Further developments in the Erlang(n) risk process

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Sparre–Andersen model

\[ U(t) = u + ct - \sum_{i=0}^{N(t)} X_i, \quad X_0 \equiv 0, \quad t \geq 0, \quad u \geq 0, \]

- \( u \), initial capital,
- \( t \), time variable,
- \( c > 0 \), loading factor,
- \( \{X_i\}_{i=1}^{\infty} \), single claim amount sequence,
- \( \{W_i\}_{i=1}^{\infty} \), interclaim time sequence,
- \( N(t) = \max\{k : W_1 + W_2 + \cdots + W_k \leq t\} \), number of claims.
Assumptions on the model:

- Sequence \( \{X_i\}_{i=1}^\infty \) i.i.d., \( X_i \sim P(x) \) with density \( p(x) \),
- \( \hat{p}(s) = \int_0^\infty e^{-sx} p(x)dx, \ s \in \mathbb{C} \) Laplace transform of \( p(x) \),
- \( \mu_k = E[X_1^k] \) the \( k \)-th moment of \( X_1 \),
- Sequence \( \{W_i\}_{i=1}^\infty \) i.i.d. and independent of \( \{X_i\}_{i=1}^\infty \),
- \( W_i \sim K(t), \ W_i \sim \text{Erlang}(n, \lambda) \), with density \( k(t) \),
- \( cE(W_1) > E(X_1) \) positive loading factor, \( cn > \lambda \mu_1 \).
Definitions

Time of ruin \( T = \inf \{ t > 0 : U(t) < 0 \}, \ u \geq 0, \)
\[
T = \infty \text{ iff } U(t) \geq 0 \ \forall t > 0.
\]

Ultimate ruin probability \( \Psi(u) = P(T < \infty), \)

Non-ruin probability \( \Phi(u) = 1 - \Psi(u). \)

Time to upcross barrier \( b \) \( \tau_b = \inf \{ t > 0 : U(0) = u, U(t) \geq b \} \)

\( \chi(u, b) = P(T > \tau_b | U(0) = u) \) probability of attaining \( b \) before ruin,
Time of the first upcrossing of the level 0 after ruin

\[ T' = \inf\{t : t > T, U(t) \geq 0\}, \ T < \infty \]

Maximum severity of ruin

\[ M_u = \sup\{|U(t)| : U(0) = u, \ T \leq t \leq T'\}, \ u \geq 0, \]

Distribution of the maximum severity

\[ J(z; u) = P(M_u \leq z \mid T < \infty), \ u, z \geq 0, \]

Probability of ruin and maximum severity smaller or equal than \( y \)

\[ G(u, y) = P(T < \infty, |U(T)| \leq y), \quad g(u, y) = \frac{d G(u, y)}{dy}. \]
\[ \delta > 0 \text{ constant force of interest.} \]

\[ D_u \text{ total amount of discounted dividends payable to shareholders before ruin.} \]

\[ V_m(u, b) = E[D_u^m], \ m \geq 0, \text{ the } m\text{-th moment of } D_u, \text{ where } V_0(u, b) \equiv 1. \]

For simplicity, \( V_1(u, b) = V(u, b). \)
Further developments in the Erlang(n) risk process

\[ U(t) \]

\[ U(T) < 0 \]

\[ U(T') = 0 \]
The Generalized Lundberg’s equation is

\[
\left( \frac{\lambda + \delta}{c} - s \right)^n = \left( \frac{\lambda}{c} \right)^n \hat{p}(s).
\] (1)

**Theorem 1** For \( \delta > 0 \) and \( n \in \mathbb{N}^+ \) the Generalized Lundberg’s equation has exactly \( n \) roots with positive real parts.

Proof in Li and Garrido (2004).
The Fundamental Lundberg’s equation is

\[ \left( \frac{\lambda}{c} - s \right)^n = \left( \frac{\lambda}{c} \right)^n \hat{\rho}(s). \]  

(2)

**Corollary 1** For \( \delta = 0 \) and \( n \in \mathbb{N}^+ \) the Fundamental Lundberg’s equation has exactly \( n - 1 \) roots with positive real parts.

We denote by the numbers \( \rho_1, \rho_2, \ldots, \rho_{n-1} \in \mathbb{C} \), the only roots of this equation which have positive real parts. Another root is \( -R \), where \( R > 0 \) is the adjustment coefficient.
Li and Dickson (2006) showed that $\chi(u, b)$ satisfies the integro-differential equation

$$
\sum_{k=0}^{n} (-1)^k \left( \frac{c}{\lambda} \right)^k \binom{n}{k} \frac{d^k v(u)}{d u^k} = \int_0^u v(u - y) p(y) dy, \quad 0 \leq u < b. \tag{3}
$$

If we find $n$ linearly independent particular solutions $v_j(u)$, $j = 1, \ldots, n$ for this equation, then we have

$$
\chi(u, b) = \vec{v}(u) [V(b)]^{-1} \vec{e}^T, \tag{4}
$$

where $\vec{v}(u) = (v_1(u), \ldots, v_n(u))$ is a $1 \times n$ vector,

$$(V(b))_{ij} = \left. \frac{d^{i-1} v_j(u)}{d u^{i-1}} \right|_{u=b}$$

is a $n \times n$ matrix and $\vec{e} = (1, 0, \ldots, 0)$ is a $1 \times n$ vector.
Li (2008) found the solutions of the integro–differential equation (3) using the roots of the fundamental Lundberg’s equation as follows

**Theorem 1** If \( \rho_1, \rho_2, \ldots, \rho_{n-1} \in \mathbb{C} \) are distinct, then the following functions are a (l.i.) set of solutions of (3)

\[
\begin{align*}
    v_1(u) &= \Phi(u), \\
    v_j(u) &= \sum_{i=1}^{j-1} a_{i,j} \int_0^u \Phi(u - y)e^{\rho_i y} \, dy, \quad j = 2, 3, \ldots, n,
\end{align*}
\]

where \( a_{i,j} = -\frac{1}{\prod_{k=1, k \neq i}^{j-1} (\rho_k - \rho_i)} \), \( i = 1, 2, \ldots, j - 1 \).
We propose a new version of Theorem 1 that, for computational purposes, is more efficient.

**Theorem 2** If $\rho_1, \rho_2, \ldots, \rho_{n-1} \in \mathbb{C}$ are distinct, then the following functions are a l.i. set of solutions of (3)

$$
\begin{align*}
\nu_1(u) &= \Phi(u), \\
\nu_j(u) &= \int_0^u \Phi(u - y)e^{\rho_j - 1 y} dy, \quad j = 2, 3, \ldots, n.
\end{align*}
$$

**Sketch of Proof** $\Phi(u)$ is solution of (3) [see Li and Garrido (2004)], its Laplace transform is given by

$$
\hat{\Phi}(s) = -\Phi(0) \left( \frac{c}{\lambda} \right)^n \prod_{i=1}^{n-1} \frac{\rho_i - s}{B(s) - \hat{p}(s)},
$$

Denote

$$
d_{\Phi}(s) = -\Phi(0) \left( \frac{c}{\lambda} \right)^n \prod_{i=1}^{n-1} (\rho_i - s).
$$
For simplicity denote

$$B(D) = \left( I - \left( \frac{c}{\lambda} \right) D \right)^n = \sum_{k=0}^{n} (-1)^k \left( \frac{c}{\lambda} \right)^k \binom{n}{k} D^k,$$

We can show, by direct computation of the derivatives of $v_j(u)$, that

$$B(D)v_j(u) = d_\Phi(\rho_{j-1}) e^{\rho_{j-1}u} + \int_0^u (B(D)\Phi(u-t)) e^{\rho_{j-1}t} dt$$

and that

$$\int_0^u v_j(u-y)p(y)dy = \int_0^u (B(D)\Phi(u-t)) e^{\rho_{j-1}t} dt.$$

Since $d_\Phi(\rho_{j-1}) = 0, j = 2, \ldots n$, we get the desired equality.

For the linear independence, given a linear combination

$$c_1 \Phi(u) + \sum_{j=2}^{n} c_j v_j(u) = 0, \forall u \geq 0,$$

we can proceed by contradiction on the cases $c_1 = 0$ and $c_1 \neq 0$. 
From Dickson (2005) we know that the distribution of the maximum severity of ruin $J(z; u)$ can be expressed as

$$J(z; u) = \frac{1}{\psi(u)} \left( \int_0^z g(u, y)v_1(z - y)dy, \ldots, \int_0^z g(u, y)v_n(z - y)dy \right) [V(z)]^{-1} \epsilon^T$$  \hspace{1cm} (5)
Examples

We have done explicit calculations for two particular cases:

• Interclaim arrivals $\text{Erlang}(3, \lambda)$ distributed, and claim amounts $\text{Exponential}(\beta)$ distributed. For simplification we will denote this case by $\text{Erlang}(3) - \text{Exponential}$.

$$W_i \sim \text{Erlang}(3, \lambda), \quad X_i \sim \text{Exponential}(\beta)$$

• Interclaim arrivals $\text{Erlang}(2, \lambda)$ distributed, and claim amounts $\text{Erlang}(2, \beta)$ distributed. We will denote this case by $\text{Erlang}(2) - \text{Erlang}(2)$.

$$W_i \sim \text{Erlang}(2, \lambda), \quad X_i \sim \text{Erlang}(2, \beta)$$
Erlang(3) – Exponential case

Considering the safety loading $c = \frac{(1 + \theta)\lambda}{3\beta}$ with $\theta > 0$, the fundamental Lundberg's equation (2) takes the form

$$
\left(1 - \left(\frac{c}{\lambda}\right)s\right)^3 - \frac{\beta}{(s + \beta)} = 0,
$$

which has four roots: $0, \rho_1, \rho_2$ and $-R$, where $0 < R < \beta$ is the adjustment coefficient, $\rho_1, \rho_2$ are complex roots with positive real parts and $\rho_2 = \bar{\rho}_1$. 
After applying our Theorem 2, the 3 solutions for the integro–
differential equation (3) are

\[ \Phi(u) = 1 - \left(1 - \frac{R}{\beta}\right) e^{-Ru} \]
\[ v_2(u) = \frac{-1}{\rho_1} + \frac{\beta - R}{\beta(R + \rho_1)} e^{-Ru} + \frac{R(\beta + \rho_1)}{\rho_1 \beta(R + \rho_1)} e^{\rho_1 u} \]
\[ v_3(u) = \frac{-1}{\rho_2} + \frac{\beta - R}{\beta(R + \rho_2)} e^{-Ru} + \frac{R(\beta + \rho_2)}{\rho_2 \beta(R + \rho_2)} e^{\rho_2 u} \]
After calculating (5) we get

\[ J(z; u) = 1 - \frac{\alpha e^{-Rz}}{1 - \gamma e^{-(\rho_1+R)z} - \delta e^{-(\rho_2+R)z} - \eta e^{-Rz}}, \]

where

\[ \alpha = \frac{R(R + \rho_1)(R + \rho_2)}{\beta(\beta + \rho_1)(\beta + \rho_2)} \]
\[ \gamma = \frac{-R(\beta - R)(R + \rho_2)}{\rho_1(\beta + \rho_1)(\rho_2 - \rho_1)} \]
\[ \delta = \frac{R(\beta - R)(R + \rho_1)}{\rho_2(\beta + \rho_2)(\rho_2 - \rho_1)} \]
\[ \eta = \frac{(\beta - R)(R + \rho_1)(R + \rho_2)}{\beta \rho_1 \rho_2} \]

with \( 0 < \alpha < 1, \ \delta = \bar{\gamma} \) and \( 0 < \eta = 1 - \alpha - \gamma - \delta. \)

Note that this expression for \( J(z; u) \) is independent from \( u. \)
Now choosing $\beta = 1$, $\lambda = 3$ and $c = 1 + \theta$ we evaluate the moments of the maximum severity for some values of $\theta$ and compare with the results from Li (2008) for the Exponential – Exponential and Erlang(2) – Exponential cases.

**Table 1:** The values of $E(M_u)$ and $s.d.(M_u)$ for $n = 1, 2, 3$; $m = 1$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$n = 1$</th>
<th>$m = 1$</th>
<th>$n = 2$</th>
<th>$m = 1$</th>
<th>$n = 3$</th>
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<tbody>
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<td>$E(M_u)$</td>
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</table>
Erlang(2) – Erlang(2) case

Considering the safety loading \( c = \frac{(1 + \theta)\lambda}{\beta} \) with \( \theta > 0 \), the fundamental Lundberg’s equation (2) takes the form

\[
\left(1 - \left(\frac{c}{\lambda}\right)s\right)^2 - \frac{\beta^2}{(s + \beta)^2} = 0,
\]

which has four real roots: 0, \(-R_1\), \(-R_2\) and \(\rho\), where \(0 < R_1 < \beta\) is the adjustment coefficient, \(R_2 > \beta\) and \(\rho > \beta\).
The 2 solutions for the integro-differential equation (3) are

\[ \Phi(u) = 1 - \frac{R_2(\beta - R_1)^2}{\beta^2(R_2 - R_1)} e^{-R_1 u} - \frac{R_1(\beta - R_2)^2}{\beta^2(R_1 - R_2)} e^{-R_2 u} \]

\[ v_2(u) = \frac{-1}{\rho} + \frac{R_1 R_2(\beta + \rho)^2}{\beta^2 \rho(\rho + R_1)(\rho + R_2)} e^{\rho u} + \frac{R_2(\beta - R_1)^2}{\beta^2(R_2 - R_1)(\rho + R_1)} e^{-R_1 u} + \frac{R_1(\beta - R_2)^2}{\beta^2(R_1 - R_2)(\rho + R_2)} e^{-R_2 u} \]
Distribution of the maximum severity of ruin

In this case the formula that we get from (5) is not independent from \( u \), we write it in the following way

\[
J(z; u) = \frac{1}{\psi(u)} \left[ \frac{R_2(\beta - R_1)^2}{\beta^2(R_2 - R_1)} e^{-R_1u} J_1(z; u) + \frac{R_1(\beta - R_2)^2}{\beta^2(R_1 - R_2)} e^{-R_2u} J_2(z; u) \right]
\]

where

\[
J_1(z; u) = \frac{1 - \gamma_1 e^{-(\rho+R_1)z} - \gamma_2 e^{-(\rho+R_2)z} - (1 - \gamma_1) e^{-R_1z} - \tau_1 e^{-R_2z} - \omega_1 e^{-(\rho+R_1+R_2)z}}{1 - \gamma_1 e^{-(\rho+R_1)z} - \gamma_2 e^{-(\rho+R_2)z} - \delta_1 e^{-R_1z} - \delta_2 e^{-R_2z} - \eta e^{-(\rho+R_1+R_2)z}}
\]

\[
J_2(z; u) = \frac{1 - \gamma_1 e^{-(\rho+R_1)z} - \gamma_2 e^{-(\rho+R_2)z} - \tau_2 e^{-R_1z} - (1 - \gamma_2) e^{-R_2z} - \omega_2 e^{-(\rho+R_1+R_2)z}}{1 - \gamma_1 e^{-(\rho+R_1)z} - \gamma_2 e^{-(\rho+R_2)z} - \delta_1 e^{-R_1z} - \delta_2 e^{-R_2z} - \eta e^{-(\rho+R_1+R_2)z}}
\]

and \( \gamma_1, \gamma_2, \delta_1, \delta_2, \tau_1, \tau_2, \omega_1, \omega_2, \eta, \alpha \) are constant which depends on \( R_1, R_2, \beta \) and \( \rho \)
Now choosing $\beta = 1$, $\lambda = 1$ and $c = 1 + \theta$ we evaluate the moments of the maximum severity for some values of $\theta$ and compare with the results from Li (2008) for the Exponential – Exponential and Erlang(2) – Exponential cases.

**Table 3:** The values of $E(M_u)$ and $s.d.(M_u)$ for $n = 1, 2$; $m = 1$ and $n = m = 2$.

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<td>2.791</td>
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Dividends: \( m \)-th moment of the expected discounted future dividends

Following Dickson and Waters (2004), in the same way we condition on the time and the amount of the first claim and we get, for \( 0 \leq u < b \)

\[
V_m(u, b) = \int_{\frac{b-u}{c}}^{\infty} g_n(t) e^{-m \delta t} \left[ \left( c \frac{\bar{s}_t - b - u}{c} \right)^m \right] + \\
\sum_{j=1}^{m} \binom{m}{j} \left( c \frac{\bar{s}_t - b - u}{c} \right)^{m-j} \int_{0}^{b} f(x) V_j(b - x, b) dx \right] dt + \\
\int_{0}^{\frac{b-u}{c}} e^{-m \delta t} g_n(t) \int_{0}^{u+ct} V_m(u + ct - x, b) p(x) dx dt, \quad m \geq 1. \quad (6)
\]

where \( \bar{s}_t = \frac{e^{\delta t} - 1}{\delta} \) in standard actuarial notation.
In particular, for $m = 1$

\[
V(u, b) = \int_0^\infty g_n(t)e^{-\delta t} \left( c \frac{s}{t - \frac{b-u}{c}} + \int_0^b f(x)V(b-x, b)dx \right) dt + \\
+ \int_0^{\frac{b-u}{c}} e^{-\delta t}g_n(t)\int_0^{u+ct} V(u+ct-x, b)p(x)dx \, dt,
\]

(7)
For an Erlang($n$) risk process the integro–differential equations satisfied by the discounted expected dividends are

\[
\left(\left(1 + \frac{\delta}{\lambda}\right) I - \frac{c}{\lambda} D\right)^n V(u, b) = \int_0^u V(u - x, b)p(x)dx \tag{8}
\]

\[
\frac{d^k V(u, b)}{du^k} \bigg|_{u=b} = \left(\frac{\delta}{c}\right)^{k-1}, \ 1 \leq k \leq n,
\]

and for a general $m$

\[
\left(\left(1 + \frac{\delta}{\lambda}\right) I - \frac{c}{\lambda} D\right)^n V_m(u, b) = \int_0^u V_m(u - x, b)p(x)dx \tag{9}
\]

\[
\frac{d^k V_m(u, b)}{du^k} \bigg|_{u=b} = \sum_{j=1}^k \frac{m!}{(m-j)!} \left\{k\right\} \left\{j\right\} \left(\frac{\delta}{c}\right)^{k-j} V_{m-j}(b, b), \ 1 \leq k \leq n,
\]

where \(\left\{k\right\} = \frac{1}{j!} \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} i^k\) denotes the Stirling numbers of the second kind. We define for convenience \(V_{m-j}(u, b) \equiv 0\), for $m < j$ in the formula above.
The solutions of integro-differential equation (8) are of the form

\[ V(u, b) = \sum_{i=1}^{n} C_i e^{\rho_i u} \beta_i(u), \]

where \( \rho_i \)'s are the roots with positive real parts of the generalized Lundberg's equation \( \beta_i(u) \)'s are solutions of the auxiliary equation

\[ (\lambda_i I - cD)^n \beta_i(u) = \lambda_i^n \int_0^u \beta_i(u - x)p_i(x)\,dx \]  

with \( \lambda_i = \lambda \hat{p}^{\frac{1}{n}}(\rho_i) \) and \( p_i(x) = \frac{e^{-\rho_i x}p(x)}{\hat{p}(\rho_i)} \) and the constants \( C_i \)'s are defined in

\[ \left. \frac{d^k V(u, b)}{du^k} \right|_{u=b} = \sum_{i=1}^{n} C_i \left. \frac{d^k (e^{\rho_i u} \beta_i(u))}{du^k} \right|_{u=b} = \left( \frac{\delta}{c} \right)^{k-1}, \quad 1 \leq k \leq n, \]

(11)
Consider an Erlang(2,2) risk model with Erlang(2,2) claim amounts. The positive loading factor is $c=1.1$ and the force of interest is $\delta = 0.03$. Compute $V(u,b)$ and $V_2(u,b)$.

- $V(u,b)$:
  We get $\rho_1 = 0.169$, $\rho_2 = 2.631$ from the generalized Lundberg’s equation

\[
(2.03 - 1.1s)^n = \frac{8}{2 + s}.
\]

Then the auxiliary equation gives

\[
\begin{align*}
\beta_1(u) &= 1 + 0.026e^{-2.954u} - 0.718e^{-0.492u}, \\
\beta_2(u) &= 1 + 0.047e^{-5.235u} - 0.108e^{-3.845u},
\end{align*}
\]
and to find the constant $C_1, C_2$ we solve the system

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \frac{d(e^{\rho_1 u} \beta_1(u))}{du} \bigg|_{u=b} & \frac{d(e^{\rho_2 u} \beta_2(u))}{du} \bigg|_{u=b} \\ \frac{d^2(e^{\rho_1 u} \beta_1(u))}{du^2} \bigg|_{u=b} & \frac{d^2(e^{\rho_2 u} \beta_2(u))}{du^2} \bigg|_{u=b} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \delta - c \end{pmatrix},$$

Then

$$V(u, b) = C_1 e^{\rho_1 u} \beta_1(u) + C_2 e^{\rho_2 u} \beta_2(u).$$
<table>
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<th>3</th>
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**Table:** Values of $V(u, b)$ for $0 \leq u, b \leq 9$
• $V_2(u, b)$:
  We get $\rho_1 = 0.273, \rho_2 = 2.654$ from the generalized Lundberg's equation
  \[
  (2.06 - 1.1s)^n = \frac{8}{2 + s}.
  \]
  Then the auxiliary equation gives
  \[
  \beta_1(u) = 1 + 0.033e^{-3.054u} - 0.636e^{-0.673u},
  \beta_2(u) = 1 + 0.047e^{-5.256u} - 0.107e^{-3.873u},
  \]
and to find the constant $C_1, C_2$ we solve the system

$$
\begin{pmatrix}
C_1 \\
C_2
\end{pmatrix} = \begin{pmatrix}
\frac{d(e^{\rho_1 u} \beta_1(u))}{du} \bigg|_{u=b} & \frac{d(e^{\rho_2 u} \beta_2(u))}{du} \bigg|_{u=b} \\
\frac{d^2(e^{\rho_1 u} \beta_1(u))}{du^2} \bigg|_{u=b} & \frac{d^2(e^{\rho_2 u} \beta_2(u))}{du^2} \bigg|_{u=b}
\end{pmatrix}^{-1} \begin{pmatrix}
2V(b, b) \\
2 + 2\left(\frac{\delta}{c}\right)V(b, b)
\end{pmatrix}.
$$

Then

$$V_2(u, b) = C_1 e^{\rho_1 u} \beta_1(u) + C_2 e^{\rho_2 u} \beta_2(u),$$
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**Table:** Values of $V_2(u, b)$ for $0 \leq u, b \leq 9$
References


Thank you for your attention!