On competing non-life insurers
Joint work with Hansjoerg Albrecher (Lausanne) and Christophe Dutang (Strasbourg)

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INTRODUCTION

- Lapse rates
- Price elasticity
- Competition: different markets, different features
- Customer behavior: a complex topic
- Classical explanatory variables: age of the policyholder / of the policy, cross-selling, sales channel, declared claims last year, ...
- Focus on some particular competition mechanism
- Impact of Solvency constraints
- Disclaimer: I’m not a game theory specialist. For questions about game theory, please ask Prof. Jean Lemaire!
FRAMEWORK

Usually, competition involves

1. market cycles models (time series)
2. lapse rates and new business (regression techniques)
3. position w.r.t. estimated market price (optimal control)
**FRAMEWORK**

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1. market cycles models (time series)
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3. position w.r.t. estimated market price (optimal control)

Game theory, prisoner’s dilemma:

- Two suspects
- if one testifies against his partner who remains silent, he is free and the other one gets a 10-year sentence.
- If both of them testify against each other, they get a 5-year sentence.
- If both of them remain silent, they get a 6-month sentence.

<table>
<thead>
<tr>
<th>P1</th>
<th>P2</th>
<th>silent (S)</th>
<th>betrays (B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>silent (S)</td>
<td>(-1/2, -1/2)</td>
<td>(-10, 0)</td>
<td></td>
</tr>
<tr>
<td>betrays (B)</td>
<td>(0, -10)</td>
<td>(-5, -5)</td>
<td></td>
</tr>
</tbody>
</table>

Each player tries to minimize its potential sentence.
⇒ (B, B) is a Nash equilibrium, i.e. a couple of strategies such that no player can decrease its sentence unilaterally.
**Simple Lapse Model**

Consider $I$ insurers on a market with $n$ policyholders. Let $x \in [\underline{x}, \overline{x}]^I$ correspond to the price vector.

The choice $C_i$ of policyholder number $i$ is given by a multinomial r.v. $\mathcal{M}_I(1, p_{j \rightarrow}(x))$ where $p_{j \rightarrow}(x) = (p_{j \rightarrow 1}(x), \ldots, p_{j \rightarrow I}(x))$ and $j$ is the previous insurer of customer $i$.

- Probability $P(C_i = k; j) = p_{j \rightarrow k}(x)$ is given by
  
  $$p_{j \rightarrow k}(x) = \begin{cases} 
  \frac{1}{1 + \sum_{l \neq j} e^{f_j(x_j, x_l)}} & \text{if } j = k, \\
  \frac{e^{f_j(x_j, x_k)}}{1 + \sum_{l \neq j} e^{f_j(x_j, x_l)}} & \text{if } j \neq k, 
  \end{cases}$$

  (1)

  where function $f_j(x_j, x_l)$ represents price sensitivity

  $$\bar{f}_j(x_j, x_l) = \mu_j + \alpha_j \frac{x_j}{x_l} \text{ et } \tilde{f}_j(x_j, x_l) = \tilde{\mu}_j + \tilde{\alpha}_j(x_j - x_l).$$

- Portfolio size of insurer $j$ is

  $$N_j(x) = B_{jj}(x) + \sum_{k=1,k \neq j}^I B_{kj}(x).$$

  where $B_{kj} \sim \mathcal{B}(n_k, p_{k \rightarrow j}(x))$ and $n_k$ is the initial portfolio size.
**SIMPLE CLAIM MODEL**

- Assume that for policyholder $i$, the aggregate claim amount is

\[
Y_i = \sum_{l=1}^{M_i} Z_{i,l},
\]

where $M_i$ is the number of claims, $(Z_{i,l})_l$ are the i.i.d. amounts and $M_i \perp (Z_{i,l})$.

- Assume that the $(Y_i)_i$ are independent.

- The total loss for insurer $j$ is

\[
S_j(x) = \sum_{i=1}^{N_j(x)} Y_i.
\]

- 2 particular cases are considered for illustrations: Poisson – Lognormal (PLN) and Negative Binomial – Lognormal (NBLN).
Simple objective function

Choice of objective function $x \mapsto O_j(x)$ is guided by

- economic criteria: for given $x_{-j}$, $x_j \mapsto O_j(x)$ must be decreasing in $x_j$ and depend on a threshold $\pi_j$,
- mathematical constraints: $x_j \mapsto O_j(x)$ must be strictly concave.

One chooses

$$O_j(x) = \frac{n_j}{n} \left( 1 - \beta_j \left( \frac{x_j}{m_j(x)} - 1 \right) \right) (x_j - \pi_j)$$

where threshold premium $\pi_j$ and average market premium $m_j(x)$ are given by

$$\pi_j = \omega_j \bar{a}_{j,0} + (1 - \omega_j) \bar{m}_0 \text{ and } m_j(x) = \frac{1}{l-1} \sum_{k \neq j} x_k.$$  

- $\bar{a}_{j,0}$, $\bar{m}_0$, $\omega_j$ respectively represent the average actuarial premium, the average market premium and the credibility factor.
- If one did not take competition into account, one would have $O_j(x) = O_j(x_j)$.
**Simple solvency constraint**

One needs \( x_j \mapsto g_j^1(x_j) \) to be closed-form and concave.

- We choose the following solvency constraint:

\[
K_j + n_j(x_j - \pi_j)(1 - e_j) \geq k_{99.5\%} \sigma(Y) \sqrt{n_j},
\]

where \( e_j \) is the commission rate and \( k_{99.5\%} \) is the coefficient such that

\[
E(Y)n_j + k_{99.5\%} \sigma(Y) \sqrt{n_j} \approx \text{VaR}_{99.5\%} \left( \sum_{i=1}^{n_j} Y_i \right).
\]

In practice we take \( k_{99.5\%} = 3 \).

- Constraint \( g_j \) is defined by

\[
\begin{align*}
g_j^1(x_j) &= \frac{K_j + n_j(x_j - \pi_j)(1 - e_j)}{k_{99.5\%} \sigma(Y) \sqrt{n_j}} - 1 \\
g_j^2(x_j) &= x_j - x \\
g_j^3(x_j) &= \bar{x} - x_j
\end{align*}
\] (3)
SIMPLE GAME SEQUENCE

During one period, the game is as follows.

1. Insurers determine their offered price after a Nash equilibrium, by solving for all \( j \in \{1, \ldots, I\} \)

\[
x_{-j} \mapsto \arg \max_{x_j, g_j(x_j) \geq 0} O_j(x_j, x_{-j}).
\]

2. Policyholders randomly choose their new insurer according to \( p_{k\rightarrow j}(x^*) : \) on obtient \( N_j(x^*) \).

3. During the year, the aggregate claim amounts \( S_j(x^*) \) are simulated according to portfolio sizes.

4. The underwriting result is determined as

\[
UW_j(x^*) = N_j(x^*)x_j^*(1 - e_j) - S_j(x^*).
\]
**PROPERTIES OF THE SIMPLE MODEL**

**Proposition (ADL(2012a))**

The game considered with objective and constraints described by (2) and (3) admits a unique Nash equilibrium.

**Elements of proof.**

\( O_j \) continuous + \( x_j \mapsto O_j(x) \) quasiconcave \( \Rightarrow \) existence,

\( x_j \mapsto O_j(x) \) strictly concave \( \Rightarrow \) uniqueness.

**Proposition (ADL(2012a))**

Let \( x^* \) be the equilibrium premium of the I-insurer game. For each player \( j \), if \( x_j^* \in ]x, \bar{x}[ \), the equilibrium premium \( x_j^* \) depends on parameters in the following manner:

- it increases in threshold premium \( \pi_j \), in Solvency constraint \( k_{99.5\%} \), in claims volatility \( \sigma(Y) \), in commission fee \( e_j \);

- and decreases in sensitivity premium \( \beta_j \) and in capital \( K_j \).

**Elements of proof.**

KKT Conditions and implicit function theorem.
**Simple model : numerical results**

Consider 3 players with 10000 customers, i.e. $n = 10000$, $I = 3$, and $(n_1, n_2, n_3) = (4500, 3200, 2300)$ as well as $K_i$ such that the coverage ratio is 133%.

<table>
<thead>
<tr>
<th>P1</th>
<th>P2</th>
<th>P3</th>
<th>market</th>
<th>$E(X)$</th>
<th>$\sigma(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PLN</td>
<td>1.129</td>
<td>1.227</td>
<td>1.029</td>
<td>1.190</td>
<td>1</td>
</tr>
<tr>
<td>NBLN</td>
<td>1.142</td>
<td>1.258</td>
<td>1.095</td>
<td>1.299</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table:** Actuarial premium $\bar{a}_{j,0}$ and market premium $\bar{m}_0$

Equilibrium premia are given in the table below.

<table>
<thead>
<tr>
<th></th>
<th>$x_1^*$</th>
<th>$x_2^*$</th>
<th>$x_3^*$</th>
<th>$\Delta \hat{N}_1$</th>
<th>$\Delta \hat{N}_2$</th>
<th>$\Delta \hat{N}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PLN-ratio</td>
<td>1.612</td>
<td>1.583</td>
<td>1.531</td>
<td>-258.5</td>
<td>-43.4</td>
<td>301.9</td>
</tr>
<tr>
<td>PLN-diff</td>
<td>1.659</td>
<td>1.621</td>
<td>1.566</td>
<td>-382.8</td>
<td>-38.4</td>
<td>421.2</td>
</tr>
<tr>
<td>NBLN-ratio</td>
<td>1.727</td>
<td>1.697</td>
<td>1.648</td>
<td>-239.4</td>
<td>-35.3</td>
<td>274.7</td>
</tr>
<tr>
<td>NBLN-diff</td>
<td>1.777</td>
<td>1.738</td>
<td>1.685</td>
<td>-385.4</td>
<td>-29.6</td>
<td>415.0</td>
</tr>
</tbody>
</table>

**Table:** Equilibrium premium
LESS SIMPLE MODEL – MAIN CHANGES

Better taking $N_j$ into account: its mean $\hat{N}_j(x)$ is given by

$$\hat{N}_j(x) = n_j \times p_{j \rightarrow j}(x) + \sum_{l \neq j} n_l \times p_{l \rightarrow j}(x).$$

We choose the following functions:

- **Objective function**

  $$\tilde{O}_j(x) = \frac{n_j p_{j \rightarrow j}(x)}{n} (x_j - \pi_j), \quad (4)$$

- **Solvency constraint**

  $$\tilde{g}_j^1(x) = \frac{K_j + n_j (x_j - \pi_j) (1 - e_j)}{k_{99.5\%} \sigma(Y) \sqrt{\hat{N}_j(x)}} - 1. \quad (5)$$
**LESS SIMPLE MODEL : PROPERTIES**

**Proposition (ADL(2012a))**

*The game with objective functions and constraints defined by (4) and (5) admits a generalized Nash equilibrium if for each player \( j = 1, \ldots, l \), \( \tilde{g}_j^1(\bar{x}) > 0 \).*
Consider premium volume $\text{GWP}_{j,t}$, portfolio size $n_{j,t}$, and capital $K_{j,t}$ for player $j$ at time $t$.

- At the beginning of each period, one determines

$$\bar{m}_{t-1} = \frac{1}{d} \sum_{u=1}^{d} \sum_{j=1}^{N} \frac{\text{GWP}_{j,t-u} \times x_{j,t-u}^*}{\text{GWP}_{.,t-u}}$$

et

$$\bar{a}_{j,t} = \frac{1}{1 - e_{j,t}} \frac{1}{d} \sum_{u=1}^{d} \frac{s_{j,t-u}}{n_{j,t-u}}.$$ 

So, one has $\pi_{j,t} = \omega_j \bar{a}_{j,t} + (1 - \omega_j) \bar{m}_{t-1}$.

- Objective functions and constraints become

$$O_{j,t}(x) = \frac{n_{j,t}}{n} \left( 1 - \beta_{j,t} \left( \frac{x_j}{m_j(x)} - 1 \right) \right) (x_j - \pi_{j,t}),$$

$$g_{j,t}^1(x_j) = \frac{K_{j,t} + n_{j,t}(x_j - \pi_{j,t})(1 - e_{j,t})}{k_{995} \sigma(Y) \sqrt{n_{j,t}}} - 1.$$
REPEATED SIMPLE MODEL ; GAME SEQUENCE

For period $t$, the game is described as follows:

1. Insurers maximize their objective function
   \[
   \sup_{x_{j,t}} O_{j,t}(x_{j,t}, x_{-j,t}) \text{ tel que } g_{j,t}(x_{j,t}) \geq 0.
   \]

2. Once equilibrium premium $x^*_t$ is determined, policyholders lapse or do not lapse. We get new values $n^*_{j,t}$ of $N_{j,t}(x^*)$.

3. The aggregate claim amount $S_{j,t}$ is simulated according to PLN or NBLN. One obtains $s_{j,t}$.

4. The underwriting result of insurer $j$ is then determined:
   \[
   UW_{j,t} = n^*_{j,t} \times x^*_{j,t} \times (1 - e_j) - s_{j,t}.
   \]

5. Eventually, the capital is given by
   \[
   K_{j,t} = K_{j,t-1} + UW_{j,t}.
   \]

Player $j$ is said to be ruined or insolvent if $K_{j,t} < 0$ or $n^*_{j,t} = 0$. 
**Repeated simple model : properties**

**Proposition (ADL(2012a))**

For the 1-period game, if $\forall k \neq j$, $x_{j,t} \leq x_{k,t}$ and $x_{j,t}(1 - e_{j,t}) \leq x_{k,t}(1 - e_{k,t})$, then underwriting results by policy $uw_{j,t}$ are stochastically ordered : $uw_{j,t} \leq_{icx} uw_{k,t}$.

**Elements of proof.**

Majorization order techniques and convex order properties.

**Proposition (ADL(2012a))**

For the game repeated infinitely many times, the probability that at least two insurers remain solvent up to time $t$ geometrically decreases in $t$.

**Elements of proof.**

Bounding probability $P(\text{Card}(I_t) > 1)$.
**REPEATED SIMPLE MODEL : ONE SAMPLE PATH**

**Figure:** NBLN Claim model and sensitivity $\tilde{f}_j$
Empirical probability to be leader

Simulation of $2^{14} \approx 16000$ sample paths over $T = 20$ periods.

<table>
<thead>
<tr>
<th></th>
<th>Ruin before $t = 10$</th>
<th>Ruin before $t = 20$</th>
<th>Leader at $t = 5$</th>
<th>Leader at $t = 10$</th>
<th>Leader at $t = 20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Insurer 1</td>
<td>6.1e-05</td>
<td>6.1e-05</td>
<td>0.593</td>
<td>0.381</td>
<td>0.331</td>
</tr>
<tr>
<td>Insurer 2</td>
<td>0</td>
<td>0</td>
<td>0.197</td>
<td>0.308</td>
<td>0.329</td>
</tr>
<tr>
<td>Insurer 3</td>
<td>0.000244</td>
<td>0.000244</td>
<td>0.21</td>
<td>0.312</td>
<td>0.34</td>
</tr>
</tbody>
</table>

**Table:** Ruin probability and probability to be leader

<table>
<thead>
<tr>
<th></th>
<th>Min.</th>
<th>1st Qu.</th>
<th>Median</th>
<th>Average</th>
<th>3rd Qu.</th>
<th>Max.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Insurer 1</td>
<td>-0.7905</td>
<td>0.2309</td>
<td>0.3617</td>
<td>0.3563</td>
<td>0.4869</td>
<td>1.2140</td>
</tr>
<tr>
<td>Insurer 2</td>
<td>-0.4340</td>
<td>0.2279</td>
<td>0.3600</td>
<td>0.3555</td>
<td>0.4869</td>
<td>1.1490</td>
</tr>
<tr>
<td>Insurer 3</td>
<td>-0.4730</td>
<td>0.2308</td>
<td>0.3627</td>
<td>0.3563</td>
<td>0.4871</td>
<td>1.0950</td>
</tr>
</tbody>
</table>

**Table:** Underwriting result by policy at $t = 20
**CYCLES**

Calibrate an AR(2) process $X_t = a_1 X_{t-1} + a_2 X_{t-2} + \varepsilon_t$. If $a_2 < 0$ and $a_1^2 + 4a_2 < 0$, then $(X_t)$ is $p$-periodic with $p = 2\pi \arccos \left( \frac{a_1}{2\sqrt{-a_2}} \right)$.

**Figure:** Market premium
SOME PERSPECTIVES

- Work in progress.

- Leader - follower models.

- Direct sales channel / UK market data.