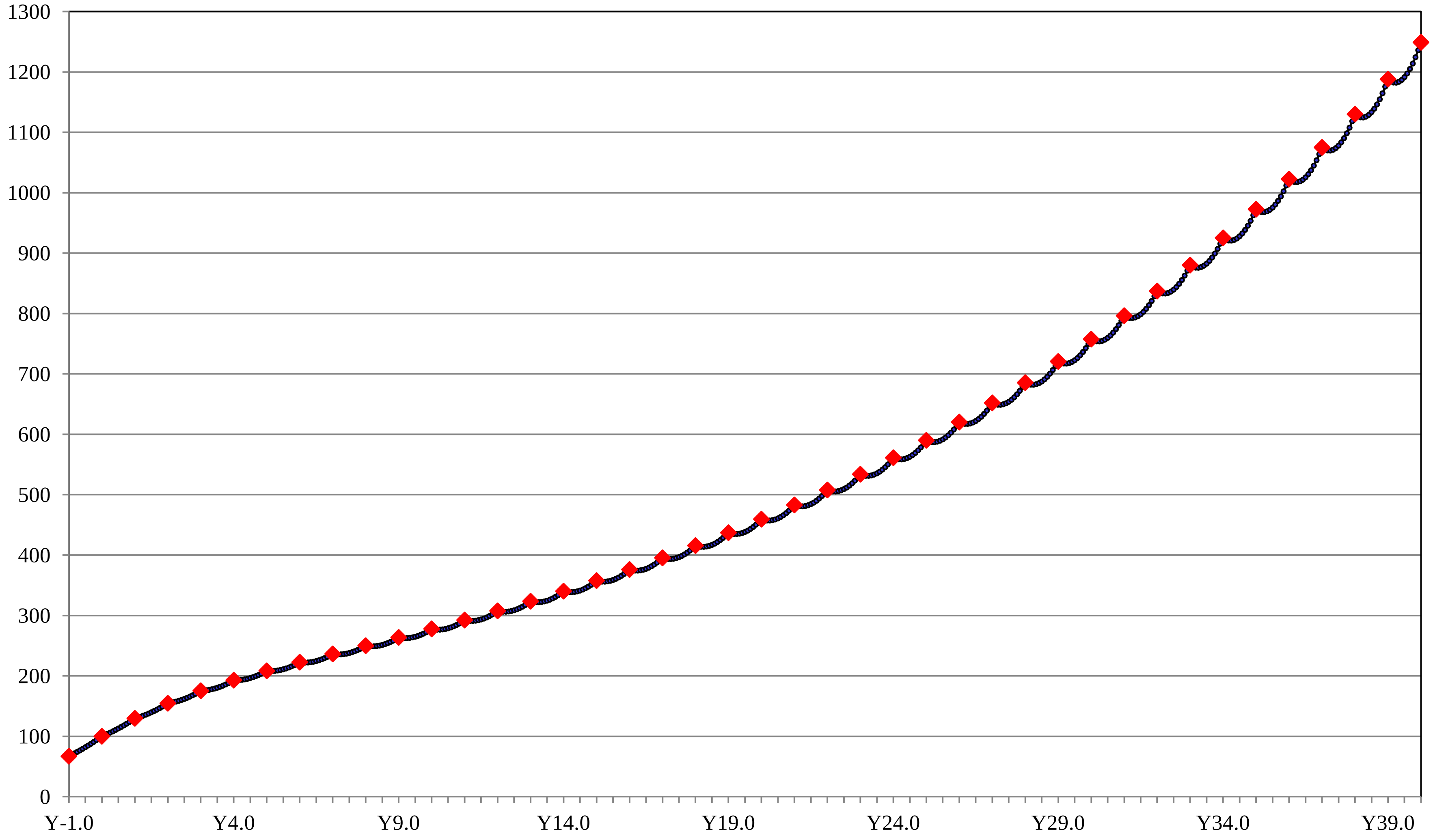
I: AR(1) and OU bridges: skeletons



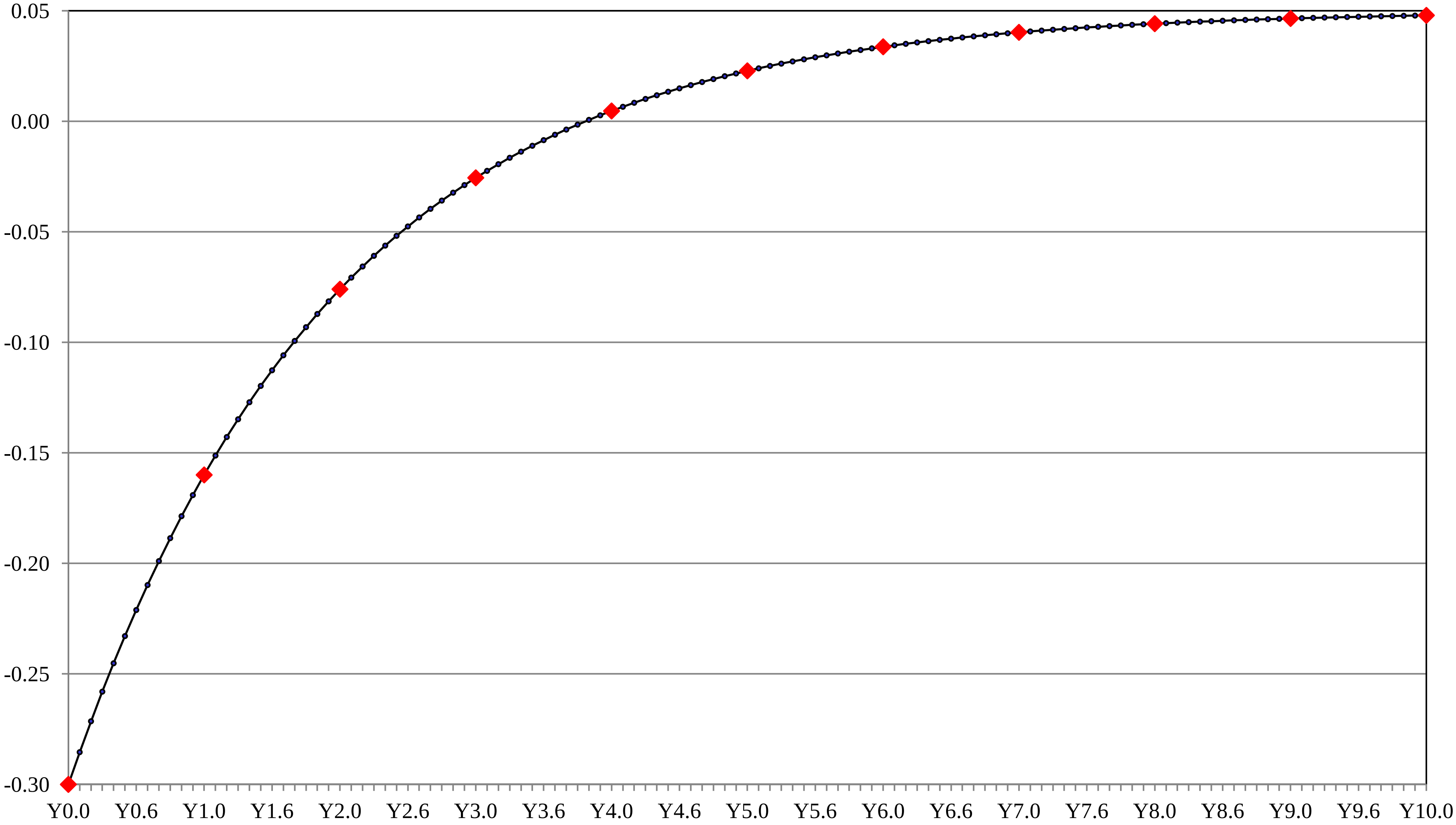
Q: Based on I: skeletons



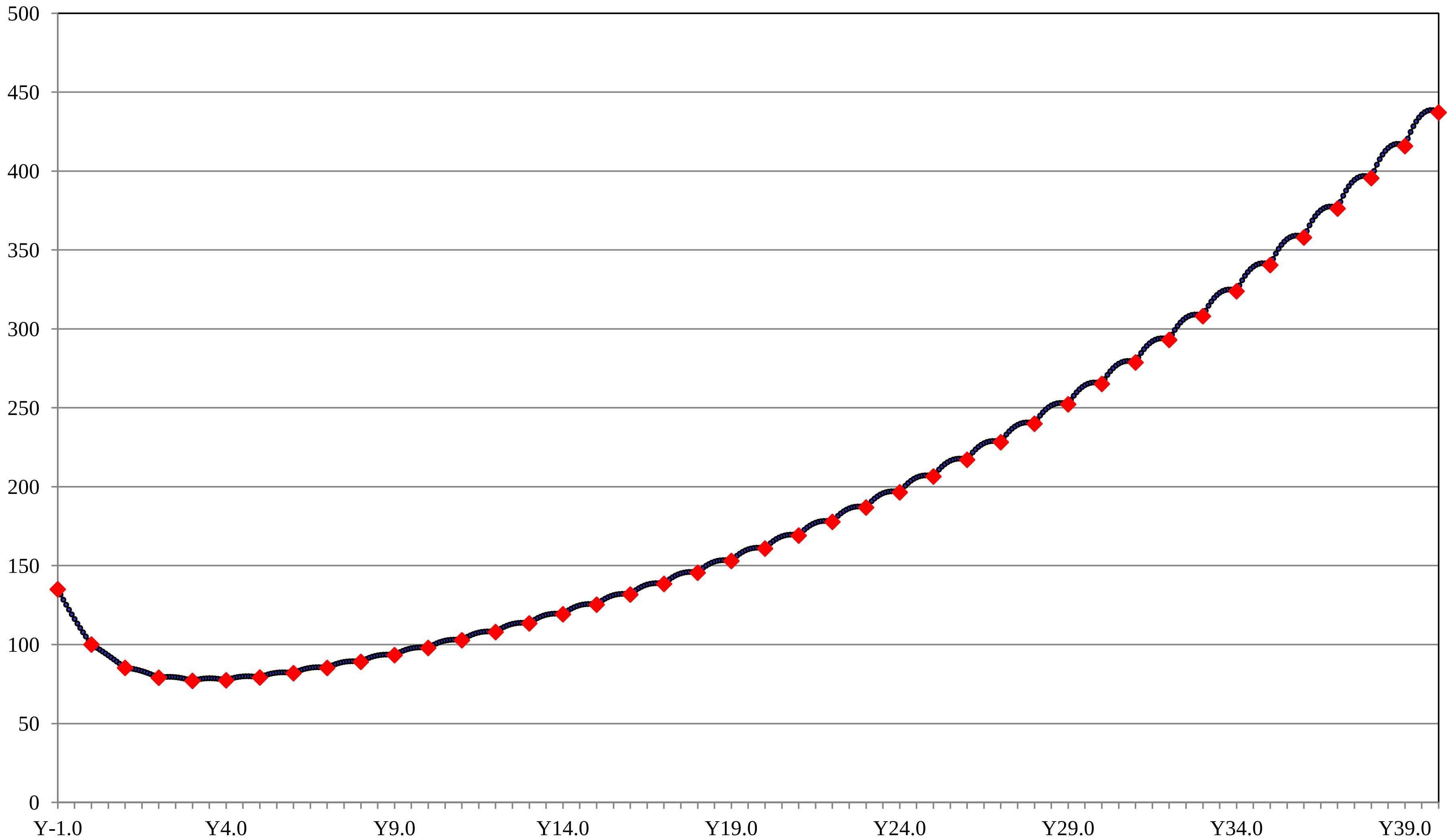
Q: Based on I: skeletons for 40 years



I: Very low initial value: skeletons



Q: Based on I: skeletons for 40 years



Oh dear!

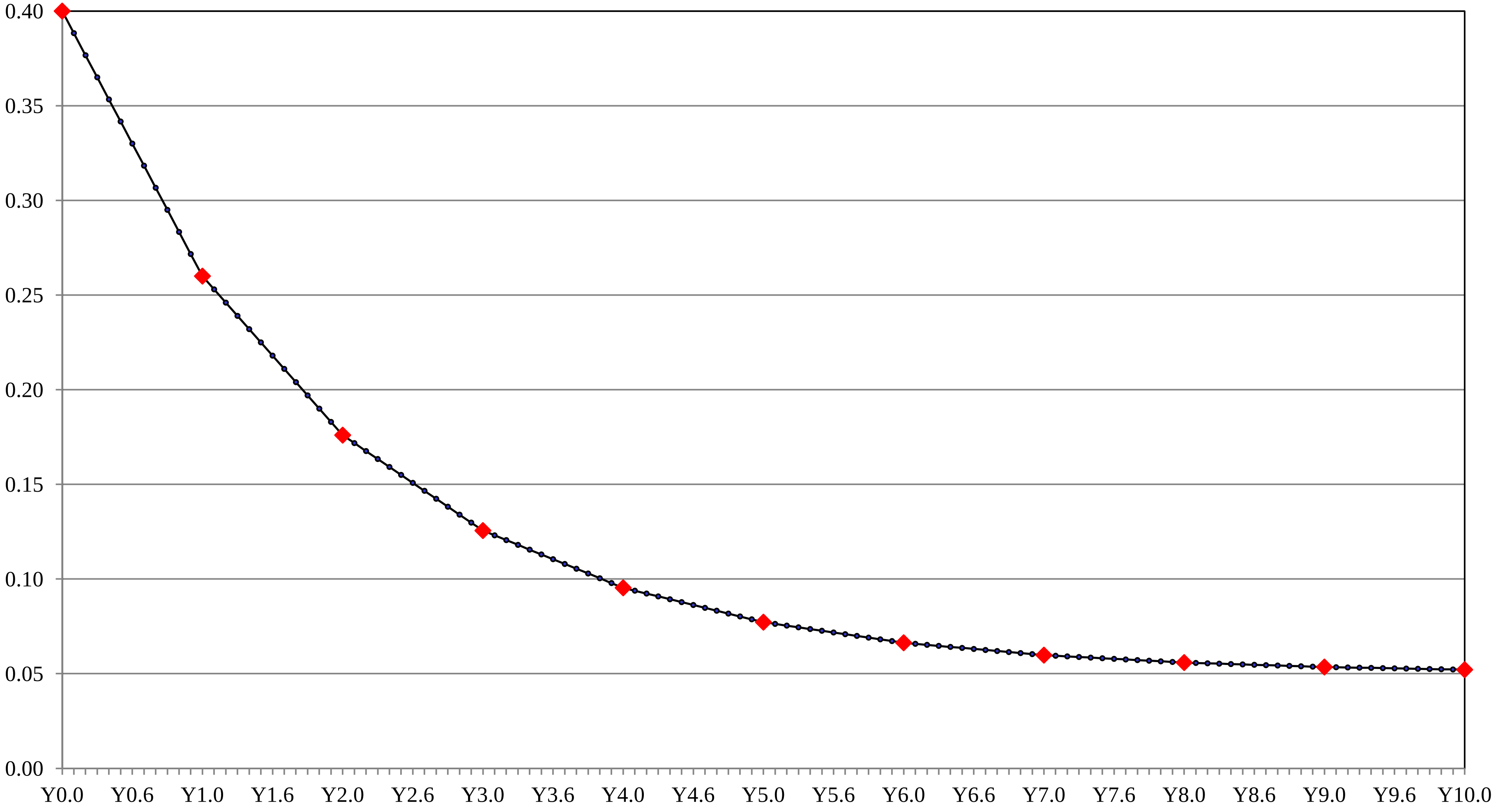
- **Why the “scallops” or “arches”?**

Curves in OU bridges for I accumulate in Q and increase asymptotically to a limit.

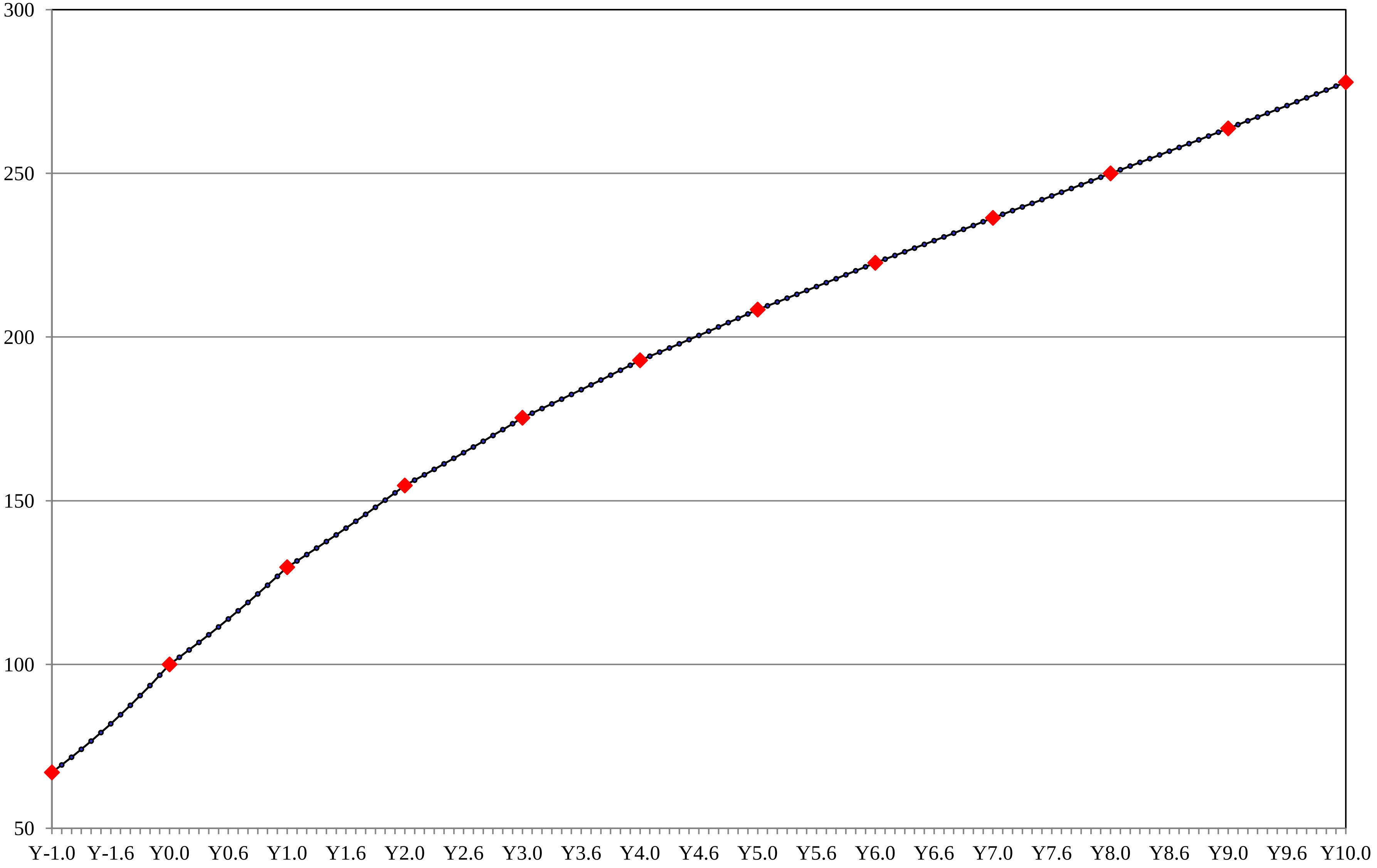
How can we get rid of them?

- **Try Brownian bridge for I.**

I: AR(1) and Brownian bridges: skeletons



Q: Based on I: skeletons

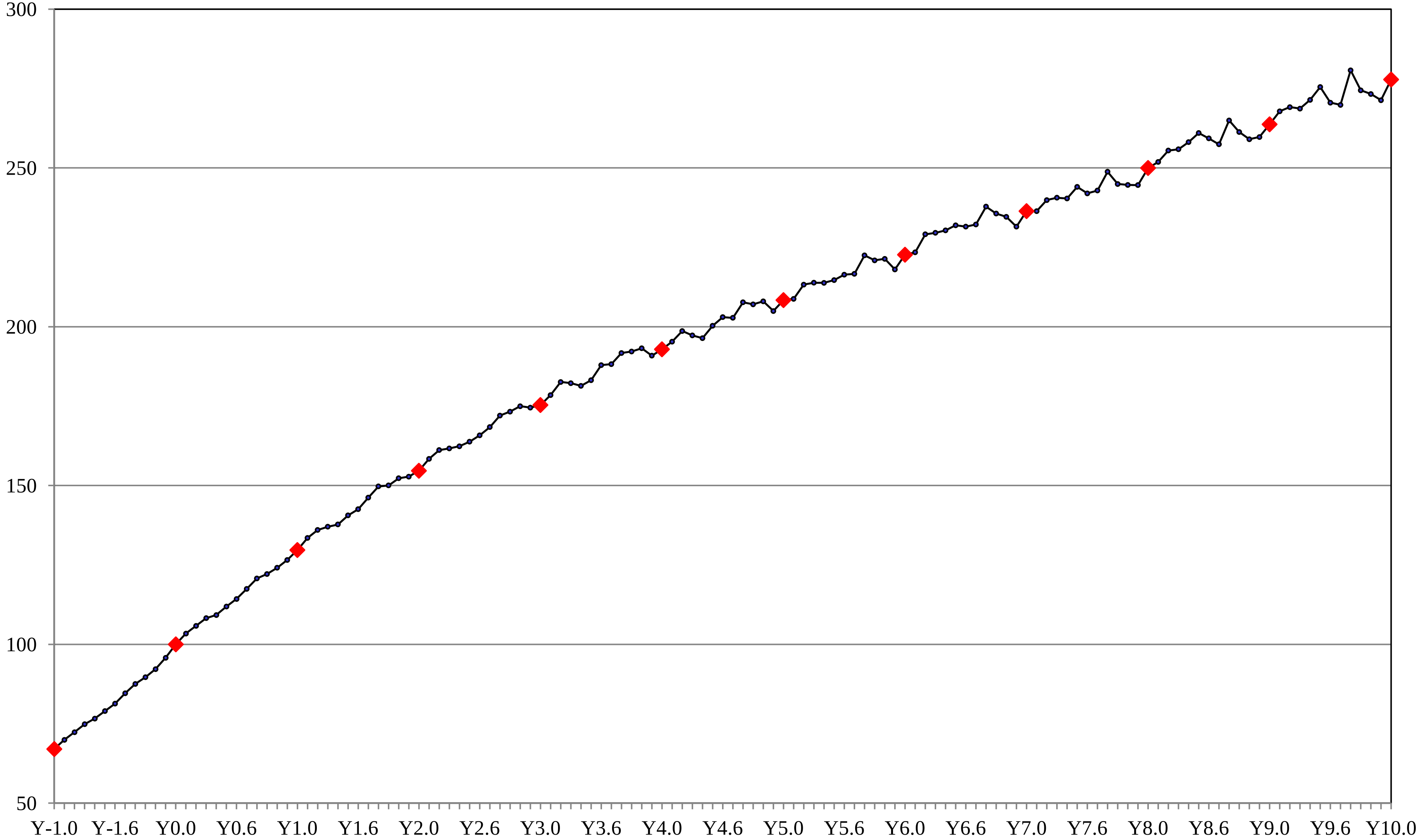


That looks better.

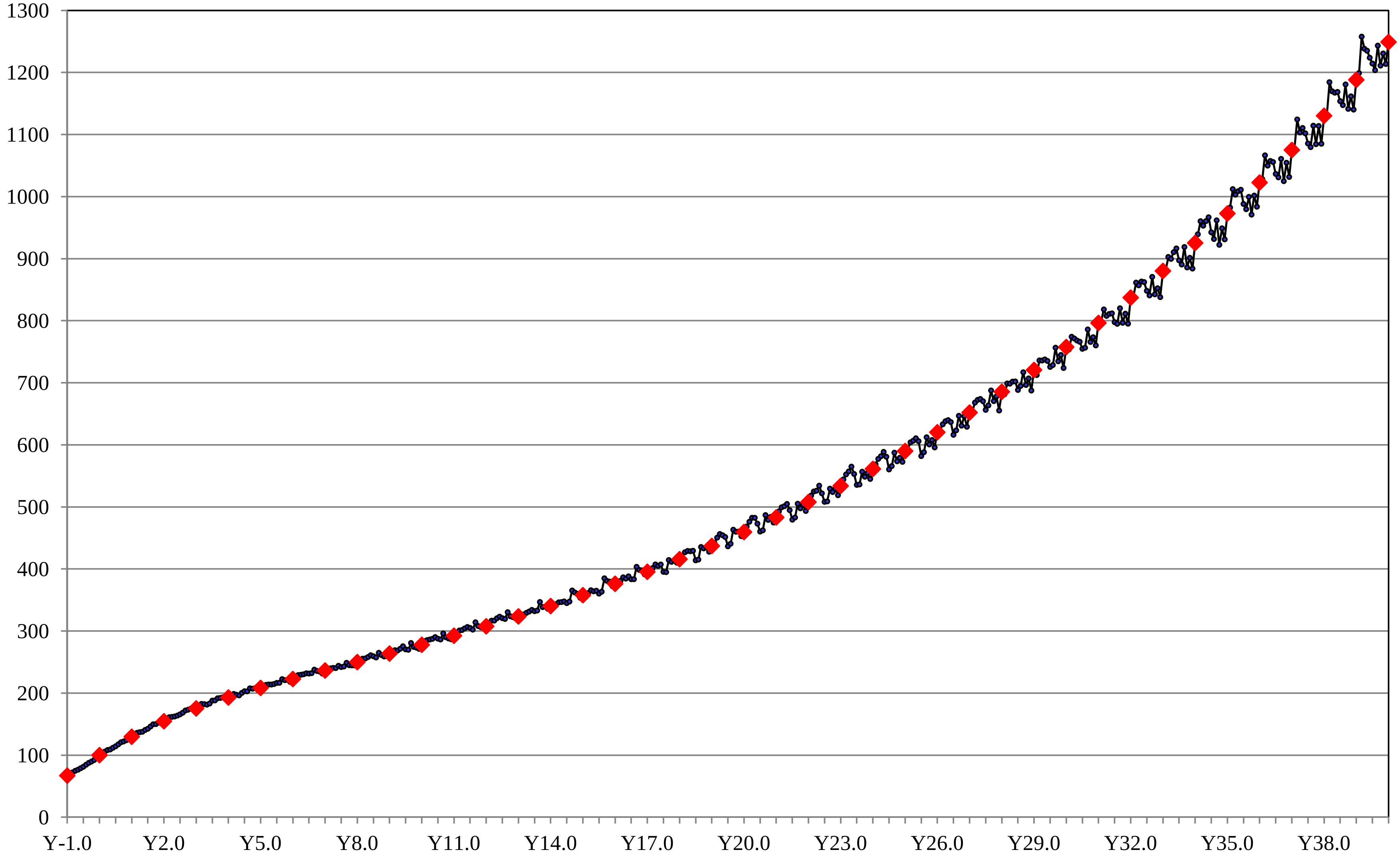
Now try stochastic bridge on yearly skeleton

I: AR(1) and BB: stochastic monthly

Q: Based on I: stochastic monthly



Q: Based on I: stochastic monthly 40 years



Oh dear again!

The values of Q are getting more spread out, and are in the same shape each year.

Each QL(t.k) is the sum of all previous values of I(u.k), u = 1 to t, and the variance of QL(t.k) is the sum of the variances, so it gets bigger.

So try Brownian bridge for QL and then calculate I from QL.

But what value to use for σ_m?

Yearly variance of I(t+1), given I(t), in AR(1) model is QSD^2.

So is yearly variance of QL(t+1), given QL(t).

But if variance of each QL(t.k) is $f(k).\sigma_m^2$ then variance of I(t.k) is $2f(k).\sigma_m^2$

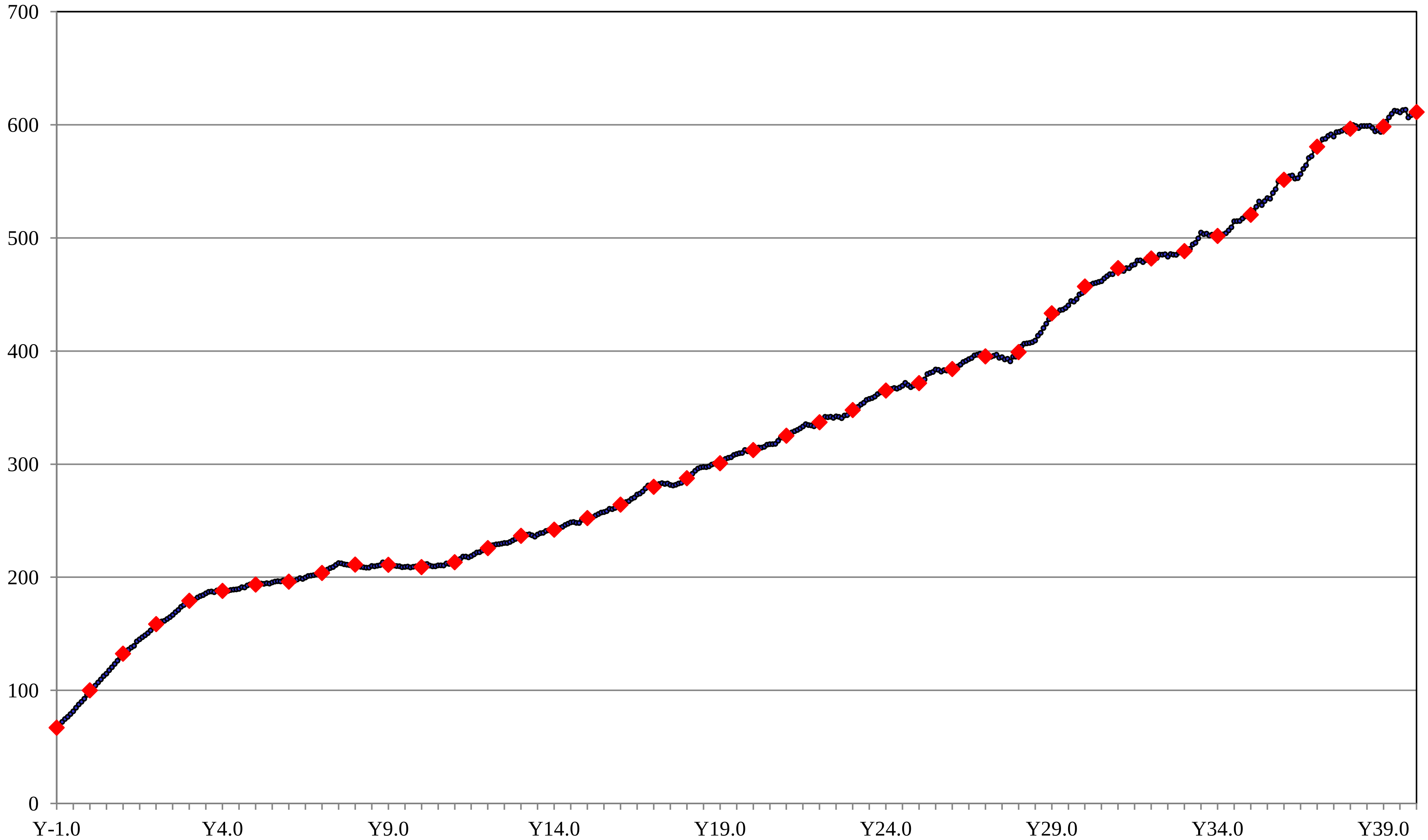
So should we use $\sigma_m^2 = QSD^2/2M$?

Alternative to go back to source data for CPI from which model was derived.

In most countries this is available monthly, so a monthly σ could be calculated.

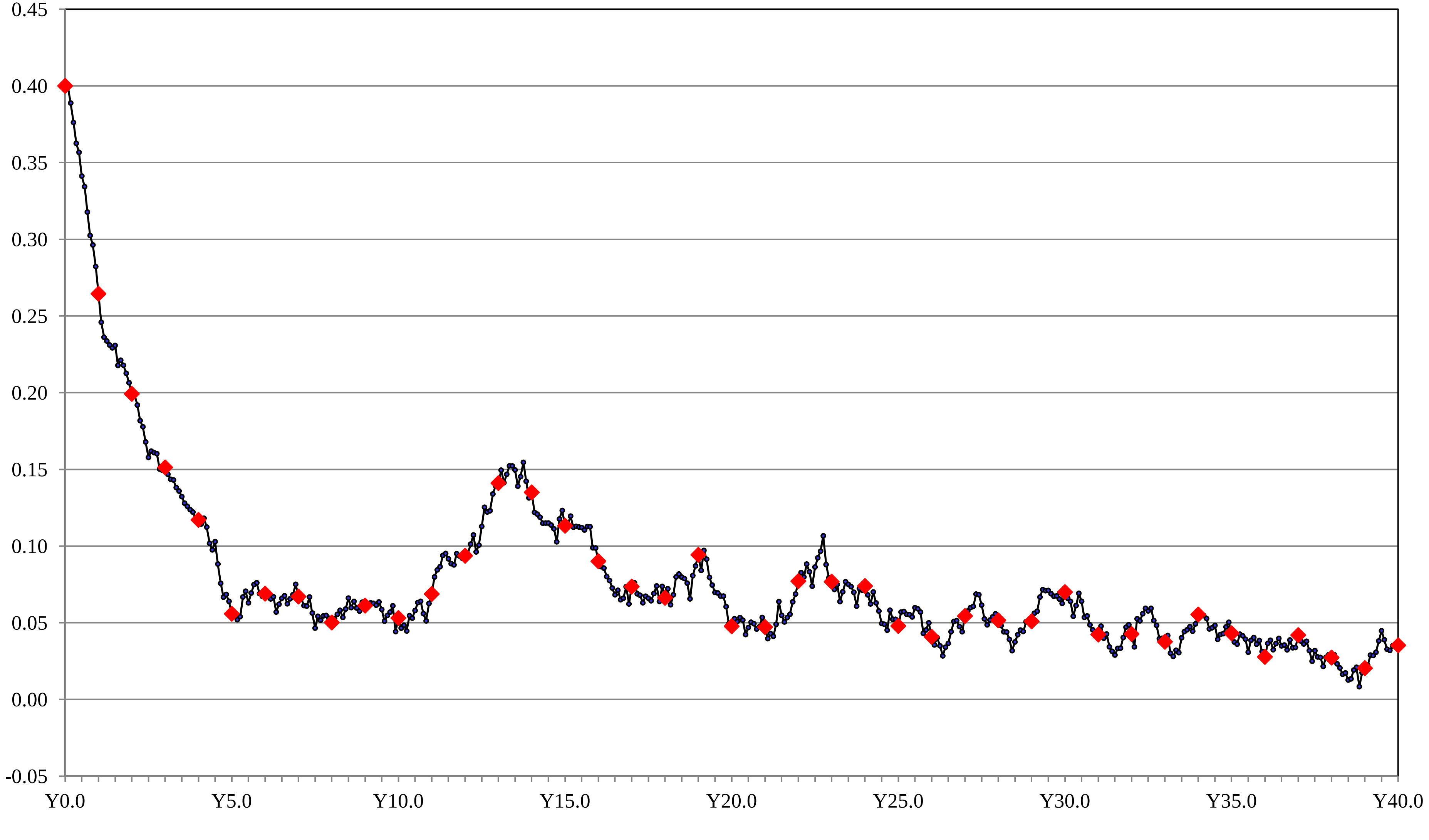
Note that in stochastic interpolation, the monthly model does not have to be consistent with yearly one.

BB for Q: fully stochastic – 40 years

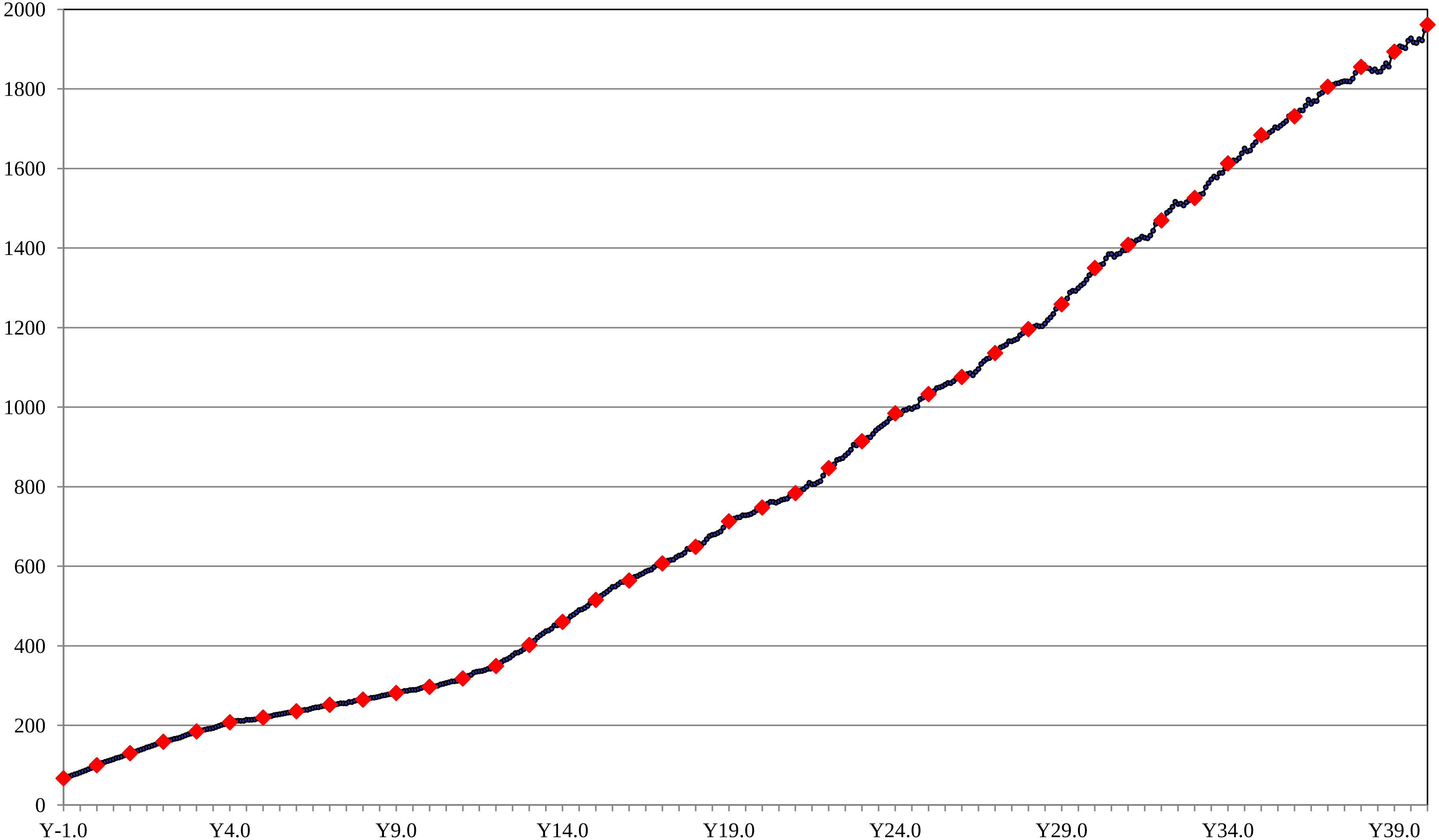


Corresponding I

I from another simulation: higher values



Q with much higher final value



Conclusions:

Some annual models can be very easily interpolated.

e.g. AR(1) models for interest rates.

Some, especially integrated (summed) models require great care:

e.g. CPI, wages, share dividends, etc.

Take care

Always check all combinations of skeletons and stochastic parts.

Try extreme parameter values
extreme starting conditions.

Good luck.

“Yet more on a stochastic economic model: Part I: Updating and refitting, 1995 to 2009”

Wilkie, Şahin, Cairns, Kleinow

Annals of Actuarial Science, 2011, Vol 5, part 1, pp 53-99

Wait for Part *X*.

End