Pricing European Options on Deferred Insurance Contracts

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Outline

- Motivation
- Interest Rate and Mortality Rate Processes
- The Option Pricing Framework
- Solution of the Density Function PDE
- Pure Endowment Contracts
- Deferred Annuity Contracts
- Options on Deferred Insurance Contracts
- Derive Delta Hedging Ratios
- Numerical Results
Motivation

- Insurers and annuity providers are exposed to the risk of ever improving mortality trends.
- At present mortality risk is non-tradable and there is no market to hedge such risk other than reinsurance.
- If mortality risk can be traded through securities such as longevity bonds and swaps then the techniques developed in financial markets can be adapted and implemented for mortality risk.
- Milevsky & Promislow (2001) value mortality contingent claims by taking the underlying securities as defaultable coupon paying bonds with the time of death as a stopping time.
- Schrager (2006) use the numeraire approach (approximation) to value guaranteed annuity options with closed-form Affine models.
The Interest Rate and Mortality Model

- We assume a single factor CIR process for interest rate dynamics

\[ dr(t) = \kappa_r (\theta_r - r(t))dt + \sigma_r \sqrt{r(t)}dW^Q. \]  

(1.1)

- Arbitrage-free price of a zero coupon bond is represented as

\[ B(t, T) = \mathbb{E}^Q \left[ e^{-\int_t^T r(u)du} | G_t \right] = e^{\alpha_r(t, T) - \beta_r(t, T)r(t)}, \]

- One factor time-inhomogeneous mortality process

\[ d\mu(t; x) = \kappa_\mu (m(x + t) - \mu(t; x))dt + \Sigma_\mu \sqrt{m(x + t)\mu(t; x)}dW(t) \]

(1.2)

- Drift follows a Weibull mortality law

\[ m(x + t) = \frac{c}{\theta^c} (x + t)^{c-1} \]

- Risk-neutral survival probability

\[ S(t, T; x) = \mathbb{E}^Q \left[ e^{-\int_t^T \mu(u)du} | G_t \right] = e^{\alpha_\mu(t, T; x) - \beta_\mu(t, T; x)\mu(x, t)} \]
Survival Curves

- We note that lower values of $\kappa_\mu$, corresponding to a slower speed of mean-reversion, produces survival curves with higher survival probabilities for a fixed $\Sigma_\mu$.
- From the second figure, we note that higher values of $\Sigma_\mu$ increase the survival probability in the older ages.

Figure: Survival Probabilities for Varying $\kappa_\mu$ and $\Sigma_\mu$
By using hedging arguments and Ito’s Lemma the value of an option, \( C(t, T, r, \mu; x) \), written on an insurance contract with payoff \( P(T, r, \mu; x) \) at time-\( T \), is the solution to the partial differential equation (PDE)

\[
\frac{\partial C}{\partial \psi}(\psi, T, r, \mu; x) = \mathcal{L}C(\psi, T, r, \mu; x) - r_x C(\psi, T, r, \mu; x),
\]

where

\[
\mathcal{L} = \kappa_r (\theta_r - r) \frac{\partial}{\partial r} + \kappa_\mu (m(t) - \mu) \frac{\partial}{\partial \mu} + \frac{1}{2} \sigma_r^2 r \frac{\partial^2}{\partial r^2} + \frac{1}{2} \Sigma_\mu m(t) \mu \frac{\partial^2}{\partial \mu^2},
\]

with \( \mathcal{L} \) being a Dynkin operator, \( 0 < r, \mu < \infty \), \( r_x = r + \mu \) and \( \psi = T - t \)

The PDE is solved subject to the terminal payoff which is specific for any given contract.
Also associated with the system of SDEs (1.1) and (1.2) is the transition density function, which we denote here as 
\( G(\psi, r, \mu; r_T, \mu_T, x) \) with \( \psi = T - t \) being the time-to-maturity.

Chiarella and Ziveyi (2011) show that SDEs like (1.1) and (1.2) satisfy the associated Kolmogorov backward PDE

\[
\frac{\partial G}{\partial \psi}(\psi, r, \mu; r_0, \mu_0, x) = \mathcal{L}G(\psi, r, \mu; r_0, \mu_0, x). \tag{1.5}
\]

Equation (1.5) is solved subject to the boundary condition

\[
G(0, r, \mu; r_0, \mu_0, x) = \delta(r - r_0)\delta(\mu - \mu_0) \tag{1.6}
\]

where \( \delta(\cdot) \) is the Dirac Delta function, \( r_0 \) is the instantaneous short-rate while \( \mu_0 \) is the instantaneous mortality rate.
The option pricing function can be expressed as

$$ C(\psi, T, r, \mu; x) = 1_{\tau > t} \mathbb{E}^Q \left[ e^{-\int_t^T [r(u) + \mu(u; x)] du} \mathbb{E}^Q [P(T, r, \mu; x)|G_T]|G_t] \right]. $$

(1.7)

The problem becomes that of finding $P(T, r, \mu; x)$, at time-$T$ discounted to time-$t$.

By making use of Duhamel’s principle, the general solution of equation (1.7) can be represented as

$$ C(\psi, T, r, \mu; x) = 1_{\tau > t} \mathbb{E}^Q \left[ e^{-\int_t^T [r(u) + \mu(u; x)] du} |G_t] \right] 
\times \int_0^\infty \int_0^\infty P(T, w_1, w_2; x) G(\psi, r, \mu; w_1, w_2) dw_1 dw_2 
= 1_{\tau > t} e^{\alpha_r(t, T) - \beta_r(t, T)r(t)} \times e^{\alpha_\mu(t, T; x) - \beta_\mu(t, T; x)\mu(x, t)} 
\times \int_0^\infty \int_0^\infty P(T, w_1, w_2; x) G(\psi, r, \mu; w_1, w_2) dw_1 dw_2, $$

(1.8)

where $G(\psi, r, \mu; w_1, w_2)$ is the solution of the transition density PDE (1.5).
Use Laplace techniques to transform (1.5) system of characteristic PDEs.

Solve the characteristic PDEs

Inverse Laplace transform to recover the transition density function

\[
G(\psi, r, \mu; r_0, \mu_0, x) = \exp\left\{ -\frac{2\kappa_r}{\sigma_r^2 (e^{\kappa_r \psi} - 1)} (r_0 e^{\kappa_r \psi} + r) - \frac{2\kappa_\mu}{\sigma_\mu^2 (e^{\kappa_\mu \psi} - 1)} (\mu_0 e^{\kappa_\mu \psi} + \mu) \right\} \\
\times \left( \frac{r_0 e^{\kappa_r \psi}}{r} \right)^{\frac{\Phi_r}{\sigma_r^2}} \frac{1}{2} \times \left( \frac{\mu_0 e^{\kappa_\mu \psi}}{\mu} \right)^{\frac{\Phi_\mu}{\sigma_\mu^2}} \frac{1}{2} \times I_2 \frac{2\Phi_r}{\sigma_r^2} - 1 \left( \frac{4\kappa_r}{\sigma_r^2 (e^{\kappa_r \psi} - 1)} (r \times r_0 e^{\kappa_r \psi}) \frac{1}{2} \right) \\
\times I_2 \frac{2\Phi_\mu}{\sigma_\mu^2} - 1 \left( \frac{4\kappa_\mu}{\sigma_\mu^2 (e^{\kappa_\mu \psi} - 1)} (\mu \times \mu_0 e^{\kappa_\mu \psi}) \frac{1}{2} \right) \times \frac{2\kappa_r e^{\kappa_r \psi}}{\sigma_r^2 (e^{\kappa_r \psi} - 1)} \frac{2\kappa_\mu e^{\kappa_\mu \psi}}{\sigma_\mu^2 (e^{\kappa_\mu \psi} - 1)}
\]

(1.9)

This is the explicit form of the transition density function.
Options on Deferred Insurance Contracts

- An option on a pure endowment contract has a pay-off at option maturity which we represent here as
  \[ P(T, r, \mu; x) = \max \left[ 0, \mathbb{E}^Q \left[ e^{-\int_T^{T_m} [r(u)+\mu(u)] du} - K \right] G_T \right] \]

- The option price on a pure endowment is
  \[ C(\psi, T, r, \mu; x) = 1_{\tau > t} e^{\alpha_r(T-t)-\beta_r(T-t)} e^{\alpha_\mu(T-t;x)-\beta_\mu(T-t;x)} \]
  \[ \times \int_0^\infty \int_0^\infty \max \left[ 0, e^{\alpha_r(T_m-T)-\beta_r(T_m-T)} \varphi_1 \times e^{\alpha_\mu(T_m-T;x)-\beta_\mu(T_m-T;x)} \varphi_2 - K \right] \]
  \[ \times G(\psi, r, \mu; \varphi_1, \varphi_2, x) d\varphi_1 d\varphi_2. \]

- The pay-off of a deferred immediate annuity can be represented as
  \[ P(T, r, \mu; x) = \max \left[ 0, \sum_{i=T}^{\omega} \mathbb{E}^Q \left[ e^{-\int_i^{T_m} [r(u)+\mu(u)] du} \right] G_T - K \right] \]

- An option on a deferred immediate annuity is
  \[ C(\psi, T, r, \mu; x) = 1_{\tau > t} e^{\alpha_r(T-t)-\beta_r(T-t)} e^{\alpha_\mu(T-t;x)-\beta_\mu(T-t;x)} \]
  \[ \times \int_0^\infty \int_0^\infty \max \left[ 0, \sum_{i=T}^{\omega} e^{\alpha_r(i-T)-\beta_r(i-T)} \varphi_1 \times e^{\alpha_\mu(i-T;x)-\beta_\mu(i-T;x)} \varphi_2 - K \right] \]
  \[ \times G(\psi, r, \mu; \varphi_1, \varphi_2, x) d\varphi_1 d\varphi_2. \]
Delta Hedging

Derive the option price sensitivity to interest rates

\[
\frac{\partial C}{\partial r} = e^{\alpha r(T-t)} - \beta_r(T-t) r(t) e^{\alpha \mu(T-t; t)} - \beta_\mu(T-t; x) \mu(x) \int_0^{\infty} \int_0^{\infty} P(T, w_1, w_2; x) \left[ - \beta_r(T-t) G(\psi, r, \mu; w_1, w_2) + \frac{dG(\psi, r, \mu; w_1, w_2)}{dr} \right] dw_1 dw_2,
\]

(1.11)

where

\[
\frac{dG}{dr} = \exp \left\{ - \frac{2\kappa_r}{\sigma_r^2(e^{\kappa r T} - 1)} (\varphi_1 e^{\kappa r T} + r) - \frac{2\kappa_\mu}{\sigma_\mu^2(e^{\kappa \mu T} - 1)} (\varphi_2 e^{\kappa \mu T} + \mu) \right\} \left( \frac{\varphi_2 e^{\kappa \mu T}}{\mu} \right) \left( \frac{\phi_\mu}{\sigma_\mu^2} - \frac{1}{2} \right) \\
\times \left( \frac{\phi_r}{\sigma_r^2} - \frac{1}{2} \right) \times \frac{2\kappa_r e^{\kappa r T}}{\sigma_r^2(e^{\kappa r T} - 1)} \frac{2\kappa_\mu e^{\kappa \mu T}}{\sigma_\mu^2(e^{\kappa \mu T} - 1)} \frac{I_2 \phi_\mu}{\sigma_\mu^2} - 1 \left( \frac{4\kappa_\mu}{\sigma_\mu^2(e^{\kappa \mu T} - 1)} (\mu \times \varphi_2 e^{\kappa \mu T}) \frac{1}{2} \right) \\
\times \left[ B_2^r \frac{\phi_r}{\sigma_r^2} - 1 - \left( \left( \frac{\phi_r}{\sigma_r^2} - \frac{1}{2} \right) \left( \frac{1}{r} \right) + \frac{2\kappa_r}{\sigma_r^2(e^{\kappa r T} - 1)} \right) \frac{I_2 \phi_r}{\sigma_r^2} - 1 \left( \frac{4\kappa_r}{\sigma_r^2(e^{\kappa r T} - 1)} (r \times \varphi_1 e^{\kappa r T}) \frac{1}{2} \right) \right],
\]

(1.12)

with

\[
B_2^r \frac{\phi_r}{\sigma_r^2} - 1 = \frac{\kappa_r}{\sigma_r^2(e^{\kappa r T} - 1)} \frac{1}{2} \left[ \frac{I_2 \phi_r}{\sigma_r^2} - 2 \left( \frac{4\kappa_r}{\sigma_r^2(e^{\kappa r T} - 1)} (r \times \varphi_1 e^{\kappa r T}) \frac{1}{2} \right) \right. \\
+ \left. \frac{I_2 \phi_r}{\sigma_r^2} \left( \frac{4\kappa_r}{\sigma_r^2(e^{\kappa r T} - 1)} (r \times \varphi_1 e^{\kappa r T}) \frac{1}{2} \right) \right]
\]

(1.13)
Delta Hedging

- Derive the option price sensitivity to mortality rates

\[
\frac{\partial C}{\partial \mu} = e^{\alpha r (T-t)} - \beta r (T-t) r(t) e^{\alpha \mu (T-t;x) - \beta \mu (T-t;x) \mu(t;x)} \int_0^\infty \int_0^\infty P(T, w_1, w_2; x) \\
\left[- \beta \mu (T-t) G(\psi, r, \mu; w_1, w_2) + \frac{dG(\psi, r, \mu; w_1, w_2)}{d\mu}\right] dw_1 dw_2,
\]

(1.14)

where

\[
\frac{dG}{d\mu} = \exp \left\{ - \frac{2 \kappa r}{\sigma_r^2 (e^{\kappa r T} - 1)} (\varphi_1 e^{\kappa r T} + r) - \frac{2 \kappa \mu}{\sigma_\mu^2 (e^{\kappa \mu T} - 1)} (\varphi_2 e^{\kappa \mu T} + \mu) \right\} \left( \frac{\varphi_2 e^{\kappa \mu T}}{\mu} \right) \left( \frac{\Phi \mu - \frac{1}{2}}{\sigma_\mu^2} \right)
\]

\[
\left( \frac{\varphi_1 e^{\kappa r T}}{r} \right) \left( \frac{\Phi_r \mu - \frac{1}{2}}{\sigma_r^2} \right) \times \frac{2 \kappa r e^{\kappa r T}}{\sigma_r^2 (e^{\kappa r T} - 1)} \frac{2 \kappa \mu e^{\kappa \mu T}}{\sigma_\mu^2 (e^{\kappa \mu T} - 1)} \frac{I_2 \varphi_r}{\sigma_r^2} - 1 \left( \frac{4 \kappa r}{\sigma_r^2 (e^{\kappa r T} - 1)} (r \times \varphi_1 e^{\kappa r T}) \frac{1}{2} \right)
\]

\times \left[ B_2 \frac{\varphi_\mu}{\sigma_\mu^2} - 1 - \left( \left( \frac{\Phi_\mu}{\sigma_\mu^2} - \frac{1}{2} \right) \left( \frac{1}{\mu} \right) + \frac{2 \kappa \mu}{\sigma_\mu^2 (e^{\kappa \mu T} - 1)} \right) \frac{I_2 \varphi_\mu}{\sigma_\mu^2} - 1 \left( \frac{4 \kappa \mu}{\sigma_\mu^2 (e^{\kappa \mu T} - 1)} (\mu \times \varphi_2 e^{\kappa \mu T}) \frac{1}{2} \right) \right]

(1.15)

with

\[
B_2 \frac{\varphi_\mu}{\sigma_\mu^2} - 1 = \frac{\kappa \mu}{\sigma_\mu^2 (e^{\kappa \mu T} - 1) \sqrt{\mu}} \left[ \frac{I_2 \varphi_\mu}{\sigma_\mu^2} - 2 \left( \frac{4 \kappa \mu}{\sigma_\mu^2 (e^{\kappa \mu T} - 1)} (\mu \times \varphi_2 e^{\kappa \mu T}) \frac{1}{2} \right) \right]
\]

\[
+ \frac{I_2 \varphi_\mu}{\sigma_\mu^2} \left( \frac{4 \kappa \mu}{\sigma_\mu^2 (e^{\kappa \mu T} - 1)} (\mu \times \varphi_2 e^{\kappa \mu T}) \frac{1}{2} \right)
\]

(1.16)

Options on Deferred Insurance Contracts
Numerical Results

- Model a cohort aged 50 at time zero
- We start by showing the behaviour of the density functions.

(a) $\kappa_\mu = 0.015, \Sigma_\mu = 0.03$

(b) $\kappa_\mu = 0.065, \Sigma_\mu = 0.15$

Figure: Probability Density Function - Age = 65
Mortality pdfs

- From the first figure, we note that the peak in the density function corresponds to the lowest value of $\kappa_\mu$.
- The peak density decreases and mortality intensity dispersion increases as $\kappa_\mu$ increases.
- For the second figure, peak is greatest at the lowest volatility level. Increasing $\Sigma_\mu$ decreases (and then increases) density level while the mortality intensity dispersion increases.

**Figure:** Probability Density Function at $T = 65$
Numerical Results for Pure Endowments

- We consider a pure endowment with an option maturing at $T = 65$ and contract maturity varying from age 65 to 100.
- The contracts with lower $\kappa\mu$’s have higher values at time-$T$.
- For option prices, the guaranteed value is the model market value of the pure endowment contract valued at time-$T$.

Figure: Contract Values and Corresponding Option Prices at Age = 50
Numerical Results for Deferred Immediate Annuities

- We consider a whole life deferred immediate annuity, with payments deferred until time $T$ which varies from 55 - 100.
- From the first figure we note that lower values of $\kappa_\mu$ have higher guarantee values as expected.

**Figure**: Contract Values and Corresponding Option Prices at Age = 50
Numerical Results

- Compare the option price to face value of the contract
- First Figure: Pure Endowment Contracts
- Second Figure: Annuity Contracts

**Figure:** Option Price to Face Value (Percentage)
Changes in the Volatility on Annuity Options

- Interest rate processes are generally known as highly volatile in the short term.
- Mortality process more volatile in the long term.

Figure: Annuity - Changing Interest and Mortality Volatility
Varying Strike Prices

- As the strike goes to zero the price of the option converges to the market value of the contract. This is in line with the behavior of option prices.

Figure: Deferred Whole Life Annuity Options with Maturities at Ages 65 and 80 Respectively
Varying Strike Prices - Hedging Ratios Age 65

- As the strike goes to the market value of the contract, the deltas converge to zero.

Figure: Deferred Whole Life Annuity Deltas with Maturity at Ages 65. Interest Delta (left) and Mortality Delta (right).
Varying Strike Prices - Hedging Ratios Age 80

- As the strike goes to the market value of the contract, the deltas converge to zero.

Figure: Deferred Whole Life Annuity Deltas with Maturity 80 Respectively. Interest Delta (left) and Mortality Delta (right).
Thank you very much!