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On the independence between financial and actuarial risks[1]

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[1] Dhaene, J., Kukush, A., Luciano, E., Schoutens, W., Stassen, B. (2012). A note on the independence between financial and actuarial risks. Working paper.

Outline

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Introduction

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- Model financial world to describe financial risks via filtered probability space

$$\left(\Omega^{(1)}, \mathcal{F}^{(1)}, \left(\mathcal{F}_t^{(1)}\right)_{0 \leq t \leq T}, \mathbb{P}^{(1)}\right)$$

- Model actuarial world to describe actuarial risks via filtered probability space

$$\left(\Omega^{(2)}, \mathcal{F}^{(2)}, \left(\mathcal{F}_t^{(2)}\right)_{0 \leq t \leq T}, \mathbb{P}^{(2)}\right)$$

- Model combined financial-actuarial world as product space

- Insurance securitization involves financial and actuarial considerations.
- Pricing via the no-arbitrage principle: $\mathbb{P} \longrightarrow \mathbb{Q}$.
- When independence between financial and actuarial risks is assumed in the real world, one often makes the same assumption in the pricing world.
- In Black & Scholes - setting: independence relations translate from $\mathbb{P}$ to $\mathbb{Q}$ and vice versa. But what happens in a general setting?

- Assume a market with 2 traded assets: a financial asset $S^{(1)}$ and an actuarial asset $S^{(2)}$.
- In the real world, the dynamics are described by

$$\frac{dS^{(i)}(t)}{S^{(i)}(t)} = \mu^{(i)} dt + \sigma^{(i)} dB^{(i)}(t), \qquad \text{for } i = 1, 2.$$

- Furthermore,

$$cov_{\mathbb{P}}\left[B^{(1)}(t), B^{(2)}(t+s)\right] = \rho t, \qquad \text{for any } t, s \geq 0.$$

- The coefficient ρ is the correlation coefficient of the couple $\left(B^{(1)}(t), B^{(2)}(t)\right)$.

Introduction

Black & Scholes setting

- For an artibrage-free market, the pricing dynamics are

$$\frac{dS^{(i)}(t)}{S^{(i)}(t)} = rdt + \sigma^{(i)} dW^{(i)}(t), \qquad \text{for } i = 1, 2.$$

- Also

$$cov_{\mathbb{Q}}\left[W^{(1)}(t), W^{(2)}(t+s)\right] = \rho t = cov_{\mathbb{P}}\left[B^{(1)}(t), B^{(2)}(t+s)\right]$$

for any $t, s \geq 0$.

Introduction

Main goal

- Our goal:
  - Can we translate an independence assumption from the real world $\mathbb{P}$ to the pricing world $\mathbb{Q}$?
  - What are the conditions needed such that $\mathbb{P}$-independence implies $\mathbb{Q}$-independence?
  - Illustration by some numerical examples.
  - General approach: see paper.

The combined financial-actuarial world: Example

- The usual set up: Consider the *financial world*

$$\left(\Omega^{(1)}, \mathcal{F}^{(1)}, \mathbb{P}^{(1)}\right)$$

- Assume interest rate $r = 0$.
- The stock market:
  - Traded non-dividend paying stock $S^{(1)}$
  - $S^{(1)}(0) = 100$
  - At time 1: $S^{(1)}(1)$ is either 50 or 150
- $\Omega^{(1)}$ describes all possible evolutions of the financial world:

$$\Omega^{(1)} = \{50, 150\}.$$

- $\mathcal{F}^{(1)}$ is the set of all subsets of $\Omega^{(1)}$.
- Real-world probability measure $\mathbb{P}^{(1)}$:

$$\begin{cases} \mathbb{P}^{(1)}[50] = p_1 \\ \mathbb{P}^{(1)}[150] = 1 - p_1 \end{cases}$$

The combined financial-actuarial world: Example

- The usual set up: Consider the *actuarial world*

$$\left(\Omega^{(2)}, \mathcal{F}^{(2)}, \mathbb{P}^{(2)}\right)$$

- Survival index $\mathcal{I}$ of a population:
  - 0: 'few' persons survive during period [0,1]
  - 1: 'many' persons survive during period [0,1]
  - At time 1: $\mathcal{I}(1)$ is 0 or 1
- $\Omega^{(2)}$ describes all possible evolutions of the actuarial world:

$$\Omega^{(2)} = \{0, 1\} .$$

- $\mathcal{F}^{(2)}$ is the set of all subsets of $\Omega^{(2)}$.
- Real-world probability measure $\mathbb{P}^{(2)}$:

$$\begin{cases} \mathbb{P}^{(2)}\left[0\right] = p_2 \\ \mathbb{P}^{(2)}\left[1\right] = 1 - p_2 \end{cases}$$

The combined financial-actuarial world: Example

- Consider the combined *financial-actuarial world*

$$(\Omega, \mathcal{F}, \mathbb{P})$$

- Cartesian product of both spaces.
- Universe Ω:

$$\Omega = \Omega^{(1)} \times \Omega^{(2)} = \left\{ (50, 0) , (50, 1) , (150, 0) , (150, 1) \right\}$$

The combined financial-actuarial world: Example

Financial and actuarial risks - The combined world

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- $\mathbb{P}$ is the 'real-world' probability measure.
- Assumption: $\boxed{\mathbb{P} \equiv \mathbb{P}^{(1)} \times \mathbb{P}^{(2)}}$
- In this case:

$$\begin{cases} \mathbb{P}\left[(50,0)\right] = p_1 \times p_2 \\ \mathbb{P}\left[(50,1)\right] = p_1 \times (1-p_2) \\ \mathbb{P}\left[(150,0)\right] = (1-p_1) \times p_2 \\ \mathbb{P}\left[(150,1)\right] = (1-p_1) \times (1-p_2) \end{cases}$$

The combined financial-actuarial world: Example

Financial and actuarial risks - Pricing

- Prices of (non-dividend paying) traded securities in the combined world follow from no-arbitrage principle.
- Replace $\mathbb{P}$ by an 'equivalent martingale measure' $\mathbb{Q}$.
- Price recipy:
  For a pay-off at time 1 equal to X, the price at time 0 under this measure $\mathbb{Q}$ is given by

$$e^{-rT} E^{\mathbb{Q}}[X].$$

The combined financial-actuarial world: Example

Financial and actuarial risks - Pricing

- Given $\mathbb{Q}$, we can construct the projections $\mathbb{Q}^{(1)}$ and $\mathbb{Q}^{(2)}$ on the financial and the actuarial world, respectively.
- Introduce the measure $\mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}$.

Definition

Independence in pricing world between financial and actuarial risks if

$$\boxed{\mathbb{Q} \equiv \mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}}$$

The combined financial-actuarial world: Example

A market with 1 actuarial and 2 financial securities

- Traded securities:
  - Risk-free bank account (assume $r = 0$)
  - Stock with price $S^{(1)}(1)$ at time 1 and $S^{(1)}(0) = 100$
  - Actuarial security:

$$S^{(2)}(1) = 100 \times \mathcal{I}(1), \text{ with } S^{(2)}(0) = 70$$

- Determining $\mathbb{Q}$:

$$\begin{cases} \mathbb{E}^{\mathbb{Q}}\left[S^{(1)}(1)\right] = 100 \\ \mathbb{E}^{\mathbb{Q}}\left[S^{(2)}(1)\right] = 70 \\ \mathbb{Q}\left[(50,0)\right] + \mathbb{Q}\left[(150,0)\right] + \mathbb{Q}\left[(50,1)\right] + \mathbb{Q}\left[(150,1)\right] = 1 \end{cases}$$

The combined financial-actuarial world: Example

A market with 1 actuarial and 2 financial securities

- Equivalent with

$$\begin{cases} \mathbb{Q}^{(1)}[50] = 0.5 \\ \mathbb{Q}^{(1)}[150] = 0.5 \\ \mathbb{Q}^{(2)}[0] = 0.3 \\ \mathbb{Q}^{(2)}[1] = 0.7 \end{cases}$$

- Particular solutions: $\overline{\mathbb{Q}}$ and $\mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}$

$$\begin{cases} \overline{\mathbb{Q}}[(50,0)] = 0.2 \\ \overline{\mathbb{Q}}[(150,0)] = 0.1 \\ \overline{\mathbb{Q}}[(50,1)] = 0.3 \\ \overline{\mathbb{Q}}[(150,1)] = 0.4 \end{cases} \quad \text{and} \quad \begin{cases} \mathbb{Q}^{(1)}[50] \times \mathbb{Q}^{(2)}[0] = 0.15 \\ \mathbb{Q}^{(1)}[150] \times \mathbb{Q}^{(2)}[0] = 0.15 \\ \mathbb{Q}^{(1)}[50] \times \mathbb{Q}^{(2)}[1] = 0.35 \\ \mathbb{Q}^{(1)}[150] \times \mathbb{Q}^{(2)}[1] = 0.35 \end{cases}$$

- The combined market is arbitrage-free but incomplete.
- Under $\overline{\mathbb{Q}}$, the independence relation is not maintained.
- There is only one solution with the independence property, namely $\mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}$.

The combined financial-actuarial world: Example

A market with a combined product

- Traded securities:
  - Risk-free bank account (assume $r = 0$)
  - Stock $S^{(1)}$
  - Actuarial security $S^{(2)}$
  - Combined product:

$$S(1) = \left(100 - S^{(1)}(1)\right)_+ \times \mathcal{I}(1), \text{ with current price } S(0)$$

- Determining $\mathbb{Q}$:

$$\begin{cases} \mathbb{E}^{\mathbb{Q}}\left[S^{(1)}(1)\right] = 100 \\ \mathbb{E}^{\mathbb{Q}}\left[S^{(2)}(1)\right] = 70 \\ \mathbb{E}^{\mathbb{Q}}\left[S(1)\right] = S(0) \\ \mathbb{Q}\left[(50,0)\right] + \mathbb{Q}\left[(150,0)\right] + \mathbb{Q}\left[(50,1)\right] + \mathbb{Q}\left[(150,1)\right] = 1 \end{cases}$$

The combined financial-actuarial world: Example

A market with a combined product

- The single solution $\widetilde{\mathbb{Q}}$ depends on $S(0)$:

$$\begin{cases} \widetilde{\mathbb{Q}}\left[(50,0)\right] = \frac{25-S(0)}{50} \\ \widetilde{\mathbb{Q}}\left[(150,0)\right] = \frac{-10+S(0)}{50} \\ \widetilde{\mathbb{Q}}\left[(50,1)\right] = \frac{S(0)}{50} \\ \widetilde{\mathbb{Q}}\left[(150,1)\right] = \frac{35-S(0)}{50} \end{cases}$$

- The combined market is now arbitrage-free AND complete.
- $\widetilde{\mathbb{Q}} = \widetilde{\mathbb{Q}}^{(1)} \times \widetilde{\mathbb{Q}}^{(2)} \iff S(0) = 17.5$
- **Conclusion**: The market can choose the measure with the independence property by setting the price of the combined product equal to $S(0) = 17.5$.

The combined financial-actuarial world: Example

An incomplete market with a combined product

- Financial world:
  - Risk-free bank account (assume $r = 0$)
  - Barometer of the economy: *Booming (B)*, *Moderate growth (M)* or *Recession (R)*
  - Financial universe $\Omega^{(1)} = \{B, M, R\}$
  - Real-world probabilities: $\mathbb{P}^{(1)}[B], \mathbb{P}^{(1)}[M]$ and $\mathbb{P}^{(1)}[R]$
- Actuarial world:
  - Universe $\Omega^{(2)}$:
    $$\Omega^{(2)} = \{0, 1\}.$$
  - Real-world probabilities: $\mathbb{P}^{(2)}[0]$ and $\mathbb{P}^{(2)}[1]$

The combined financial-actuarial world: Example

An incomplete market with a combined product

- Combined financial-actuarial world:
  - Universe Ω:

$$\Omega = \{(B,0),(M,0),(R,0),(B,1),(M,1),(R,1)\}.$$

  - **Assumption**: under $\mathbb{P}$, financial and actuarial risks are independent

$$\boxed{\mathbb{P} \equiv \mathbb{P}^{(1)} \times \mathbb{P}^{(2)}}$$

- Traded securities:
  - Risk-free bank account (assume $r = 0$)
  - Financial asset with current price $\overline{S}^{(1)}(0) = 50$

$$\overline{S}^{(1)}(1) = \begin{cases} 100 & \text{if the economy is } \textit{Booming} \\ 0 & \text{otherwise} \end{cases}$$

  - Actuarial security $S^{(2)}$ with current price $S^{(2)}(0) = 70$
  - Combined product $\overline{S}$ with pay-off

$$\overline{S}(1) = \overline{S}^{(1)}(1) \times (1 - \mathcal{I}(1)) \text{ and current price } \overline{S}(0)$$

The combined financial-actuarial world: Example

An incomplete market with a combined product

- Determining $\mathbb{Q}$:

$$\begin{cases} \mathbb{E}^{\mathbb{Q}}\left[\overline{S}^{(1)}(1)\right] = 50 \\ \mathbb{E}^{\mathbb{Q}}\left[S^{(2)}(1)\right] = 70 \\ \mathbb{E}^{\mathbb{Q}}\left[\overline{S}(1)\right] = \overline{S}(0) \\ \mathbb{Q}\left[(B,0)\right] + \mathbb{Q}\left[(B,1)\right] + \mathbb{Q}\left[(M,0)\right] + \mathbb{Q}\left[(M,1)\right] + \mathbb{Q}\left[(R,0)\right] + \mathbb{Q}\left[(R,1)\right] = 1 \end{cases}$$

- Equivalent with:

$$\begin{cases} \mathbb{Q}\left[(B,0)\right] = \frac{\overline{S}(0)}{100} \\ \mathbb{Q}\left[(B,1)\right] = \frac{50-\overline{S}(0)}{100} \\ \mathbb{Q}\left[(M,0)\right] + \mathbb{Q}\left[(R,0)\right] = \frac{30-\overline{S}(0)}{100} \\ \mathbb{Q}\left[(M,1)\right] + \mathbb{Q}\left[(R,1)\right] = \frac{20+\overline{S}(0)}{100} \end{cases}$$

- More than one solution $\Longrightarrow$ the market is incomplete.

The combined financial-actuarial world: Example

An incomplete market with a combined product

- We find that:

$$\mathbb{Q}^{(1)}[B] = 0.5 \qquad \text{and} \qquad \mathbb{Q}^{(2)}[0] = 0.3.$$

- For every $\mathbb{Q}$ satisfying the system of equations, one has that

$$\mathbb{Q}[(B,0)] = \mathbb{Q}^{(1)}[B] \times \mathbb{Q}^{(2)}[0] \Longleftrightarrow \overline{S}(0) = 15,$$

  which is a necessary condition for the independence property to hold.

- **Conclusion**: The market can choose an equivalent martingale measure with the independence property by setting the current price of the combined product equal to $\overline{S}(0) = 15$.

A general combined financial-actuarial world

Financial and actuarial risks - The financial world

- Consider the filtered probability space

$$\left(\Omega^{(1)}, \mathcal{F}^{(1)}, \left(\mathcal{F}_t^{(1)}\right)_{0\leq t\leq T}, \mathbb{P}^{(1)}\right)$$

- Pricing financial assets by the no-arbitrage principle.
- $\left(S^{(1)}(t)\right)_{0\leq t\leq T}$: financial stochastic process
- $(r(t))_{0\leq t\leq T}$: instantaneous risk-free interest rate process

- Consider the filtered probability space

$$\left(\Omega^{(2)}, \mathcal{F}^{(2)}, \left(\mathcal{F}_t^{(2)}\right)_{0\leq t\leq T}, \mathbb{P}^{(2)}\right)$$

- Used to describe the biometrical evolutions in a time interval $[0, T]$. E.g. future mortality of a group of insureds
- $\left(S_{(x)}^{(2)}(t)\right)_{0\leq t\leq T}$: biometrical stochastic process of (x)
- $S_{(x)}^{(2)}(t) = \begin{cases} 1 & \text{if } (x) \text{ is alive at } t \\ 0 & \text{otherwise} \end{cases}$

A general combined financial-actuarial world

Financial and actuarial risks - The combined world

- Consider the filtered probability space

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P}) = \left(\Omega^{(1)} \times \Omega^{(2)}, \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}, \left(\mathcal{F}_t^{(1)} \otimes \mathcal{F}_t^{(2)}\right)_{0\le t\le T}, \mathbb{P}\right)$$

- To model combined financial - actuarial risks and stochastic processes.
- Pricing securities of which the pay-off depends on financial and actuarial quantities.
  E.g. mortality-linked securities

A general combined financial-actuarial world

Financial and actuarial risks - Pricing

- An equivalent martingale measure $\mathbb{Q}$ is needed for pricing purposes.
- Price recipy:
  The price of a traded mortality-linked security at time t with pay-off H at fixed term $u \geq t$ is equal to

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^u r(\tau)\, d\tau}\ H \mid \mathcal{F}_t \right].$$

- Financial assets can be priced in $\mathbb{Q}$ or $\mathbb{Q}^{(1)}$.
- Actuarial assets can be priced in $\mathbb{Q}^{(2)}$ only if r is deterministic.

A general combined financial-actuarial world

From real-world independence to pricing independence

- Again, we assume

$$\boxed{\mathbb{P} \equiv \mathbb{P}^{(1)} \times \mathbb{P}^{(2)}}$$

- For an equivalent pricing measure $\mathbb{Q}$, we determine the projections $\mathbb{Q}^{(1)}$ and $\mathbb{Q}^{(2)}$ and define

$$\mathbb{Q}' = \mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}$$

- **Question**: Is $\mathbb{Q}$ independent or equivalently, is $\mathbb{Q} \equiv \mathbb{Q}'$?

A general combined financial-actuarial world

From real-world independence to pricing independence

Consider a combined world where the following assumptions hold:

- *In the $\mathbb{P}$-world, financial and actuarial risks are independent,*
- *The risk-free interest rate is deterministic,*
- *No combined assets are traded in the market,*
- *The combined market is arbitrage-free in the sense that there exists a pricing measure.*

In such a combined world, it is always possible to construct an equivalent martingale measure $\mathbb{Q}$ under which financial and actuarial risks are independent.

Conclusion

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- A real-world independence between both risks may seem reasonable.
- Real-world independence does not imply pricing world independence.
- This wrong implication is sometimes made in the literature.
- Often, the market can choose a martingale measure $\mathbb{Q}$ for which the financial and actuarial risks are independent.

Further research

- In a market where combined assets are traded, what are the extra conditions to guarantee the existence of a $\mathbb{Q}$-measure with the independence property?
- Can we extent this theory to other copulas, besides the independent copula?
- What conditions are required to maintain a stochastic order relation when moving from $\mathbb{P}$ to $\mathbb{Q}$?

References

[1] Delbaen, F. and Schachermayer, W. (2008). *The mathematics of arbitrage.* Springer.

[2] Denuit, M., Dhaene, J., Goovaerts, M.J. and Kaas, R. (2005). *Actuarial theory for dependent risks: measures, orders and models.* Wiley.

[3] Dhaene, J., Denuit, M., Goovaerts, M. J., Kaas, R. and Vyncke, D. (2002a). The concept of comonotonicity in actuarial science and finance: theory. *Insurance: Mathematics & Economics*, 31(1):3-33.

[4] Dhaene, J., Denuit, M., Goovaerts, M. J., Kaas, R. and Vyncke, D. (2002b). The concept of comonotonicity in actuarial science and finance: applications. *Insurance: Mathematics & Economics*, 31(2):133-161.

[5] Gorvett, R. W. (1999). Insurance securitization: The development of a new asset class. *Casualty Actuarial Society*, pages 133-172.

[6] Shiryaev, A. N., Kabanov, Y. M., Kramkov, O. D. and Melnikov, A. V. (1994). Towards the theory of pricing of options of both european and american types. i. discrete time. *Theory of Probability and Its Applications*, 39(1):23-79.

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