EXPERIENCED-BASED STOCHASTIC MORTALITY FOR INTERNAL MODELS *

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Introduction

Purpose: To perform a stochastic assessment of mortality/longevity risks, calibrating the stochastic mortality model to market or portfolio experience

Both process and aggregate mortality/longevity risks are addressed

Other risks (i.e.: market, operational, and so on) are disregarded

Reference is to a life annuity portfolio

Application: Within an internal model, for the assessment of the capital required to face the specific risks of an insurance portfolio

The idea:
We extend some classical results about the modeling of the number of deaths joint to the modeling of parameter uncertainty. Calibration is based on the best-estimate life table and, through inference, on portfolio experience
Representing the number of deaths: the process risk

Conditional on the best-estimate (BE) life table

One cohort
Assuming independence among the lifetimes:

\[ [D_{x,t} | q^*_x,t; n_{x,t-1}] \sim \text{Bin}(n_{x,t-1}, q^*_x,t) \]

\( n_{x,t-1} \): annuitants age \( x \) at time \( t - 1 \)
\( q^*_x,t-1 \): BE mortality rate

Possibly approximated as:

\[ [D_{x,t} | q^*_x,t; n_{x,t-1}] \sim \text{Poi}(n_{x,t-1} q^*_x,t) \]

More than one cohort
If we accept the Poisson approximation (and independence among cohorts):

\[ [D_t | \{q^*_x,t\}; \{n_{x,t-1}\}] \sim \text{Poi}(\sum_{x=x_0}^{\omega} n_{x,t-1} q^*_x,t) \]

However:
due to uncertainty in mortality dynamics, the mortality rate is random
Modeling the mortality rate

Two approaches (at least)

1. model the mortality rate $Q_{x,t}$ directly
2. model an adjustment to the BE mortality rate
Modeling the mortality rate (cont.)

First refer to one cohort only

Take the multiplicative model: \( Q_{x,t} = q_{x,t}^* Z_{x,t} \)

where \( Z_{x,t} (Z_{x,t} > 0, \text{ but such that } 0 \leq Q_{x,t} \leq 1) \) is a (random) coefficient expressing deviations in aggregate mortality/longevity

Note that: \( \mathbb{E}[Q_{x,t}] = q_{x,t}^* \mathbb{E}[Z_{x,t}] \)

A particular choice:

\( Z_{x,t} \sim \text{Gamma}(\alpha_{x,t}, \beta_{x,t}) \) \Rightarrow \( Q_{x,t} \sim \text{Gamma} \left( \alpha_{x,t}, \frac{\beta_{x,t}}{q_{x,t}^*} \right) \)

Then: \( \mathbb{E}[Z_{x,t}] = \frac{\alpha_{x,t}}{\beta_{x,t}} \Rightarrow \mathbb{E}[Q_{x,t}] = q_{x,t}^* \frac{\alpha_{x,t}}{\beta_{x,t}} \)
Modeling the mortality rate (cont.)

Against the Poisson-Gamma assumption

approximation at old ages (from Poisson assumption)
but: possibly of poor impact when more than one cohort are referred to
the range of possible values of the mortality rate (Gamma distribution)
but: possibly negligible when more than one cohort are addressed

Advantages of the Poisson-Gamma assumption

generalizations to the case of more than one cohort are straightforward
correlation in time among the mortality rates can be modeled naturally, without explicit assumptions
possible extension: accounting for the rate of decrease of the annual payout (rather than simply the mortality rate)
Number of deaths, including aggregate longevity risk

For one cohort only

Setting $Q_{x,t} = q_{x,t}$ we take: $[D_{x,t} \mid q_{x,t} ; n_{x,t-1}] \sim \text{Poi}(n_{x,t-1} q_{x,t})$

Mortality rate: $Q_{x,t} = q^*_{x,t} Z_{x,t}$

$Z_{x,t} \sim \text{Gamma}(\alpha_{x,t}, \beta_{x,t}) \Rightarrow Q_{x,t} \sim \text{Gamma}(\alpha_{x,t}, \frac{\beta_{x,t}}{q^*_{x,t}})$

Then: $[D_{x,t} \mid n_{x,t-1}] \sim \text{NBin}(\alpha_{x,t}, \frac{\theta_{x,t}}{\theta_{x,t}+1})$

$\theta_{x,t} = \frac{\beta_{x,t}}{n_{x,t-1} q^*_{x,t}}$
Number of deaths, including aggregate longevity risk (cont.)

More than one cohort

Setting $Q_{x,t} = q_{x,t}$ for all ages $x$

we let: $[D_t | \{q_{x,t}\}; \{n_{x,t-1}\}] \sim \text{Poi} \left( \sum_{x=x_0}^\omega n_{x,t-1} q_{x,t} \right)$

We assume: $Q_{x,t} = q_{x,t}^* Z_t$ for all ages $x$

**REMARK**: the adjustment coefficient $Z_t$ is common to all the cohorts in-force at time $t$ ⇒ it accounts for period effects only; cohort effects are not addressed

Then: $[D_t | \{zq_{x,t}^*\}; \{n_{x,t-1}\}] \sim \text{Poi} \left( z \sum_{x=x_0}^\omega n_{x,t-1} q_{x,t}^* \right)$

Taking $Z_t \sim \text{Gamma} (\alpha_t, \beta_t)$ we have:

$[D_t | \{n_{x,t-1}\}] \sim \text{NBin} \left( \alpha_t, \frac{\theta_t}{\theta_t+1} \right)$

$\theta_t = \sum_{x=x_0}^\omega \frac{\beta_t}{n_{x,t-1} q_{x,t}}$
Updating the parameters to experience

Due to the underlying trend, deviations in respect of the BE mortality rate should be (positively) correlated in time

Assumption: the mortality experience from the portfolio is reliable as an evidence of the trend of the cohort (or the population)

⇒ An inferential procedure can be defined for updating the parameters of the pdf of $Q_{x,t}$ (or the parameters of the distribution of the number of deaths) to experience
Updating the parameters to experience (cont.)

One cohort

Valuation at time 0

Issue time; no previous experience available

$n_{x_0,0}$ annuitants at time 0

\[ Z_{x,t} \sim \text{Gamma}(\bar{\alpha}, \bar{\beta}) \]

for all times $t$ (and ages $x = x_0 + t$)

\[ D_{x_0,1} \sim \text{NBin}\left(\alpha_{x_0,1}, \frac{\theta_{x_0,1}}{\theta_{x_0,1} + 1}\right) \]

\[ \alpha_{x_0,1} = \bar{\alpha} \]

\[ \theta_{x_0,1} = \frac{\bar{\beta}}{n_{x_0,0} q^{*}_{x_0,1}} \]
Valuation at time 1

Let $D_{x,0,1} = d_{x,0,1}$ be the observed number of deaths in $(0, 1)$

We can calculate the posterior pdf of $Q_{x,0,1}$, conditional on $D_{x,0,1} = d_{x,0,1}$. It turns out

$$[Q_{x,0,1} | D_{x,0,1} = d_{x,0,1}] \sim \text{Gamma} \left( \bar{\alpha} + d_{x,0,1}, \frac{\bar{\beta}}{q_{x,0,1}^*} + n_{x,0} \right)$$

and hence:

$$[Z_{x,t} | D_{x,0,1} = d_{x,0,1}] \sim \text{Gamma}(\bar{\alpha} + d_{x,0,1}, \bar{\beta} + n_{x,0} q_{x,0,1}^*)$$

Expected aggregate deviation:

$$\mathbb{E}[Z_{x,t} | D_{x,0,1} = d_{x,0,1}] = \frac{\bar{\alpha} + d_{x,0,1}}{\bar{\beta} + n_{x,0} q_{x,0,1}^*}$$

\[ \begin{array}{c|c}
\text{updated} & \bar{\alpha} \\
\hline
\text{prior} & \bar{\beta} \\
\end{array} \]
Number of deaths:

\[
[D_{x_0+1,2} | n_{x_0,0}, D_{x_0,1} = d_{x_0,1}] \sim \text{NBin} \left( \alpha_{x_0+1,2}, \frac{\theta_{x_0+1,2}}{\theta_{x_0+1,2} + 1} \right)
\]

\[
\alpha_{x_0+1,2} = \bar{\alpha} + d_{x_0,1} \quad \theta_2 = \frac{\bar{\beta} + n_{x_0,0} q^*_{x_0,1}}{n_{x_0+1,1} q^*_{x_0+1,2}}
\]

Valuation at time \( t \): . . . (similar results follow)
More than one cohort

Similar inferential procedure, with the advantage of relying on a wider data set (i.e. on the number of deaths observed in the population)

At time 0:

\[
[D_1|\{n_{x,0}\}] \sim \text{NBin} \left( \alpha_1, \frac{\theta_1}{\theta_1 + 1} \right)
\]

\[\alpha_1 = \bar{\alpha} \quad \theta_1 = \frac{\bar{\beta}}{\sum_{x=x_0} \omega_n x,0 q_{x,0,1}^*}
\]

At time 1:

Let \(D_1 = d_1 = \sum_{x=x_0} d_{x,1}\) be the observed number of deaths in \((0, 1)\)

Then:

\[
[D_2|\{n_{x,0}\}, D_1 = d_1] \sim \text{NBin} \left( \alpha_2, \frac{\theta_2}{\theta_2 + 1} \right)
\]

\[\alpha_2 = \bar{\alpha} + d_1 \quad \theta_2 = \frac{\bar{\beta} + \sum_{x=x_0} \omega_n x,0 q_{x,0,1}^*}{\sum_{x=x_0} \omega_n x,1 q_{x,0,2}^*}
\]

At time \(t\): ... (similar results follow)
Updating the parameters to experience (cont.)

Some numerical findings:

Expected systematic deviation $\mathbb{E}[Z_t \mid \{d_s\}_{s=1,2,...,t}]$

Best-estimate life table ($\{q^*_{x,t}\}$): IPS55 (Italian projected life table, cohort 1955)
Initial age: $x_0 = 65$, males
Initial value of the parameters of the pdf of $Z_t$:

$$\bar{\alpha} = \bar{\beta} \Rightarrow \mathbb{E}[Q_{x,t}] = \frac{\bar{\alpha}}{\bar{\beta}} q^*_{x,t} = q^*_{x,t}$$

(justified by the meaning of best-estimate life table)

$$\bar{\beta} = 100 \Rightarrow \text{CV}[Q_{x,t}] = \frac{\sqrt{\text{Var}[Q_{x,t}]} }{\mathbb{E}[Q_{x,t}]} = \frac{1}{\sqrt{\bar{\alpha}}} = 10\%$$

(based on expert’s opinion on trend volatility)
Updating the parameters to experience (cont.)

Expected systematic deviation $\mathbb{E}[Z_t | \{d_s\}_{s=1,2,...,t}]$

$\bar{\alpha} = \bar{\beta}, \bar{\beta} = 100$

deaths = 75% of best-estimate ($d_{x,s} = 0.75 n_{x,s-1} q^*_{x,s}$)

Left panel: one cohort. Right panel: multiple cohorts
Updating the parameters to experience (cont.)

Expected systematic deviation $\mathbb{E}[Z_t | \{d_s\}_{s=1,2,\ldots,t}]$

$\bar{\alpha} = \bar{\beta}; \ n_{65,s} = 1\ 000$

deaths $= 75\%$ of best-estimate $(d_{x,s} = 0.75 \ n_{x,s-1} q_{x,s}^*)$

Left panel: one cohort. Right panel: multiple cohorts
Application

The mortality model is experience-based
⇒ suitable for internal models
Particular application: capital required to face mortality/longevity risks

Implementation through a

deterministic . . .
the expected numbers of deaths
(or the expected mortality rates) only are involved

stochastic . . .
the numbers of deaths are stochastically simulated

Deterministic via stochastic . . .
short-cut formulae, constructed according to simulated findings
Deterministic implementation

Refer to the Solvency 2 standard formula

Capital charge (within the SCR – Solvency Capital Requirement) for (aggregate) longevity risk: change in the net value of assets minus liabilities ($\Delta$NAV) against a permanent 20% decrease in mortality rates for each age

Under our assumptions, this reduces to

$$\text{Life}_{\text{long}, t} = V_t^{(\Pi)[-20\%]} - V_t^{(\Pi)[BE]}$$

expected present value of future payments in portfolio $\Pi$, under the shock assumption (BE $-20\%$) expected present value of future payments in portfolio $\Pi$, under BE assumptions (or: portfolio reserve, net of the risk margin)
At time $t = 0$: shock scenario $-20\%$

\[ \mathbb{E}[Z_{x,t}] = \mathbb{E}[Z_t] = \frac{\bar{\alpha}}{\bar{\beta}} = 0.80 \]

\( \Rightarrow \bar{\alpha} = 0.80 \bar{\beta}, \bar{\beta} = 100 \) (experts’ opinion)

At time $t = 1$: shock scenario updated to experience

\[
\alpha_1 = \bar{\alpha} \rightarrow \alpha_2 = \bar{\alpha} + \begin{cases} d_{x_0,1} & \text{one cohort} \\ d_1 = \sum_x d_{x,1} & \text{multiple cohorts} \end{cases}
\]

\[
\beta_1 = \bar{\beta} \rightarrow \beta_2 = \bar{\beta} + \begin{cases} n_{x_0,0} q_{x_0,1}^* & \text{one cohort} \\ \sum_x n_{x,0} q_{x,1}^* & \text{multiple cohorts} \end{cases}
\]

\( \Rightarrow \) shock scenario: 

\[
1 - \begin{cases} \mathbb{E}[Z_{x,t} \mid d_{x_0,1}] = 1 - \frac{\alpha_{x_0+1,2}}{\beta_{x_0+1,2}} \\ \mathbb{E}[Z_t \mid d_{1}] = 1 - \frac{\alpha_2}{\beta_2} \end{cases}
\]

At time $t$ . . .
Example:
One cohort; initial age: \( x_0 = 65 \); males
Best estimate life table: IPS55

(Initial) parameters of the pdf of \( Z_{x,t} \): \( \bar{\alpha} = 0.75 \bar{\beta}, \bar{\beta} = 100 \)

Experienced mortality
as the shock scenario
(# deaths \( \simeq 75\% \) of what expected)

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{time } t
\end{array}
\end{array}
\end{array}
\]

as the BE scenario
(# deaths \( \simeq 100\% \) of what expected)

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{time } t
\end{array}
\end{array}
\end{array}
\]

\( n_{65,0} = 10,000 \)
\( n_{65,0} = 1,000 \)
\( n_{65,0} = 100 \)
Stochastic implementation

Reference to one cohort only

Let $M_t$ be the assets above the portfolio reserve

Default at time $t$: $M_t < 0$ (i.e.: unfunded liabilities)

Objective: at the valuation time $z$, given the accepted default probability $\varepsilon$, set $M_z$ such that

$$P[(M_{z+1} \geq 0) \cap (M_{z+2} \geq 0) \cap \cdots \cap (M_{\omega - x_0 - z} \geq 0) | n_{x_0 + z, z}] = 1 - \varepsilon$$
Example

One cohort; initial age: $x_0 = 65$; males
Best-estimate life table: IPS55
Maximum age: $\omega = 119$, whence the maturity of the portfolio at time $z$ is:
$\omega - x_0 - z = 119 - 65 - z$
Initial parameters of the pdf of $Z_{x,t}$: $\bar{\alpha} = 0.75\bar{\beta}$, $\bar{\beta} = 100$
Risk-free rate and investment yield: 3% p.a.
Annual amount: $b = 1$
Internal model with default probability: $\varepsilon = 0.005$
Application (cont.)

(0) Solvency 2: \[ \frac{M^{[\text{Solv2}]}}{V_z^{(\Pi)[BE]}} = \frac{\text{Life}_{\text{long},z} + RM_z}{V_z^{(\Pi)[BE]}} \]

(1)–(2) Internal model: \[ \frac{M^{[\text{IM}]}}{V_z^{(\Pi)[BE]}} \]

(1) experience as the best-estimate life table
(2) experience as the Solvency 2 stress scenario (BE–25%)
Application (cont.)

![Graph showing initial size and valuation time]

Initial size $n_{65,0} = 1000$

**Solvency 2:**

$$\frac{M_{[\text{Solv2}]}^{[\text{Solv2}]}}{V_z^{(\Pi)[\text{BE}]}} = \frac{\text{Life}_{\text{long},z} + RM_z}{V_z^{(\Pi)[\text{BE}]}}$$

**Internal model:**

$$\frac{M_{[\text{IM}]}^{[\text{IM}]}}{V_z^{(\Pi)[\text{BE}]}}$$

1. Experience as the best-estimate life table
2. Experience as the Solvency 2 stress scenario (BE $-25\%$)
Further developments

Statistical tests, especially for setting the initial volatility parameter

Possible extensions: Reduction of the total annual payout, instead of the number of deaths, in order to account for adverse-selection; introduction of cohort specificity; MCMC methods to approximate posterior distributions

Short-cut (factor-based) formulae for capital requirements within an internal model

\ldots \ldots \ldots \ldots
References


Awarded the AFIR/ERM Bob Alting von Geusau Prize 2010.
