ELLIPTICAL SYMMETRY OF REAL RETURNS IN SOUTH AFRICA AND ITS IMPLICATIONS FOR LONG-TERM ACTUARIAL MODELLING

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AFIR/ERM Colloquium

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AGENDA

- Introduction
- Tests for elliptical symmetry
- Empirical tests
- Summary & conclusions

INTRODUCTION

- Why the CAPM is useful for actuaries
- Common misconceptions about the CAPM
- Elliptical symmetry
- Risk measures

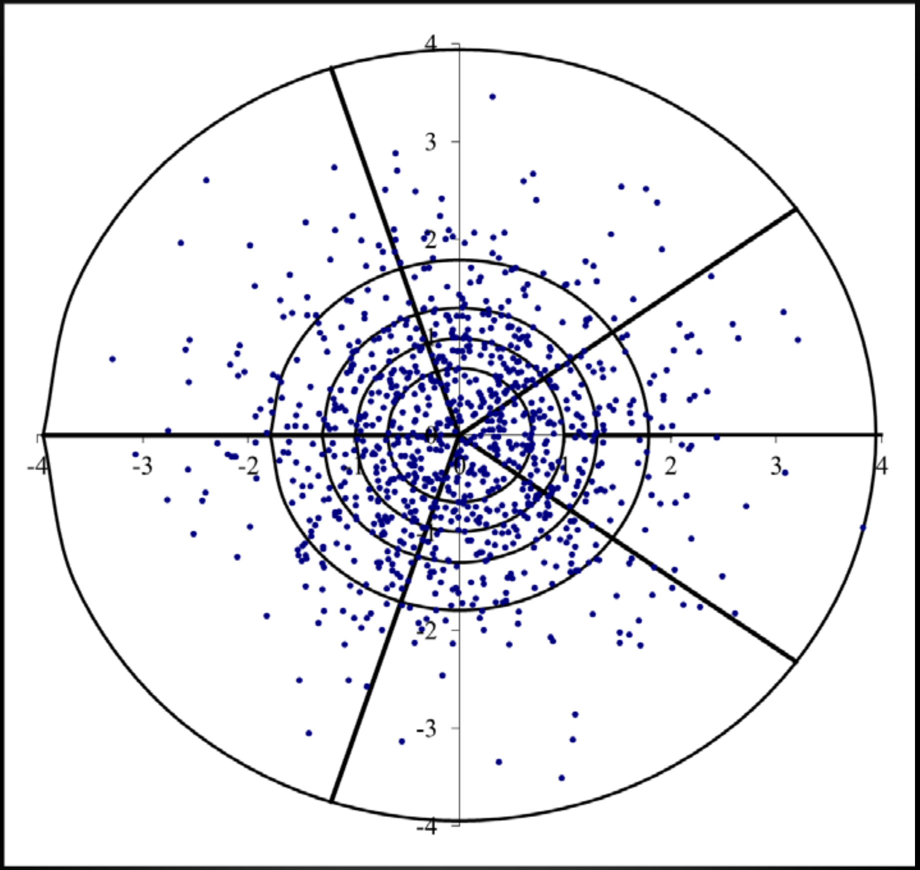
INTRODUCTION: WHY THE CAPM IS USEFUL FOR ACTUARIES

- Assumes equilibrium; so:
  - market-consistent
  - can be used for incomplete markets
- Simple
- Can be used for stochastic modelling; in particular:
  - market-consistent pricing of liabilities
  - benchmark portfolios for investment-management mandates;
  - capital-adequacy requirements

INTRODUCTION: COMMON MISCONCEPTIONS ABOUT THE CAPM

The misconception	**The simple truth**
It assumes that the distribution of returns is multivariate normal (no fat tails)	It assumes that the distribution of returns is elliptically symmetric (tails may be fat)
It assumes that investors have quadratic utility functions	It assumes that investors' risk measures are positive-homogeneous and translation-invariant
or that they are expected-utility maximisers	or that they choose to express their preferences in mean–variance space

INTRODUCTION: CIRCULAR SYMMETRY



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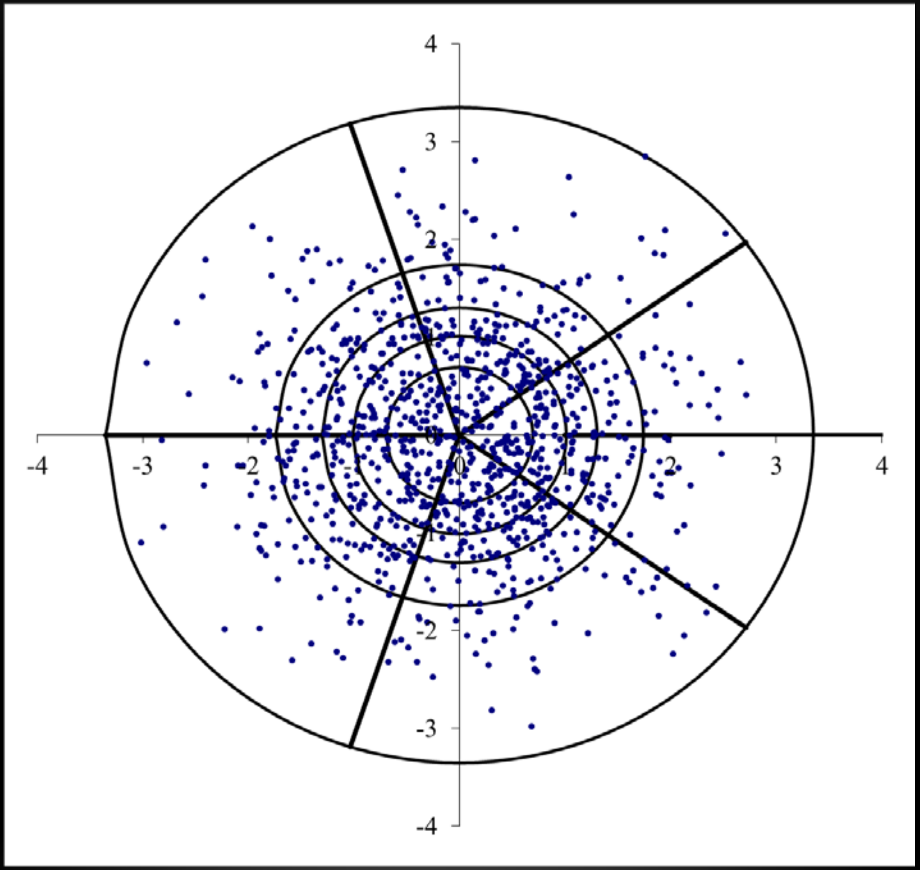

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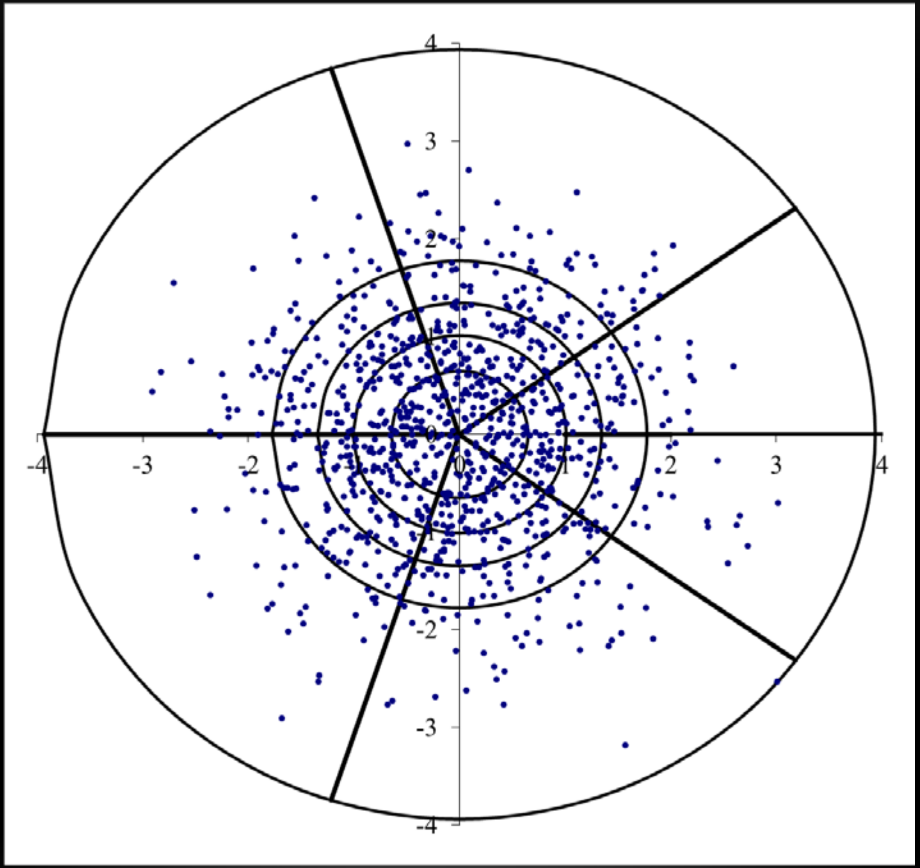


INTRODUCTION:
ELLIPTICAL SYMMETRY

- “far superior models to the multivariate normal for daily and weekly US stock-return data”
  - McNeil, Frey & Embrechts, 2005
- mainly because of fat tails
- still allows us to use the CAPM

INTRODUCTION:
RISK MEASURES

- Positive-homogeneous:
  - market participant’s capital required is proportional to the scale of the distribution of losses
- Translation-invariant:
  - extra capital required for a deterministic shift in the distribution of losses is equal to the shift

TESTS FOR ELLIPTICAL SYMMETRY: STANDARDISATION

$$X = \hat{\Sigma}^{-\frac{1}{2}}\left(R - \hat{\mu}\right)$$

TESTS FOR ELLIPTICAL SYMMETRY: USEFUL PROPERTIES

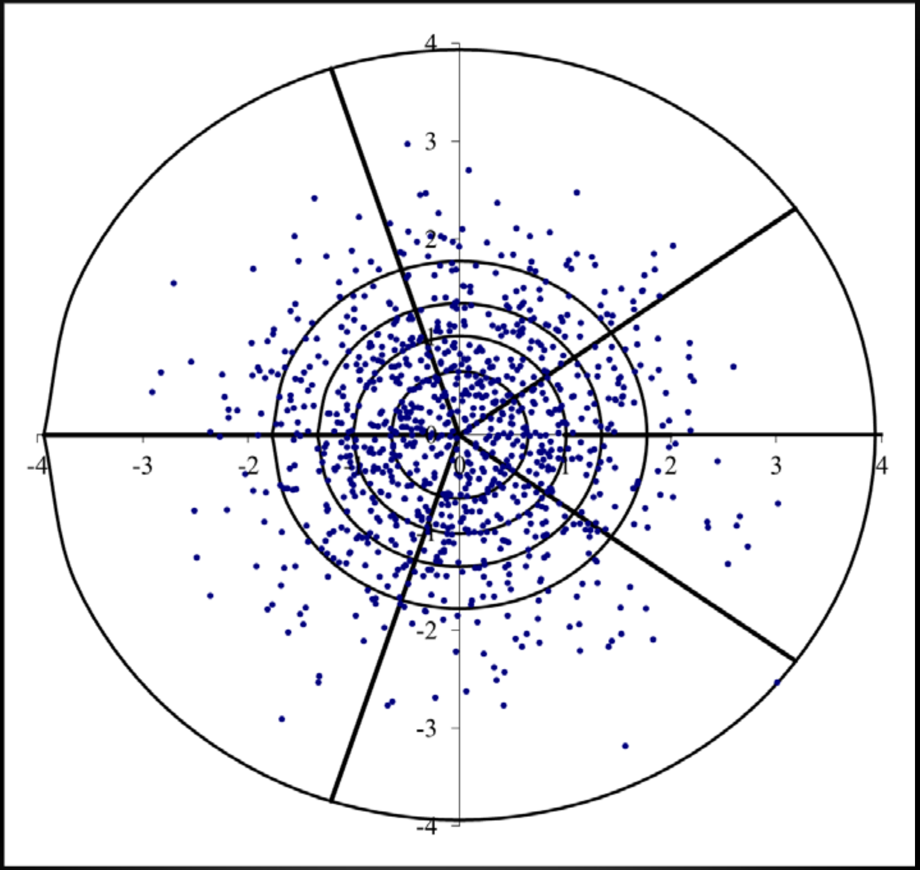
- $U = \frac{Z}{|Z|}$ is uniformly distributed on the unit hypersphere;

and

- ***U*** and |*Z*| are independent.

Here |*Z*| is the Euclidean norm (i.e. the radial length) of the vector ***Z*** and ***U*** is the radial projection of that vector ***Z*** onto the unit hypersphere.

TESTS FOR ELLIPTICAL SYMMETRY: SEGMENTATION



Huffer & Park, 2007

TESTS FOR ELLIPTICAL SYMMETRY: SPHERICAL HARMONICS

- Statistics with various desirable properties
- e.g. rotation-invariance

e.g. Baringhaus, 1991

TESTS FOR ELLIPTICAL SYMMETRY: DISTANCE

$$K\left(X_i, Z_i\right)$$

where:

$$\boldsymbol{Z}_i = \frac{\boldsymbol{Y}_i}{\left|\boldsymbol{Y}_i\right|}\left|\boldsymbol{X}_i\right|$$

$$Y_i \sim N(0,1)$$

Diks & Tong, 1999

TESTS FOR ELLIPTICAL SYMMETRY: A GENERAL PROBLEM

- All tests assume μ and Σ known

EMPIRICAL TESTS:
DATA: TOTAL REAL FORCES OF RETURN

Quarters	Period	Equities (all shares)	Conventional bonds (3, 10, 20 yrs.)	Inflation-linked bonds (3, 10, 20 yrs.)
[1,33]	10/1964–12/1972	×	×	
[34,85]	1/1973–12/1985	×	×	
[86,100]	1/1986–9/1989	×	×	
[101,152]	10/1989–9/2002	×	×	
[153,185]	10/2002–12/2010	×	×	×

EMPIRICAL TESTS: PROBLEMS WITH SEGMENTATION

- 7 dimensions, 3 sectors ⇒ $2^7 \times 3 = 378$ cells
- Period of 33 quarters ⇒ 33 observations

EMPIRICAL TESTS:
PROBLEMS WITH SPHERICAL HARMONICS

- Statistics lack intuitive or heuristic appeal
- Tests insufficiently powerful

EMPIRICAL TESTS:
THE DIKS–TONG TEST: TEST STATISTIC

$$T = \frac{2}{n(n-1)} \sum_{t=1}^{n-1} \sum_{u=t+1}^{n} K\left(\boldsymbol{X}_t, \boldsymbol{X}_u\right)$$

where:

$$K(\boldsymbol{x}, \boldsymbol{y}) = \exp\left\{ -\frac{\left(\boldsymbol{x}-\boldsymbol{y}\right)'\left(\boldsymbol{x}-\boldsymbol{y}\right)}{4d^2} \right\}$$

$$\boldsymbol{X}_t = \hat{\boldsymbol{\Sigma}}^{-\frac{1}{2}}\left(\boldsymbol{R}_t - \hat{\boldsymbol{\mu}}\right)$$

EMPIRICAL TESTS: THE DIKS–TONG TEST: DISTRIBUTION

$$T_v = \frac{2}{n(n-1)} \sum_{t=1}^{n-1} \sum_{u=t+1}^{n} K\left(\boldsymbol{X}_t, \boldsymbol{Z}_{vu}\right)$$

- where:

$$\boldsymbol{Z}_{vt} = \frac{\boldsymbol{Y}_{vt}}{\left|\boldsymbol{Y}_{vt}\right|} \left|\boldsymbol{X}_t\right| \text{ for } v = 1, \ldots, N;\ t = 1, \ldots, n;$$

$$Y_{vt} \sim N(\boldsymbol{0}, \boldsymbol{I})$$

EMPIRICAL TESTS:
THE DIKS–TONG TEST: P-VALUE

$$P\left(T \geq T_\nu \mid \boldsymbol{\mu} = \hat{\boldsymbol{\mu}}, \boldsymbol{\Sigma} = \hat{\boldsymbol{\Sigma}}, \nu = 1, \ldots, N\right) = \frac{k}{N+1}$$

where *k* is the number of values in

$$\bigcup_{\nu=1}^{N} \{T_\nu\} \cup \{T\}$$

such that

$$k \geq T$$

EMPIRICAL TESTS:
THE DIKS–TONG TEST: A PROBLEM

Whereas:

$$\sum_{t=1}^{n-1}\sum_{u=i+1}^{n} K\left(\boldsymbol{X}_t, \boldsymbol{X}_u\right) = \sum_{t=1}^{n-1}\sum_{u=i+1}^{n} K\left(\boldsymbol{X}_u, \boldsymbol{X}_t\right);$$

in general:

$$\sum_{t=1}^{n-1}\sum_{u=t+1}^{n} K\left(\boldsymbol{X}_t, \boldsymbol{Z}_{vu}\right) \neq \sum_{t=1}^{n-1}\sum_{u=t+1}^{n} K\left(\boldsymbol{Z}_{vu}, \boldsymbol{X}_t\right)$$

EMPIRICAL TESTS:
THE DIKS–TONG TEST: A PROBLEM

Whereas:

$$\sum_{t=1}^{n-1}\sum_{u=i+1}^{n} K\left(\boldsymbol{X}_t, \boldsymbol{X}_u\right) = \sum_{t=1}^{n-1}\sum_{u=i+1}^{n} K\left(\boldsymbol{X}_u, \boldsymbol{X}_t\right);$$

← no problem

in general:

$$\sum_{t=1}^{n-1}\sum_{u=t+1}^{n} K\left(\boldsymbol{X}_t, \boldsymbol{Z}_{vu}\right) \neq \sum_{t=1}^{n-1}\sum_{u=t+1}^{n} K\left(\boldsymbol{Z}_{vu}, \boldsymbol{X}_t\right)$$

← problem

EMPIRICAL TESTS:
THE DIKS–TONG TEST: THE SOLUTION

Diks–Tong	**Thomson**
$T = \frac{2}{n(n-1)} \sum_{t=1}^{n-1} \sum_{u=i+1}^{n} K\left(\boldsymbol{X}_t, \boldsymbol{X}_u\right)$	$T = \frac{1}{n(n-1)} \sum_{\substack{t,u=1 \\ u \neq t}}^{n} K\left(\boldsymbol{X}_t, \boldsymbol{X}_u\right)$
$T_v = \frac{2}{n(n-1)} \sum_{t=1}^{n-1} \sum_{u=t+1}^{n} K\left(\boldsymbol{X}_t, \boldsymbol{Z}_{vu}\right)$	$T_v = \frac{1}{n(n-1)} \sum_{\substack{t,u=1 \\ u \neq t}}^{n-1} K\left(\boldsymbol{X}_t, \boldsymbol{Z}_{vu}\right)$

EMPIRICAL TESTS:
THE DIKS–TONG TEST: GENERAL PROBLEM

- All tests assume μ and Σ known

EMPIRICAL TESTS:
THE DIKS–TONG TEST: THE SOLUTION

- All tests assume μ and Σ known
- Use a bootstrapping procedure to derive conditional distributions of the *p*-value given bootstrapped estimates of μ and Σ and hence an unconditional *p*-value.

EMPIRICAL TESTS: THE BOOTSTRAPPED DIKS–TONG TEST

For $\upsilon = 1, \ldots, M$:

(a) sample $\boldsymbol{R}^{*}_{\upsilon 1}, \ldots, \boldsymbol{R}^{*}_{\upsilon n}$ with replacement from $\boldsymbol{R}_1, \ldots, \boldsymbol{R}_n$;

(b) calculate the bootstrapped parameters:

$$\hat{\boldsymbol{\mu}}^{*}_{\upsilon} = \frac{1}{n}\sum_{t=1}^{n} \boldsymbol{R}^{*}_{\upsilon t} \text{ and } \hat{\boldsymbol{\Sigma}}^{*}_{\upsilon} = \frac{1}{n} \boldsymbol{R}^{*\prime}_{\upsilon} \boldsymbol{R}^{*}_{\upsilon}$$

where:

$$\boldsymbol{R}^{*}_{\upsilon} = \begin{pmatrix} \boldsymbol{R}^{*\prime}_{\upsilon 1} - \hat{\boldsymbol{\mu}}^{*\prime}_{\upsilon} \\ \vdots \\ \boldsymbol{R}^{*\prime}_{\upsilon n} - \hat{\boldsymbol{\mu}}^{*\prime}_{\upsilon} \end{pmatrix};$$

(c) calculate:

$$\boldsymbol{X}^{*}_{\upsilon t} = \hat{\boldsymbol{\Sigma}}^{*\,-\frac{1}{2}}_{\upsilon} \left(\boldsymbol{R}^{*}_{\upsilon t} - \hat{\boldsymbol{\mu}}^{*}_{\upsilon} \right) \text{ for } t = 1, \ldots, n$$

EMPIRICAL TESTS:
THE BOOTSTRAPPED DIKS–TONG TEST

For $\upsilon = 1, \ldots, M$:

…

(c) calculate:

$$\boldsymbol{X}^*_{\upsilon t} = \hat{\boldsymbol{\Sigma}}^{*\,-\frac{1}{2}}_{\upsilon} \left(\boldsymbol{R}_t - \hat{\boldsymbol{\mu}}^*_{\upsilon} \right) \text{ for } t = 1, \ldots, n$$

(d) calculate the test statistic:

$$T^*_{\upsilon} = \frac{1}{n(n-1)} \sum_{\substack{t,u=1 \\ u \neq t}}^{n-1} K\left(\boldsymbol{X}^*_{\upsilon t}, \boldsymbol{X}^*_{\upsilon u} \right)$$

EMPIRICAL TESTS:
THE BOOTSTRAPPED DIKS–TONG TEST

For $\upsilon = 1, \ldots, M$:

…

(d) calculate the test statistic:

$$T_\upsilon^* = \frac{1}{n(n-1)} \sum_{\substack{t,u=1 \\ u \neq t}}^{n-1} K\left(\boldsymbol{X}_{\upsilon t}^*, \boldsymbol{X}_{\upsilon u}^*\right)$$

(e) generate a pseudo-random sample by determining:

$$\boldsymbol{Z}_{\upsilon v t}^* = \frac{\boldsymbol{Y}_{\upsilon v t}}{\left|\boldsymbol{Y}_{\upsilon v t}\right|} \left|\boldsymbol{X}_{\upsilon t}^*\right| \text{ for } v = 1, \ldots, N;\ t = 1, \ldots, n;$$

where $Y_{vt} \sim N(\boldsymbol{0}, \boldsymbol{I})$

and hence, for $v = 1, \ldots, N$, calculate:

$$T = \frac{1}{n(n-1)} \sum_{\substack{t,u=1 \\ u \neq t}}^{n} K\left(\boldsymbol{X}_t, \boldsymbol{X}_u\right)$$

EMPIRICAL TESTS:
THE BOOTSTRAPPED DIKS–TONG TEST

For $\upsilon = 1, \ldots, M$:

…

(d) calculate the test statistic:

$$T_{\upsilon}^{*} = \frac{1}{n(n-1)} \sum_{\substack{t,u=1 \\ u \neq t}}^{n-1} K\left(\boldsymbol{X}_{\upsilon t}^{*}, \boldsymbol{X}_{\upsilon u}^{*} \right)$$

(e) generate a pseudo-random sample by determining:

$$\boldsymbol{Z}_{\upsilon \nu t}^{*} = \frac{\boldsymbol{Y}_{\upsilon \nu t}}{\left|\boldsymbol{Y}_{\upsilon \nu t}\right|} \left|\boldsymbol{X}_{\upsilon t}^{*}\right| \text{ for } \nu = 1, \ldots, N;\ t = 1, \ldots, n;$$

By substituting $Y_{\upsilon\nu t}$ for $Z^*_{\upsilon\nu t}$ the test becomes a test for a multivariate normal distribution

where $Y_{\nu t} \sim N(\boldsymbol{0}, \boldsymbol{I})$

and hence, for $\nu = 1, \ldots, N$, calculate:

$$T_{\upsilon \nu}^{*} = \frac{1}{n(n-1)} \sum_{\substack{t,u=1 \\ u \neq t}}^{n} K\left(\boldsymbol{X}_{\upsilon t}^{*}, \boldsymbol{Z}_{\upsilon \nu u}^{*} \right)$$

EMPIRICAL TESTS: THE BOOTSTRAPPED DIKS–TONG TEST

For $\upsilon = 1, \dots, M$:

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and hence, for $\nu = 1, \dots, N$, calculate:

$$T^*_{\upsilon\nu} = \frac{1}{n(n-1)} \sum_{\substack{t,u=1 \\ u \neq t}}^{n} K\left(\boldsymbol{X}^*_{\upsilon t}, \boldsymbol{Z}^*_{\upsilon\nu u}\right)$$

(f) calculate the p-value of the test statistic for bootstrap sample υ as:

$$p^*_\upsilon = P\left(T^*_\upsilon \geq T^*_{\upsilon\nu} \mid \boldsymbol{\mu} = \hat{\boldsymbol{\mu}}^*_\upsilon, \boldsymbol{\Sigma} = \hat{\boldsymbol{\Sigma}}^*_\upsilon, \nu = 1, \dots, N\right) = \frac{k^*_\upsilon}{N+1}$$

where k^*_υ is the number of values in $\bigcup_{\nu=1}^{N}\left\{T^*_{\upsilon\nu}\right\} \cup \left\{T^*_\upsilon\right\}$ that are greater than or equal to T^*_υ.

EMPIRICAL TESTS: THE BOOTSTRAPPED DIKS–TONG TEST

For $\upsilon = 1,\ldots,M$:

…

(f) calculate the *p*-value of the test statistic for bootstrap sample υ as:

$$p_\upsilon^* = P\left(T_\upsilon^* \geq T_{\upsilon\nu}^* \mid \boldsymbol{\mu} = \hat{\boldsymbol{\mu}}_\upsilon^*, \boldsymbol{\Sigma} = \hat{\boldsymbol{\Sigma}}_\upsilon^*, \nu = 1,\ldots,N\right) = \frac{k_\upsilon^*}{N+1}$$

where k_υ^* is the number of values in $\bigcup_{\nu=1}^{N} \left\{T_{\upsilon\nu}^*\right\} \cup \left\{T_\upsilon^*\right\}$ that are greater than or equal to T_υ^*.

Determine the *p*-value for all bootstrap samples combined as:

$$\sum_{\upsilon=1}^{M} P\left(\boldsymbol{\mu} = \hat{\boldsymbol{\mu}}_\upsilon^*, \boldsymbol{\Sigma} = \hat{\boldsymbol{\Sigma}}_\upsilon^* \mid \upsilon = 1,\ldots,M\right) P\left(T_\upsilon^* \geq T_{\upsilon\nu}^* \mid \boldsymbol{\mu} = \hat{\boldsymbol{\mu}}_\upsilon^*, \boldsymbol{\Sigma} = \hat{\boldsymbol{\Sigma}}_\upsilon^*, \nu = 1,\ldots,N\right) = \sum_{\upsilon=1}^{M} \frac{1}{M} p_\upsilon^*$$

$\frac{1}{M}$ p_υ^*

EMPIRICAL TESTS: RESULTS

Period	***p*-value**	
	elliptical symmetry	normal
excluding inflation-linked bonds (p = 4)		
[1,33] (n = 33)	0,00	0,00
[34,85] (n = 52)	0,17	0,02
[86,100] (n = 15)	0,80	0,91
[101,152] (n = 52)	0,37	0,08
[153,185] (n = 33)	0,34	0,29
all (n = 185)	0,00	0,00
[34,185] (n = 152)	0,20	0,00
including inflation-linked bonds (p = 7)		
[153,185] (n = 33)	0,83	0,88
risk premia (p = 7)		
[153,185] (n = 33)	0,76	0,80

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SUMMARY & CONCLUSIONS

- Since 1973 the joint distribution of quarterly returns on equities and bonds has been elliptically symmetrical.
- This distribution explains these returns better than the multivariate normal.
- If market participants' risk measures are positive-homogeneous and translation-invariant, the CAPM may be used with a suitable elliptically symmetric distribution.

SUMMARY & CONCLUSIONS

- Normative risk measures may become descriptive.
- Otherwise the client can outperform the market on a risk-adjusted basis.
- Equilibrium: reality versus market-consistency
- Strategic investment decision-making versus market-consistent prices, market-consistent investment-performance benchmarks and capital adequacy requirements.
- For further research: the development of stochastic models using elliptically symmetric distributions.