

Superhedging in illiquid markets

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September 5, 2009

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1. Much of trading consists of exchanging sequences of cash flows: coupon payments, dividends, insurance premiums, swaps, . . .
 - Payment schedule matters since wealth cannot be transferred quite freely in time (there is no numeraire).
2. Trading costs are nonlinear: transaction costs, double auction markets, . . .
 - Nonlinearities imply decreasing returns to scale in the hedging of contingent claims.

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1. A market model for an agent with
 - nonlinear trading costs,
 - portfolio constraints,
 - no market power.
2. Superhedging and pricing in terms of claim and premium **processes** with cash delivery.
3. Dual expressions in terms of **stochastic term structures** that capture both the time value of money as well as uncertainty in payments.

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- Let J be a finite set of assets traded at $t = 0, \dots, T$.
- Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$ be a filtered probability space.

Definition 1 A *convex cost process* is a sequence

$S = (S_t)_{t=0}^T$ of real-valued functions on $\mathbb{R}^J \times \Omega$ such that

1. the function $S_t(\cdot, \omega)$ is lower semicontinuous, convex and vanishes at 0 for every $\omega \in \Omega$,
2. S_t is $\mathcal{B}(\mathbb{R}^J) \otimes \mathcal{F}_t$ -measurable.

A cost process S is nonlinear, nondecreasing, polyhedral, sublinear, linear, ... if the functions $S_t(\cdot, \omega)$ have the corresponding property for every $t = 0, \dots, T$ and $\omega \in \Omega$.

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Example 2 *If $s = (s_t)_{t=0}^T$ is an $(\mathcal{F}_t)_{t=0}^T$ -adapted \mathbb{R}^J -valued price process, then*

$$S_t(x, \omega) = s_t(\omega) \cdot x$$

*defines a **linear** cost process in the sense of Definition 1.*

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Example 3 (Jouini and Kallal, 1995) If $\bar{s} = (\bar{s}_t)_{t=0}^T$ and $\underline{s} = (\underline{s}_t)_{t=0}^T$ are $(\mathcal{F}_t)_{t=0}^T$ -adapted \mathbb{R}^J -valued price processes with $\underline{s} \leq \bar{s}$, then

$$S_t(x, \omega) = \sum_{j \in J} S_t^j(x^j, \omega),$$

where

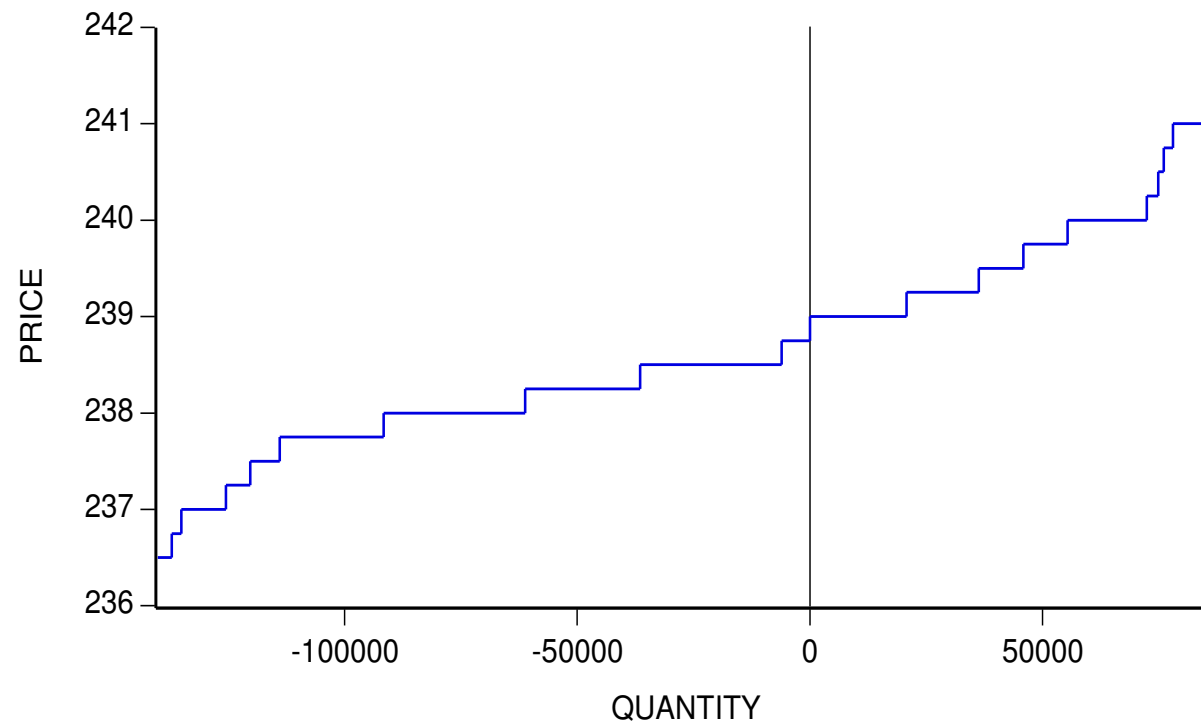
$$S_t^j(x^j, \omega) = \begin{cases} \bar{s}_t^j(\omega)x^j & \text{if } x^j \geq 0, \\ \underline{s}_t^j(\omega)x^j & \text{if } x^j \leq 0 \end{cases}$$

defines a *sublinear* cost process in the sense of Definition 1.

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Example 4 (Double auction markets) *In double auction markets, the cost of a market order is obtained by integrating the order book.*



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Definition 5 A *portfolio constraint process* is a sequence $D = (D_t)_{t=0}^T$ of set-valued mappings from Ω to \mathbb{R}^J such that

1. $D_t(\omega)$ is closed, convex and $0 \in D_t(\omega)$ for every $\omega \in \Omega$,
2. the set-valued mapping $\omega \mapsto D_t(\omega)$ is \mathcal{F}_t -measurable.

A portfolio constraint process is said to be polyhedral, conical, ... if the sets $D_t(\omega)$ have the corresponding property for every $t = 0, \dots, T$ and $\omega \in \Omega$.

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Example 6 *The unconstrained case corresponds to $D_t(\omega) = \mathbb{R}^J$ for every $t = 0, \dots, T$ and $\omega \in \Omega$.*

Example 7 (Cvitanić and Karatzas, 1992, ...) *Given a closed convex set $K \subset \mathbb{R}^J$ containing the origin, the sets $D_t(\omega) = K$ define a convex portfolio constraint process.*

Example 8 (Napp, 2003) *Given a closed convex cone $K \subset \mathbb{R}^L$ and an $(\mathcal{F}_t)_{t=0}^T$ -adapted sequence $(M_t)_{t=0}^T$ of real $L \times J$ matrices, the sets*

$$D_t(\omega) = \{x \in \mathbb{R}^J \mid M_t(\omega)x \in K\},$$

define a convex conical portfolio constraint process.

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- A (contingent) claim process is a real-valued stochastic process $c = (c_t)_{t=0}^T$ that is adapted to $(\mathcal{F}_t)_{t=0}^T$.
- A premium process is a real-valued adapted stochastic process $p = (p_t)_{t=0}^T$ of cash flows that the seller receives in exchange for delivering a claim.
- Claims and premiums live in the same space

$$\mathcal{M} = \{(c_t)_{t=0}^T \mid c_t \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R})\}.$$

- Traditionally in mathematical finance,

$$p = (p_0, 0, \dots, 0) \quad \text{and} \quad c = (0, \dots, 0, c_T).$$

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- $p \in \mathcal{M}$ is a **superhedging premium** for $c \in \mathcal{M}$ if there exists an adapted \mathbb{R}^J -valued **portfolio process** $x = (x_t)_{t=0}^T$ with $x_T = 0$ such that

$$S_t(x_t - x_{t-1}) + c_t \leq p_t \quad P\text{-a.s.} \quad t = 0, \dots, T.$$

- Here $x_{-1} = 0$ and $S_t(x_t - x_{t-1})$ denotes the \mathcal{F}_t -measurable function

$$\omega \mapsto S_t(x_t(\omega) - x_{t-1}(\omega), \omega).$$

- The above is a numeraire-free way of writing the superhedging property.

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Example 9 *If there is a **numeraire**, the superhedging condition can be written as*

$$\sum_{t=0}^T c_t + \sum_{t=0}^T S_t(x_t - x_{t-1}) \leq \sum_{t=0}^T p_t,$$

*so the payout schedule does not matter. If in addition, S is **linear**, i.e. $S_t(x) = s_t \cdot x$, the condition can be written in terms of a stochastic integral as*

$$\sum_{t=0}^T c_t \leq \sum_{t=0}^T p_t + \sum_{t=0}^{T-1} x_t \cdot \Delta S_{t+1}.$$

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Given a premium process $p \in \mathcal{M}$, we define the **superhedging cost** $\pi(c)$ of a claim $c \in \mathcal{M}$ as the smallest multiple of p that is sufficient for superhedging c . That is,

$$\pi(c) = \inf\{\alpha \mid \exists x \in \mathcal{N}_0 : x_t \in D_t, S_t(x_t - x_{t-1}) + c_t \leq \alpha p_t\}.$$

- If $p = (1, 0, \dots, 0)$ then $\pi(c)$ gives the least **initial investment** required to superhedge c .
- In swap contracts, $\pi(c)$ is the **swap rate**.
- In pension insurance, $\pi(c)$ is the **contribution rate**.

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Proposition 10 *The following properties are always valid.*

1. π is convex,
2. π is monotone: $\pi(c) \leq \pi(c')$ if $c \leq c'$,
3. $\pi(c + \lambda p) = \pi(c) + \lambda$ for all $\lambda \in \mathbb{R}$ and $c \in \mathcal{M}$,
4. $\pi(0) \leq 0$.

If S is sublinear and D is conical, then

5. π is positively homogeneous.

- $\pi(0)$ is the smallest multiple of p one needs in order to find a riskless position in the market.
- If one has to deliver a claim $c \in \mathcal{M}$, one needs $\pi(c) - \pi(0)$ units of p more.

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- More generally, the **superhedging selling price** of a $c \in \mathcal{M}$ for an agent with **initial liabilities** $\bar{c} \in \mathcal{M}$ is

$$P(\bar{c}; c) = \pi(\bar{c} + c) - \pi(\bar{c}).$$

- Analogously, the **superhedging buying price** is

$$\pi(\bar{c}) - \pi(\bar{c} - c) = -P(\bar{c}; -c).$$

- By convexity, $P(\bar{c}; c) \geq -P(\bar{c}; -c)$, i.e. agents with identical liabilities and market expectations should not trade if they aim at superhedging their positions.
- However, even completely risk averse agents may trade if they have different liabilities.

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Arbitrage has little to do with pricing in illiquid markets.

Proposition 11 *Under mild conditions, the conditions*

- (a) $\pi(c) > -\infty$ for some $c \in \mathcal{M}$,
- (b) $\pi(c) > -\infty$ for every $c \in \mathcal{M}$,
- (c) $\pi(0) > -\infty$

are equivalent and imply that the infimum in $\pi(c)$ is attained. If S is sublinear and D is conical, then (c) is equivalent to

- (e) $\pi(0) \geq 0$.

When looking for compensation $p \in \mathcal{M}$ for delivering a $c \in \mathcal{M}$ you shouldn't ask for something that is freely available in the market at unlimited quantities.

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The condition $\pi(0) \geq 0$ is well-known for $p = (1, 0, \dots, 0)$.

- In perfectly liquid markets, it means that there is no **arbitrage of the second kind** in the sense of [Ingersoll, 1987].
- In the fixed income market model of [Dermody and Rockafellar, 1991, 1995], it was called the **weak no arbitrage** condition.
- While $\pi(0) \geq 0$ means that it is not possible to superhedge the zero claim when starting from strictly negative initial wealth, the **law of one price** means that it is not possible to **replicate** the zero claim when starting with strictly negative wealth.

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- Bond prices can be expressed in terms of **zero curves**.
 - In the presence of a cash account, the price of a random claim can be expressed in terms of **martingale measures**.
 - In liquid markets or markets with proportional transaction costs, prices can be expressed in terms of the same dual variables that characterize the no arbitrage condition.
1. For random claims in illiquid markets, we need richer dual objects that encompass both the time value of money as well as the randomness.
 2. In the presence of nonlinearities, arbitrage has little to do with pricing and the corresponding dual variables.

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- Multiple payout dates and nonlinear hedging costs may be attributed to illiquidity.
- Hedging problems often involve premium processes.
- Dual representations involve **stochastic term structures** that capture uncertainty as well as time value of money.
- Martingale measures are related to the existence of a perfectly liquid asset.
- Arbitrage theory is not that important.

What is missing:

- In reality, one rarely looks for superhedging strategies when trading in practice.
- Instead, one (more or less quantitatively) sets bounds on **acceptable levels of risk** when taking positions in the market and when quoting prices.