EXPERIENCED-BASED STOCHASTIC MORTALITY FOR INTERNAL MODELS *

ANNAMARIA OLIVIERI
University of Parma (Italy)
annamaria.olivieri@unipr.it

ERMANNO PITACCO
University of Trieste (Italy)
ermanno.pitacco@econ.units.it

IAA FINANCIAL RISKS AND ERM SECTION
SECTION DES RISQUES FINANCIERS ET GRE DE L'AAI

October 1st, 2012

* Mainly based on: OLIVIERI, A. and PITACCO, E. (2009) Stochastic mortality: the impact on target capital. *ASTIN Bulletin*, vol. 39(2), pp. 541–563

Awarded the AFIR/ERM Bob Alting von Geusau Prize 2010

Introduction

Purpose: To perform a stochastic assessment of mortality/longevity risks, calibrating the stochastic mortality model to market or portfolio experience

- Both process and aggregate mortality/longevity risks are addressed
- Other risks (i.e.: market, operational, and so on) are disregarded

Reference is to a life annuity portfolio

Application: Within an internal model, for the assessment of the capital required to face the specific risks of an insurance portfolio

The idea:
We extend some classical results about the modeling of the number of deaths joint to the modeling of parameter uncertainty.
Calibration is based on the best-estimate life table and, through inference, on portfolio experience

Representing the number of deaths: the process risk

Conditional on the best-estimate (BE) life table

One cohort

Assuming independence among the lifetimes:

$$\boxed{[D_{x,t}|\ q^*_{x,t}; n_{x,t-1}] \sim \text{Bin}(n_{x,t-1}, q^*_{x,t})}$$

$n_{x,t-1}$: annuitants age x at time $t-1$

$q^*_{x,t-1}$: BE mortality rate

Possibly approximated as: $\boxed{[D_{x,t}|q^*_{x,t}; n_{x,t-1}] \sim \text{Poi}(n_{x,t-1}\, q^*_{x,t})}$

More than one cohort

If we accept the Poisson approximation (and independence among cohorts): $\boxed{[D_t|\{q^*_{x,t}\}; \{n_{x,t-1}\}] \sim \text{Poi}(\sum_{x=x_0}^{\omega} n_{x,t-1}\, q^*_{x,t})}$

However:

due to uncertainty in mortality dynamics, the *mortality rate is random*

Modeling the mortality rate

Two approaches (at least)

1. model the mortality rate $Q_{x,t}$ directly
2. model an adjustment to the BE mortality rate

First refer to one cohort only

Take the multiplicative model: $Q_{x,t} = q^*_{x,t}\, Z_{x,t}$

where $Z_{x,t}$ ($Z_{x,t} > 0$, but such that $0 \le Q_{x,t} \le 1$) is a (random) coefficient expressing deviations in aggregate mortality/longevity

Note that: $\mathbb{E}[Q_{x,t}] = q^*_{x,t}\, \underbrace{\mathbb{E}[Z_{x,t}]}$

↪ expected aggregate deviation

A particular choice:

$$Z_{x,t} \sim \text{Gamma}(\alpha_{x,t}, \beta_{x,t}) \Rightarrow Q_{x,t} \sim \text{Gamma}\left(\alpha_{x,t}, \frac{\beta_{x,t}}{q^*_{x,t}}\right)$$

Then: $\mathbb{E}[Z_{x,t}] = \frac{\alpha_{x,t}}{\beta_{x,t}} \quad \Rightarrow \quad \mathbb{E}[Q_{x,t}] = q^*_{x,t}\, \frac{\alpha_{x,t}}{\beta_{x,t}}$

Against the Poisson-Gamma assumption

- approximation at old ages (from Poisson assumption)
  but: possibly of poor impact when more than one cohort are referred to
- the range of possible values of the mortality rate (Gamma distribution)
  but: possibly negligible when more than one cohort are addressed

Advantages of the Poisson-Gamma assumption

- generalizations to the case of more than one cohort are straightforward
- correlation in time among the mortality rates can be modeled naturally, without explicit assumptions
- possible extension: accounting for the rate of decrease of the annual payout (rather than simply the mortality rate)

Number of deaths, including aggregate longevity risk

For one cohort only

Setting $Q_{x,t} = q_{x,t}$ we take: $[D_{x,t} | \, q_{x,t}; n_{x,t-1}] \sim \text{Poi}(n_{x,t-1} \, q_{x,t})$

Mortality rate: $Q_{x,t} = q^*_{x,t} \, Z_{x,t}$

$$Z_{x,t} \sim \text{Gamma}(\alpha_{x,t}, \beta_{x,t}) \Rightarrow Q_{x,t} \sim \text{Gamma}\left(\alpha_{x,t}, \frac{\beta_{x,t}}{q^*_{x,t}}\right)$$

Then: $\boxed{[D_{x,t} | \, n_{x,t-1}] \sim \text{NBin}\left(\alpha_{x,t}, \frac{\theta_{x,t}}{\theta_{x,t}+1}\right)}$

$$\theta_{x,t} = \frac{\beta_{x,t}}{n_{x,t-1} \, q^*_{x,t}}$$

More than one cohort

Setting $Q_{x,t} = q_{x,t}$ for all ages x

we let: $$[D_t | \{q_{x,t}\}; \{n_{x,t-1}\}] \sim \mathrm{Poi}(\sum_{x=x_0}^{\omega} n_{x,t-1}\, q_{x,t})$$

We assume: $$Q_{x,t} = q^*_{x,t}\, Z_t$$ for all ages x

REMARK: the adjustment coefficient Z_t is common to all the cohorts in-force at time $t \Rightarrow$ it accounts for period effects only; cohort effects are not addressed

Then: $[D_t | \{z q^*_{x,t}\}; \{n_{x,t-1}\}] \sim \mathrm{Poi}(z\, \sum_{x=x_0}^{\omega} n_{x,t-1}\, q^*_{x,t})$

Taking $$Z_t \sim \mathrm{Gamma}(\alpha_t, \beta_t)$$ we have:

$$[D_t | \ \{n_{x,t-1}\}] \sim \mathrm{NBin}\left(\alpha_t, \frac{\theta_t}{\theta_t + 1}\right)$$

$$\theta_t = \frac{\beta_t}{\sum_{x=x_0}^{\omega} n_{x,t-1}\, q^*_{x,t}}$$

Updating the parameters to experience

Due to the underlying trend, deviations in respect of the BE mortality rate should be (positively) correlated in time

Assumption: the mortality experience from the portfolio is reliable as an evidence of the trend of the cohort (or the population)

$\Rightarrow$ *An inferential procedure can be defined for updating the parameters of the pdf of $Q_{x,t}$ (or the parameters of the distribution of the number of deaths) to experience*

One cohort

Valuation at time 0

Issue time; no previous experience available

$n_{x_0,0}$ annuitants at time 0

$$Z_{x,t} \sim \mathrm{Gamma}(\bar{\alpha}, \bar{\beta})$$

for all times t (and ages $x = x_0 + t$)

$$D_{x_0,1} \sim \mathrm{NBin}\left(\alpha_{x_0,1}, \frac{\theta_{x_0,1}}{\theta_{x_0,1}+1}\right)$$

$$\alpha_{x_0,1} = \bar{\alpha}$$

$$\theta_{x_0,1} = \frac{\bar{\beta}}{n_{x_0,0}\, q^*_{x_0,1}}$$

Valuation at time 1

Let $D_{x_0,1} = d_{x_0,1}$ be the observed number of deaths in $(0,1)$

We can calculate the posterior pdf of $Q_{x_0,1}$, conditional on $D_{x_0,1} = d_{x_0,1}$. It turns out

$$[Q_{x_0,1} | D_{x_0,1} = d_{x_0,1}] \sim \mathrm{Gamma}\left(\bar{\alpha} + d_{x_0,1}, \frac{\bar{\beta}}{q^*_{x_0,1}} + n_{x_0,0}\right)$$

and hence:

$$[Z_{x,t} | D_{x_0,1} = d_{x_0,1}] \sim \mathrm{Gamma}(\bar{\alpha} + d_{x_0,1}, \bar{\beta} + n_{x_0,0}\, q^*_{x_0,1})$$

Expected aggregate deviation:

$$\mathbb{E}[Z_{x,t} |\; D_{x_0,1} = d_{x_0,1}] = \underbrace{\frac{\bar{\alpha} + d_{x_0,1}}{\bar{\beta} + n_{x_0,0}\, q^*_{x_0,1}}}_{\text{updated}} \gtreqless \underbrace{\frac{\bar{\alpha}}{\bar{\beta}}}_{\text{prior}}$$

Number of deaths:

$$[D_{x_0+1,2} | \, n_{x_0,0}, D_{x_0,1} = d_{x_0,1}] \sim \text{NBin}\left(\alpha_{x_0+1,2}, \frac{\theta_{x_0+1,2}}{\theta_{x_0+1,2}+1}\right)$$

$$\alpha_{x_0+1,2} = \bar{\alpha} + d_{x_0,1} \qquad \theta_2 = \frac{\bar{\beta} + n_{x_0,0}\, q^*_{x_0,1}}{n_{x_0+1,1}\, q^*_{x_0+1,2}}$$

Valuation at time t: . . . (similar results follow)

More than one cohort

Similar inferential procedure, with the advantage of relying on a wider data set (i.e. on the number of deaths observed in the population)

At time 0: $[D_1 | \{n_{x,0}\}] \sim \text{NBin}\left(\alpha_1, \frac{\theta_1}{\theta_1+1}\right)$

$\alpha_1 = \bar{\alpha}$ $\qquad \theta_1 = \frac{\bar{\beta}}{\sum_{x=x_0}^{\omega} n_{x,0}\, q^*_{x_0,1}}$

At time 1:
Let $D_1 = d_1 = \sum_{x=x_0}^{\omega} d_{x,1}$ be the observed number of deaths in $(0, 1)$

Then: $[D_2 | \{n_{x,0}\}; D_1 = d_1] \sim \text{NBin}\left(\alpha_2, \frac{\theta_2}{\theta_2+1}\right)$

$\alpha_2 = \bar{\alpha} + d_1$ $\qquad \theta_2 = \frac{\bar{\beta} + \sum_{x=x_0}^{\omega} n_{x,0}\, q^*_{x,1}}{\sum_{x=x_0}^{\omega} n_{x,1}\, q^*_{x_0,2}}$

At time t: ... (similar results follow)

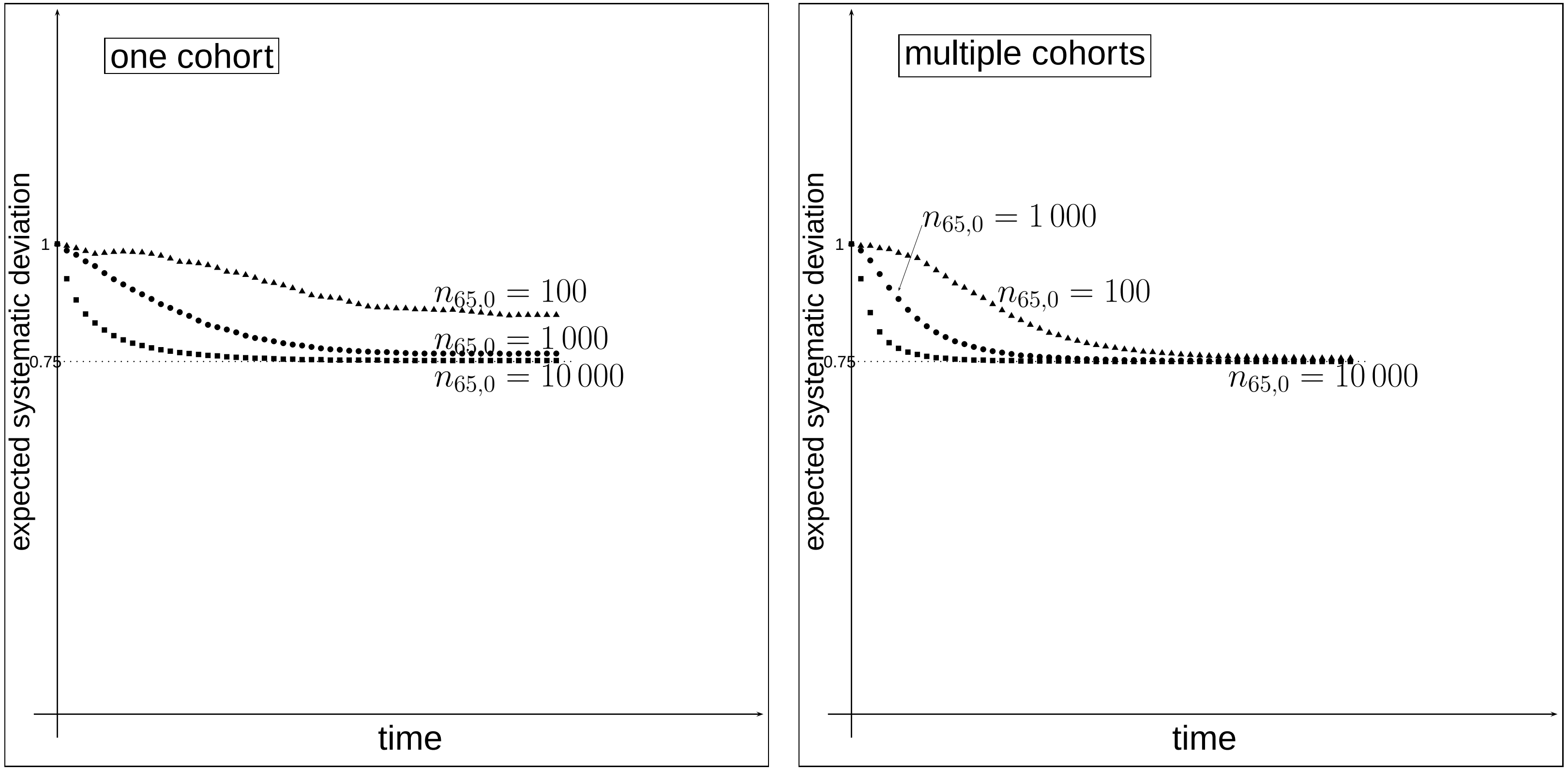
Some numerical findings:

Expected systematic deviation $\mathbb{E}[Z_t|\{d_s\}_{s=1,2,\dots,t}]$

Best-estimate life table ($\{q^*_{x,t}\}$): IPS55 (Italian projected life table, cohort 1955)
Initial age: $x_0 = 65$, males
Initial value of the parameters of the pdf of Z_t:

$\bar{\alpha} = \bar{\beta} \Rightarrow \mathbb{E}[Q_{x,t}] = \frac{\bar{\alpha}}{\bar{\beta}}\, q^*_{x,t} = q^*_{x,t}$
(justified by the meaning of best-estimate life table)

$\bar{\beta} = 100 \Rightarrow \mathbb{CV}[Q_{x,t}] = \frac{\sqrt{\mathbb{V}\mathrm{ar}[Q_{x,t}]}}{\mathbb{E}[Q_{x,t}]} = \frac{1}{\sqrt{\bar{\alpha}}} = 10\%$
(based on expert's opinion on trend volatility)



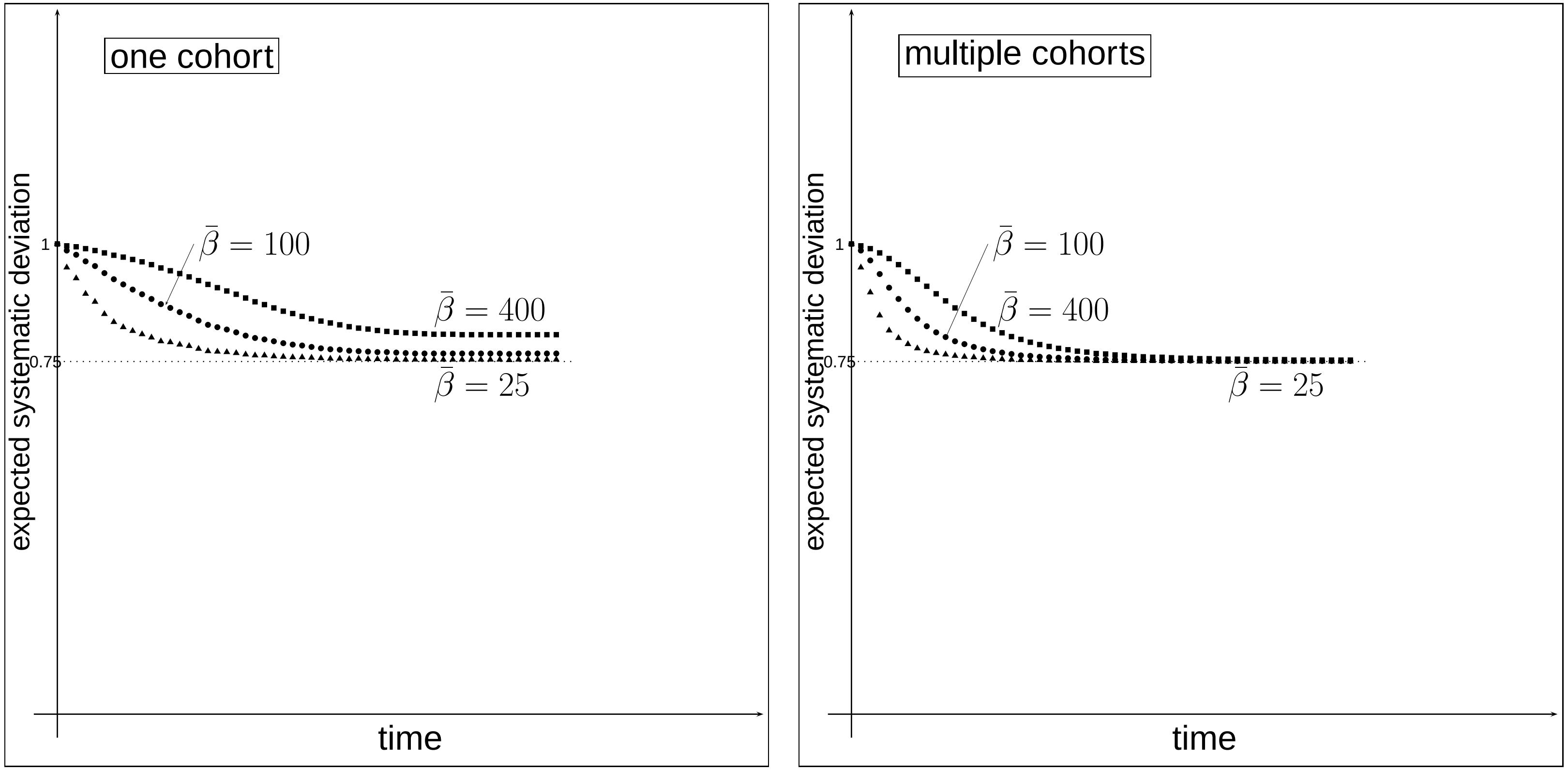
Expected systematic deviation $\mathbb{E}[Z_t | \{d_s\}_{s=1,2,\ldots,t}]$

$\bar{\alpha} = \bar{\beta}$, $\bar{\beta} = 100$

deaths = 75% of best-estimate ($d_{x,s} = 0.75\, n_{x,s-1}\, q^*_{x,s}$)

Left panel: one cohort. Right panel: multiple cohorts

Updating the parameters to experience (*cont.*)



Expected systematic deviation $\mathbb{E}[Z_t | \{d_s\}_{s=1,2,\ldots,t}]$
$\bar{\alpha} = \bar{\beta}$; $n_{65,s} = 1\,000$
deaths $=$ 75% of best-estimate ($d_{x,s} = 0.75\, n_{x,s-1}\, q^*_{x,s}$)
Left panel: one cohort. Right panel: multiple cohorts

Application

The mortality model is experience-based
$\Rightarrow$ suitable for internal models
Particular application: capital required to face mortality/longevity risks

Implementation through a

deterministic . . .
the expected numbers of deaths
(or the expected mortality rates) only are involved

stochastic . . .
the numbers of deaths are stochastically
simulated

. . . **approach**

deterministic via stochastic . . .
short-cut formulae, constructed according to
simulated findings

Deterministic implementation

Refer to the Solvency 2 standard formula

Capital charge (within the SCR – Solvency Capital Requirement) for (aggregate) longevity risk: change in the net value of assets minus liabilities (ΔNAV) against a permanent 20% decrease in mortality rates for each age

Under our assumptions, this reduces to

$$\text{Life}_{\text{long},t} = V_t^{(\Pi)[-20\%]} - V_t^{(\Pi)[\text{BE}]}$$

$V_t^{(\Pi)[-20\%]}$: expected present value of future payments in portfolio Π, under the shock assumption (BE -20%)

$V_t^{(\Pi)[\text{BE}]}$: expected present value of future payments in portfolio Π, under BE assumptions (or: portfolio reserve, net of the risk margin)

At time $t = 0$: shock scenario -20%

$$\underbrace{\mathbb{E}[Z_{x,t}]}_{\text{one cohort}} = \underbrace{\mathbb{E}[Z_t]}_{\text{multiple cohorts}} = \frac{\bar{\alpha}}{\bar{\beta}} = 0.80$$

$\Rightarrow \bar{\alpha} = 0.80\bar{\beta}$, $\bar{\beta} = 100$ (experts' opinion)

At time $t = 1$: shock scenario updated to experience

$$\alpha_1 = \bar{\alpha} \rightarrow \alpha_2 = \bar{\alpha} + \begin{cases} d_{x_0,1} & \text{one cohort} \\ d_1 = \sum_x d_{x,1} & \text{multiple cohorts} \end{cases}$$

$$\beta_1 = \bar{\beta} \rightarrow \beta_2 = \bar{\beta} + \begin{cases} n_{x_0,0}\, q^*_{x_0,1} & \text{one cohort} \\ \sum_x n_{x,0}\, q^*_{x,1} & \text{multiple cohorts} \end{cases}$$

$\Rightarrow$ shock scenario: $1 - \begin{cases} \mathbb{E}[Z_{x,t}|d_{x_0,1}] & = 1 - \frac{\alpha_{x_0+1,2}}{\beta_{x_0+1,2}} \\ \mathbb{E}[Z_t|d_1] & = 1 - \frac{\alpha_2}{\beta_2} \end{cases}$

At time t ...

Example:

One cohort; initial age: $x_0 = 65$; males

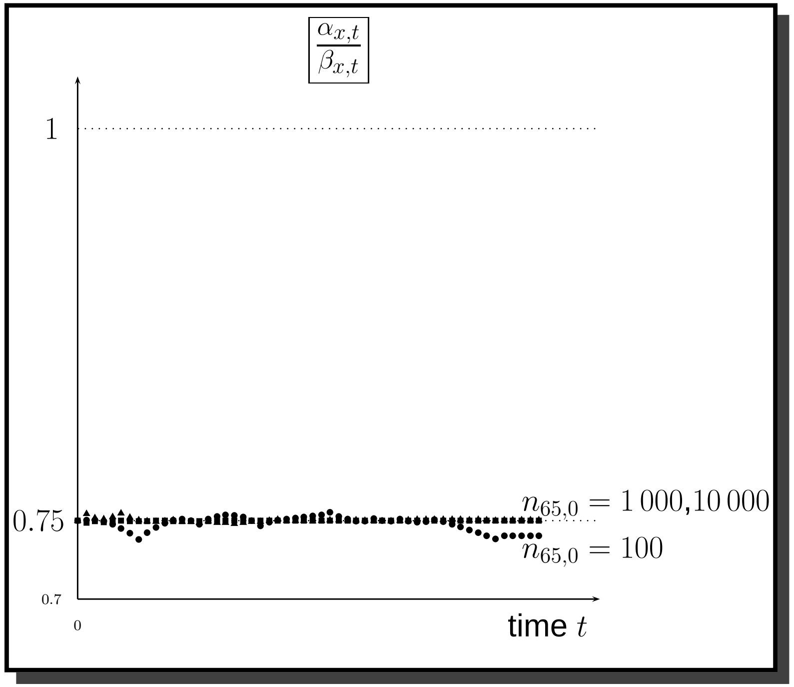
Best estimate life table: IPS55

(Initial) parameters of the pdf of $Z_{x,t}$: $\bar{\alpha} = 0.75\bar{\beta}$, $\bar{\beta} = 100$

Experienced mortality

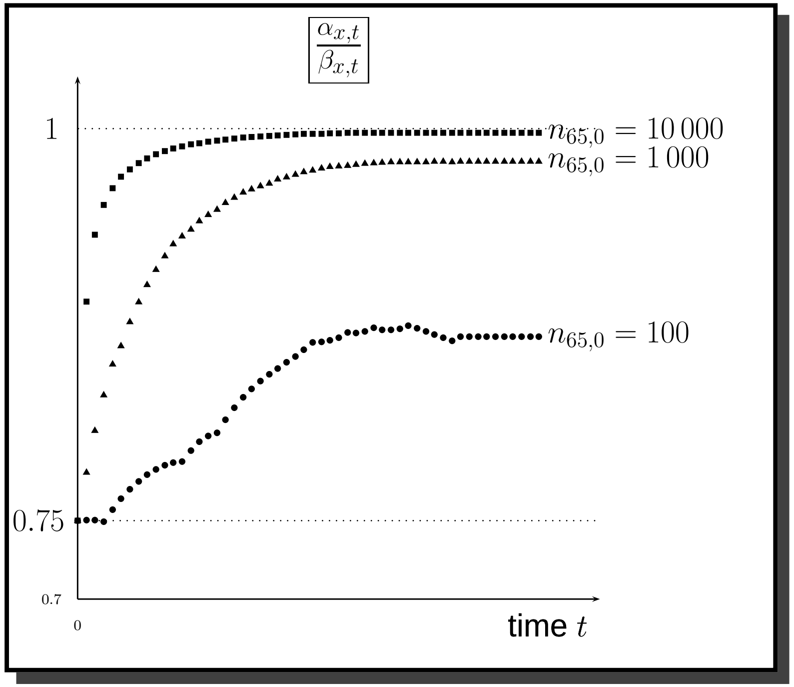
as the shock scenario

(# deaths $\simeq 75\%$ of what expected)



as the BE scenario

(# deaths $\simeq 100\%$ of what expected)



Stochastic implementation

Reference to one cohort only

Let M_t be the assets above the portfolio reserve

Default at time t: $M_t < 0$ (i.e.: unfunded liabilities)

Objective: at the valuation time z, given the accepted default probability ε, set M_z such that

$$\mathbb{P}[(M_{z+1} \geq 0) \cap (M_{z+2} \geq 0) \cap \cdots \cap (M_{\omega-x_0-z} \geq 0) | n_{x_0+z,z}] = 1 - \varepsilon$$

Example

One cohort; initial age: $x_0 = 65$; males

Best-estimate life table: IPS55

Maximum age: $\omega = 119$, whence the maturity of the portfolio at time z is:

$\omega - x_0 - z = 119 - 65 - z$

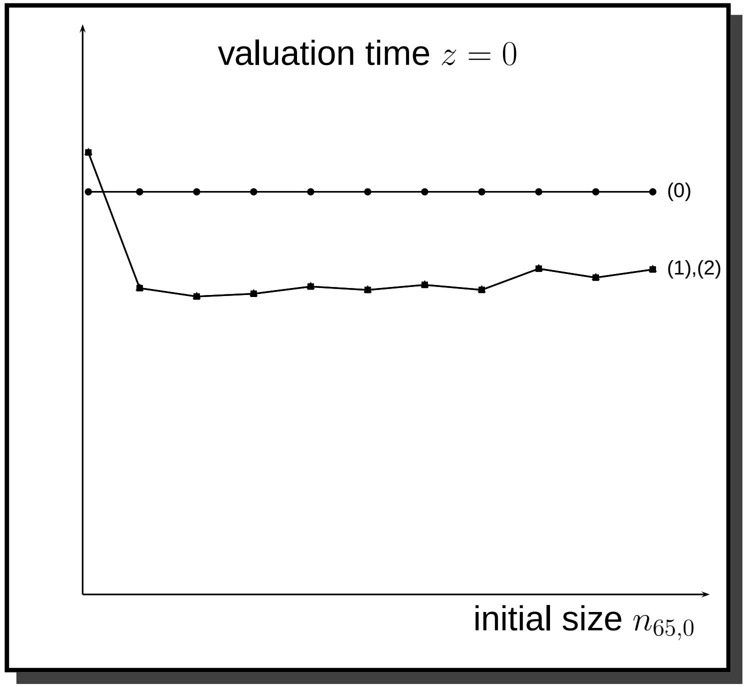
Initial parameters of the pdf of $Z_{x,t}$: $\bar{\alpha} = 0.75\bar{\beta}$, $\bar{\beta} = 100$

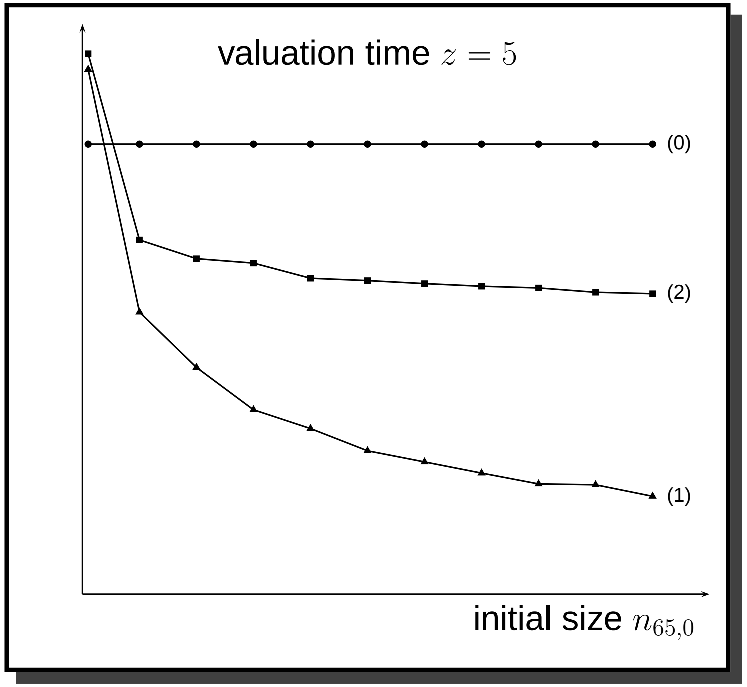
Risk-free rate and investment yield: 3% p.a.

Annual amount: $b = 1$

Internal model with default probability: $\varepsilon = 0.005$

Application (*cont.*)





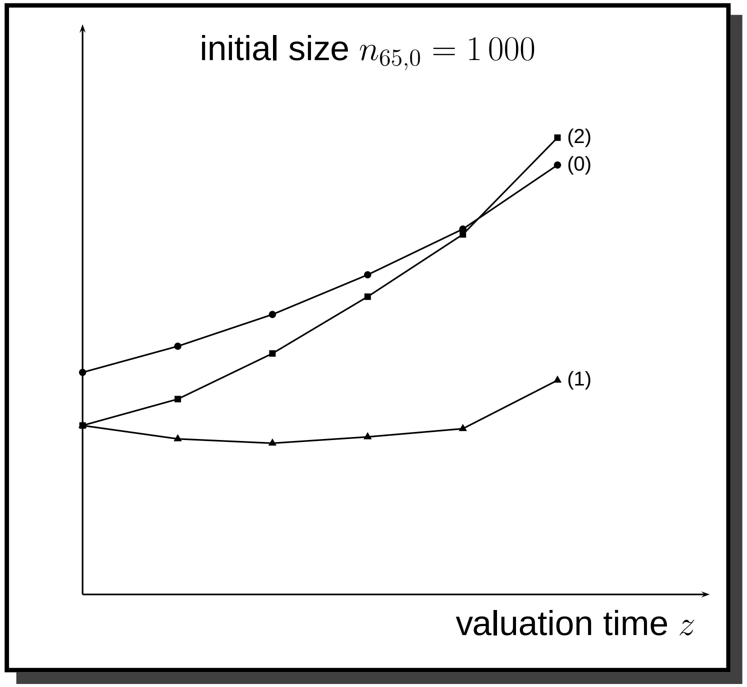
(0) Solvency 2: $\frac{M_z^{[\text{Solv2}]}}{V_z^{(\Pi)[\text{BE}]}} = \frac{\text{Life}_{\text{long},\,z} + RM_z}{V_z^{(\Pi)[\text{BE}]}}$

(1)–(2) Internal model: $\frac{M_z^{[\text{IM}]}}{V_z^{(\Pi)[\text{BE}]}}$

(1) experience as the best-estimate life table

(2) experience as the Solvency 2 stress scenario (BE-25%)

Application (*cont.*)



(0) Solvency 2: $\frac{M_z^{[\text{Solv2}]}}{V_z^{(\Pi)[\text{BE}]}} = \frac{\text{Life}_{\text{long},z} + RM_z}{V_z^{(\Pi)[\text{BE}]}}$

(1)–(2) Internal model: $\frac{M_z^{[\text{IM}]}}{V_z^{(\Pi)[\text{BE}]}}$

(1) experience as the best-estimate life table

(2) experience as the Solvency 2 stress scenario (BE-25%)

Further developments

Statistical tests, especially for setting the initial volatility parameter

Possible extensions: Reduction of the total annual payout, instead of the number of deaths, in order to account for adverse-selection; introduction of cohort specificity; MCMC methods to approximate posterior distributions

Short-cut (factor-based) formulae for capital requirements within an internal model

... ...

References

OLIVIERI, A. and PITACCO, E. (2009) Stochastic mortality: the impact on target capital. *ASTIN Bulletin*, vol. 39(2), pp. 541–563.

Awarded the AFIR/ERM Bob Alting von Geusau Prize 2010.

OLIVIERI, A. (2011) Stochastic mortality: experience-based modeling and application issues consistent with Solvency 2. *European Actuarial Journal*, European Actuarial Journal, vol. 1 (Suppl. 1), pp. S101–S125.

OLIVIERI, A. and PITACCO, E. (2012) Life tables in actuarial models: from the deterministic setting to a Bayesian approach. *AStA Advances in Statistical Analysis*, vol. 96, pp. 127–153.