The Impact of Stochastic Volatility on Pricing, Hedging, and Hedge Efficiency of Variable Annuity Guarantees

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Abstract

We analyze different types of guaranteed withdrawal benefits for life, the latest guarantee feature within Variable Annuities. Besides an analysis of the impact of different product features on the clients' payoff profile, we focus on pricing and hedging of the guarantees. In particular, we investigate the impact of stochastic (implied) equity volatilities on pricing and hedging. We consider different dynamic hedging strategies for delta and vega risks and compare their performance. We also examine the effects if the hedging model (with deterministic volatilities) differs from the data-generating model (with stochastic volatilities). This is an indication for the risk an insurer takes by assuming constant volatilities in the hedging model whilst in the real world, volatilities are stochastic.

Keywords

Variable Annuities, Guaranteed Minimum Benefits, Pricing, Hedging, Hedge Performance, Stochastic Volatility

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1 Introduction

Variable Annuities are fund-linked annuities. Such products were introduced in the 1970es in the United States. In the 1990es, insurers started to include certain guarantees in such policies, so-called guaranteed minimum death benefits (GMDB) as well as guaranteed minimum survival benefits that can be categorized in three main groups: guaranteed minimum accumulation benefits (GMAB), guaranteed minimum income benefits (GMIB) and guaranteed minimum withdrawal benefits (GMWB). GMAB and GMIB type guarantees provide the policyholder some guaranteed maturity value or some guaranteed annuity benefit, respectively.

The third and currently most popular type of guaranteed minimum living benefits are GMWB. Under certain conditions, the insured can withdraw money from their account, even if the value of the account is zero. Such withdrawals are guaranteed as long as both, the amount that is withdrawn within each policy year and the total amount that is withdrawn over the term of the policy stay within certain limits. Recently, insurers started to include additional features in GMWB products. The most prominent is called "GMWB for life": guaranteed lifelong annual withdrawals. The total amount of such withdrawals is not limited as long as each annual withdrawal amount does not exceed some maximum value and the insured is still alive. For these lifelong withdrawal guarantees, annual withdrawals of about 5% of the (single initial) premium are commonly guaranteed for insured aged 60+. At the same time, the insured can at any time access the remaining value of the underlying funds (if positive) by surrendering the contract. Also, in case of death any remaining fund value is paid to the insured's dependants. Usually, the policyholder can choose from a variety of different mutual funds. Therefore, from an insurer's point of view, these products contain an interesting combination of financial risk and longevity risk that is difficult to hedge. As a compensation for the guarantee, the insurer usually charges a guarantee fee that is deducted from the policy's fund value.

Due to the significant financial risk that is inherent within the insurance contracts sold, risk management strategies such as dynamic hedging are commonly applied. During the recent financial crisis, insurers have suffered from inefficient hedge portfolios within their books.[1] Among other effects, volatilities have significantly increased leading to a tremendous increase in option values. In particular for insurers with no or no sufficient vega hedge (i.e. a hedge against the risk of changing volatility), the hedge portfolio did not increase accordingly leading to a loss for existing business (and less attractive conditions, i.e. higher guarantee fees, for new contracts).

There already exists some literature on the pricing of different guaranteed minimum benefits and in particular GMWB: Valuation methods have been proposed by e.g., Milevsky and Posner (2001) for the GMDB-Option, Milevsky and Salisbury (2006) for the GMWB-Option, and Holz et al. (2007) for a GMWB for life. Bauer et al. (2008) have presented a general model framework that allows for the simultaneous and consistent pricing and analysis of different variable annuity guarantees. They also give a comprehensive analysis over non-pricing related literature on variable annuities. To our knowledge, there exists little literature on the performance of different strategies for hedging the market risk of variable annuity

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[1] Cf. e.g. different articles and papers in "Life and Pensions": "A challenging environment" (June 2008), "Variable Annuities – Flawed product design costs Old Mutual £150m" (September 2008), "Variable annuities – Milliman denies culpability for clients' hedging losses" (October 2008), "Variable Annuities – Axa injects $3bn into US arm" (January 2009).

guarantees. Coleman et al. (2005 and 2007) provide such analyses for death benefit guarantees under different hedging and data-generating models. However, to our knowledge, the performance of different hedging strategies for GMWB for Life contracts under stochastic equity volatility has not yet been analyzed. The present paper fills this gap.

The remainder of this paper is organized as follows. In Section 2, we describe different designs of GMWB for Life contracts that will be analyzed in the numerical section and describe the model framework for insurance liabilities used for our analyses. The liability model we describe is akin to the one presented by Bauer et al. (2008).

In Section 3, we provide the framework for the numerical analyses, starting with a description of the asset models used for pricing and hedging of insurance liabilities. For the sake of comparison, we use the classic Black-Scholes model (with deterministic volatility) as a reference and the Heston model for the evolution of an underlying under stochastic volatility. We also describe the financial instruments involved in the hedging strategies described below, and how we determine their fair prices and sensitivities under both models.

The numerical results of our contract analyses are provided in Section 4, starting with the determination of the fair guaranteed withdrawal rate in Section 4.1 for different GMWB for Life products under different model assumptions, first under the Black-Scholes model with deterministic interest rates and volatility, and, secondly, under the Heston model with stochastic volatility. We proceed with an analysis of the distribution of withdrawal amounts in Section 4.2 and trigger times, i.e. the point of time when guaranteed benefits are paid for the first time, in Section 4.3 and finally analyze the so called Greeks in Section 4.4.

In Section 5, we first give an overview over different dynamic and semi-static hedging strategies that can be used to manage the risks emerging from the financial market and analyze and compare the hedging performance of the strategies mentioned above under both asset models. We also examine the effects if the hedging model differs from the data-generating model.

2 Model framework

In Bauer et al. (2008), a general framework for modeling and a valuation of variable annuity contracts was introduced. Within this framework, any contract with one or several living benefit guarantees and/or a guaranteed minimum death benefit can be represented. In their numerical analysis however, only contracts with a rather short finite time horizon were considered. Within the same model framework, Holz et al. (2008) describe how GMWB for Life products can be included in this model. In what follows, we introduce this model framework focusing on the peculiarities of the contracts considered within our numerical analyses. We refer to Bauer et al. (2008) as well as Holz et al. (2008) for the explanation of other living benefit guarantees and more details on the model.

2.1 High-level description of the considered insurance contracts

Variable Annuities are fund linked products. The single premium P is invested in one or several mutual funds. We call the value of the insured's individual portfolio the *account value* and denote it by AV_t. All fees are taken from the account value by cancellation of fund units. Furthermore, the insured has the possibility to surrender the contract or, of course, to withdraw a portion of the account value.

Products with a GMWB option give the policyholder the possibility of guaranteed withdrawals. In this paper, we focus on the case where such withdrawals are guaranteed lifelong (GMWB for life or Guaranteed Lifetime Withdrawal Benefits, GLWB). The guaranteed withdrawal amount is usually a certain percentage x_{WL} of the single premium P. Any remaining account value at the time of death is paid to the beneficiary as death benefit. If, however, the account value of the policy drops to zero while the insured is still alive, the insured can still continue to withdraw the guaranteed amount annually until death. The insurer charges a fee for this guarantee which is usually a pre-specified annual percentage of the account value.

Often, GLWB products contain certain features that lead to an increase of the guaranteed withdrawal amount if the underlying funds perform well. Usually, on every policy anniversary, the current account value of the client is compared to a certain withdrawal benefit base. Whenever the account value exceeds that withdrawal benefit base either the guaranteed annual withdrawal amount is increased (withdrawal step-up) or (a part of) the difference is paid out to the client (surplus distribution). In our numerical analyses in Sections 4 and 5, we have a closer look on four different product designs that can be observed in the market:

- **No Ratchet**: The first and simplest alternative is one where no ratchets or surplus exist at all. In this case, the guaranteed annual withdrawal is constant and does not depend on market movements.

- **Lookback Ratchet**: The second alternative is a ratchet mechanism where a withdrawal benefit base at outset is given by the single premium paid. During the contract term, on each policy anniversary date the withdrawal benefit base is increased to the account value if the account value exceeds the previous withdrawal benefit base. The guaranteed annual withdrawal is increased accordingly to x_{WL} multiplied by the new withdrawal benefit base. This effectively means that the fund performance needs to compensate for policy charges and annual withdrawals in order to increase the guaranteed annual withdrawals.

- **Remaining WBB Ratchet**: With the third ratchet mechanism, the withdrawal benefit base at outset is also given by the single premium paid. The withdrawal benefit base is however reduced by every guaranteed withdrawal. On each policy anniversary where the current account value exceeds the current withdrawal benefit base, the withdrawal benefit base is increased to the account value. The guaranteed annual withdrawal is increased by x_{WL} multiplied by the difference between the account value and the previous withdrawal benefit base. This effectively means that the fund performance needs to compensate for policy charges only but not for annual withdrawals in order to increase guaranteed annual withdrawals. This ratchet mechanism is therefore c.p. somewhat "richer" than the Lookback Ratchet.

- **Performance Bonus:** For this alternative the withdrawal benefit base is defined exactly as in the Remaining WBB ratchet. However, on each policy anniversary where the current account value is greater than the current withdrawal benefit base, 50% of the difference is paid out immediately as a so called performance bonus. The guaranteed annual withdrawals remain constant over time. For the calculation of the withdrawal benefit base, only guaranteed annual withdrawals are subtracted from the benefit base and not the performance bonus payments.

2.2 *Model of the liabilities*

Throughout the paper, we assume that administration charges and guarantee charges are deducted at the end of each policy year as a percentage φ^{admin} and $\varphi^{\text{guarantee}}$ of the account value. Additionally, we allow for upfront acquisition charges $\varphi^{\text{acquisition}}$ that are charges as a percentage of the single premium *P*. This leads to $AV_0 = P \cdot \left(1 - \varphi^{\text{acquisition}}\right)$.

We denote the guaranteed withdrawal amount at time *t* by $W_t^{\text{guaranteed}}$ and the withdrawal benefit base by WBB_t. At inception, for each of the considered products, the initial guaranteed withdrawal amount is given by $W_0^{\text{guaranteed}} = x_{WL} \cdot WBB_0 = x_{WL} \cdot P$. The amount actually withdrawn by the client is denoted by W_t.[2] Thus, the *state vector* $y_t = \left(AV_t, WBB_t, W_t, W_t^{\text{guaranteed}}\right)$ at time *t* contains all information about the contract at that point in time.

Since we restrict our analyses to single premium contracts, policyholder actions during the life of the contract are limited to withdrawals, (partial) surrender and death.

During the year, all processes are subject to capital market movements. For the sake of simplicity, we allow for withdrawals at policy anniversaries only. Also, we assume that death benefits are paid out at policy anniversaries if the insured person has died during the previous year. Thus, at each policy anniversary $t = 1, 2, \ldots, T$, we have to distinguish between the value of a variable in the state vector $(\cdot)_t^-$ immediately before and the value $(\cdot)_t^+$ after withdrawals and death benefit payments.

In what follows, we describe the development between two policy anniversaries and the transition at policy anniversaries for different contract designs. From these, we are finally able to determine all benefits for any given policy holder strategy and any capital market path. This allows for an analysis of such contracts in a Monte-Carlo framework.

2.2.1 Development between two Policy Anniversaries

We assume that the annual fees φ^{admin} and $\varphi^{\text{guarantee}}$ are deducted from the policyholder's account value at the end of each policy year. Thus, the development of the account value between two policy anniversaries is given by the development of the underlying fund S_t after deduction of the guarantee fee, i.e.

$$AV_{t+1}^- = AV_t^+ \cdot \frac{S_{t+1}}{S_t} \cdot e^{-\varphi^{admin} - \varphi^{guarantee}} \,. \tag{1}$$

At the end of each year, the different ratchet mechanism or the performance bonus are applied after charges are deducted and before any other actions are taken. Thus $W_t^{\text{guaranteed}}$ develops as follows:

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[2] Note that the client can chose to withdraw less than the guaranteed amount, thereby increasing the probability of future ratchets. If the client wants to withdraw more than the guaranteed amount, any exceeding withdrawal would be considered a partial surrender.

- **No Ratchet:** $WBB_{t+1}^{-} = WBB_t^{+} = P$ and $W_{t+1}^{\text{guaranteed -}} = W_t^{\text{guaranteed +}} = x_{WL} \cdot P$.

- **Lookback Ratchet:** $WBB_{t+1}^{-} = \max\left\{WBB_t^{+}; AV_{t+1}^{-}\right\}$ and $W_{t+1}^{\text{guaranteed -}} = x_{WL} \cdot WBB_{t+1}^{-} = \max\left\{W_t^{\text{guaranteed +}}; x_{WL} \cdot AV_{t+1}^{-}\right\}$.

- **Remaining WBB Ratchet**: Since withdrawals are only possible on policy anniversaries, the withdrawal benefit base during the year develops like in the Lookback Ratchet case. Thus, we have $WBB_{t+1}^{-} = \max\left\{WBB_t^{+}; AV_{t+1}^{-}\right\}$ and $W_{t+1}^{\text{guaranteed -}} = x_{WL} \cdot WBB_{t+1}^{-} = \max\left\{W_t^{\text{guaranteed +}}; x_{WL} \cdot AV_{t+1}^{-}\right\}$.

- **Performance Bonus:** For this alternative a withdrawal benefit base is defined similarly to the one in the Remaining WBB Ratchet: $WBB_{t+1}^{-} = WBB_t^{+}$ and $W_{t+1}^{\text{guaranteed -}} = W_t^{\text{guaranteed +}} = x_{WL} \cdot P$. Additionally, 50% of the difference between the account value and the withdrawal benefit base is paid out as a performance bonus. Thus, we have $W_{t+1}^{\text{guaranteed -}} = x_{WL} \cdot P + 0.5 \cdot \max\left\{0; AV_{t+1}^{-} - WBB_t^{+}\right\}$.

2.2.2 Transition at a Policy Anniversary *t*

At the policy anniversaries, we have to distinguish the following four cases:

a) The insured has died within the previous year (*t*-1,*t*]

If the insured has died within the previous policy year, the account value is paid out as death benefit. With the payment of the death benefit, the insurance contract matures. Thus, $AV_t^{+} = 0$, $WBB_t^{+} = 0$, $W_t^{+} = 0$, and $W_t^{\text{guaranteed +}} = 0$.

b) The insured has survived the previous policy year and does not withdraw any money from the account at time *t*

If no death benefit is paid out to the policyholder and no withdrawals are made from the contract, i.e. $W_t^{+} = 0$, we get $AV_t^{+} = AV_t^{-}$, $WBB_t^{+} = WBB_t^{-}$, and $W_t^{\text{guaranteed +}} = W_t^{\text{guaranteed -}}$. In the Performance Bonus product, the guaranteed annual withdrawal amount is reset to its original level since $W_t^{\text{guaranteed -}}$ might have contained performance bonus payments. Thus, for this alternative we have $W_t^{\text{guaranteed +}} = x_{WL} \cdot P$.

c) The insured has survived the previous policy year and at the policy anniversary withdraws an amount within the limits of the withdrawal guarantee

If the insured has survived the past year, no death benefits are paid. Any withdrawal W_t below the guaranteed annual withdrawal amount $W_t^{\text{guaranteed -}}$ reduces the account value by the withdrawn amount. Of course, we do not allow for negative policyholder account values and thus get $AV_t^{+} = \max\left\{0; AV_t^{-} - W_t\right\}$.

For the alternatives "No Ratchet" and "Lookback Ratchet", the withdrawal benefit base and the guaranteed annual withdrawal amount remain unchanged, i.e. $WBB_t^{+} = WBB_t^{-}$, and

$W_t^{\text{guaranteed +}} = W_t^{\text{guaranteed -}}$. For the alternative "Remaining WBB Ratchet", the withdrawal benefit base is reduced by the withdrawal taken, i.e. $WBB_t^+ = \max\{0; WBB_t^- - W_t\}$ and the guaranteed annual withdrawal amount remains unchanged, i.e. $W_t^{\text{guaranteed +}} = W_t^{\text{guaranteed -}}$. For the alternative "Performance Bonus", the withdrawal benefit base is at a maximum reduced by the initially guaranteed withdrawal amount (without performance bonus), i.e. $WBB_t^+ = \max\{0; WBB_t^- - \min\{W_t; x_{WL} \cdot P\}\}$ and the guaranteed annual withdrawal amount is set back to its original level, i.e. $W_t^{\text{guaranteed +}} = x_{WL} \cdot P$.

<u>d) The insured has survived the previous policy year and at the policy anniversary withdraws an amount exceeding the limits of the withdrawal guarantee</u>

In this case again, no death benefits are paid. For the sake of brevity, we only give the formulae for the case of full surrender, since partial surrender is not analyzed in what follows. In case of full surrender, the complete account value is withdrawn, we then set $AV_t^+ = 0$, $WBB_t^+ = 0$, $W_t^+ = AV_t^-$, and $W_t^{\text{guaranteed +}} = 0$ and the contract terminates.

2.3 Contract valuation

We denote by x_0 the insured's age at the start of the contract, ${}_t p_{x_0}$ the probability for a x_0-year old to survive the next t years, q_{x_0+t} the probability for a $(x_0 + t)$-year old to die within the next year, and let ω be the limiting age of the mortality table, i.e. the age beyond which survival is impossible. The probability that an insured aged x_0 at inception passes away in the year $(t,t+1]$ is thus given by ${}_t p_{x_0} \cdot q_{x_0+t}$. The limiting age ω allows for a finite time horizon $T = \omega - x_0$. In our numerical analyses below, we assume that mortality within the population of insured happens exactly according to these probabilities.

Assuming independence between financial markets and mortality and risk-neutrality of the insurer with respect to mortality risk, we are able to use the product measure of the risk-neutral measure of the financial market and the mortality measure. In what follows, we denote this product measure by Q. In this setting, contracts can be priced as follows:

We already mentioned that for the contracts considered within our analysis, policyholder actions during the life of the policyholder are limited to withdrawals and (partial) surrender. In our numerical analyses in Sections 4 and 5, we do not consider partial surrender. To keep notation simple, we therefore here only give formulae for the considered cases (cf. Bauer et al. for formulae for the other cases). We denote by s the point of time at which the policyholder surrenders if the insured is still alive and let $s=T$ for a policyholder that does not surrender. For any given value of s, and under the assumption that the insured dies in year $t \in \{1,2,\ldots,\omega - x_0\}$, all contractual cash flows and thus all guarantee payments $Y_i(t;s)$ at times $i \in \{1,2,\ldots,t\}$ and all guarantee fees $Z_i(t;s)$ at times $i \in \{1,2,\ldots,t\}$ are specified for each capital market path. By $\Phi_\tau(t;s)$, we denote the so called time τ option value, i.e. the value of all future guarantee payments $Y_i(t;s)$ minus guarantee fees $Z_i(t;s)$ in this case:

$$V_\tau(t;s) = E_Q\left[\sum_{i=\tau+1}^{t} Y_i(t;s) e^{-r(i-\tau)} \middle| F_\tau\right] - E_Q\left[\sum_{i=\tau+1}^{t} Z_i(t;s) e^{-r(i-\tau)} \middle| F_\tau\right] \tag{2}$$

Thus, the time τ value of the option assuming the mortality rates defined above (still for a given time of surrender) is

$$V_{\tau}\left(s\right)=\sum_{t=\tau+1}^{T}{}_{t-1}p_{x_0+\tau}\cdot q_{x_0+t-1}\cdot V_{\tau}\left(t;s\right). \tag{3}$$

We finally assume that policyholders surrender their contracts with certain surrender probabilities per year and denote the probability that a policyholder surrenders at time s by p_S. Then, the time τ value of the option is given by

$$V_{\tau}=\sum_{s=1}^{T}p_S V_{\tau}\left(s\right). \tag{4}$$

3 Framework for the Numerical Analysis

3.1 Models

For our analyses we assume two primary tradable assets: the fund's underlying, whose spot price we will denote by $S(\cdot)$, and the money-market account, denoted by $B(\cdot)$. We assume the interest spread to be zero and the money-market account to evolve at a constant risk-free rate of interest r:

$$\begin{aligned}&dB(t)=rB(t)dt\\ \Rightarrow\ &B(t)=B(0)\exp(rt)\end{aligned} \tag{5}$$

For the dynamics of $S(\cdot)$, we will use two different models: first we will assume the equity volatility to be deterministic and constant over time, and hence use the Black-Scholes model for our simulations. To allow for a more realistic equity volatility model, we will use the Heston model, in which both, the underlying itself and its volatility, are modeled by stochastic processes.

3.1.1 Black-Scholes Model

In the Black-Scholes (1973) model, the underlying's spot price $S(\cdot)$ follows a geometric Brownian motion whose dynamics under the real-world measure (also called physical measure) P are given by the following stochastic differential equation (SDE)

$$dS(t)=\mu S(t)dt+\sigma_{BS}S(t)dW(t),\quad S(0)\geq 0\ , \tag{6}$$

where μ is the (constant) drift of the underlying, σ_{BS} its constant volatility and $W(\cdot)$ denotes a P-Brownian motion. By Itō's lemma, S(·) has the solution

$$S(t)=S(0)\exp\left(\left(\mu-\frac{\sigma_{BS}^{2}}{2}\right)t+\sigma_{BS}W(t)\right),\quad S(0)\geq 0\ . \tag{7}$$

3.1.2 Heston Model

There are various extensions to the Black-Scholes model that allow for a more realistic modeling of the underlying's volatility. We use the Heston (1993) model in our analyses where the instantaneous (or local) volatility of the asset is stochastic. Under the Heston model, the market is assumed to be driven by two stochastic processes: the underlying's price $S(\cdot)$, and its instantaneous variance $V(\cdot)$, which is assumed to follow a one-factor square-root process identical to the one used in the Cox-Ingersoll-Ross (1985) interest rate model. The dynamics of the two processes under the real-world measure P are given by the following system of stochastic differential equations

$$dS(t) = \mu S(t)dt + \sqrt{V(t)}S(t)\left(\rho dW_1(t) + \sqrt{1-\rho^2}\,dW_2(t)\right), \quad S(0) \geq 0 \tag{8}$$

$$dV(t) = \kappa\left(\theta - V(t)\right)dt + \sigma_v\sqrt{V(t)}dW_1(t), \quad V(0) \geq 0, \tag{9}$$

where μ again is the drift of the underlying, V(t) is the local variance at time t, κ is the speed of mean reversion, θ is the long-term average variance, σ_v is the so-called "vol of vol", or (more precisely) the volatility of the variance, ρ denotes the correlation between the underlying and the volatility, and $W_{1/2}$ are P-Wiener processes. The condition $2\kappa\theta \geq \sigma_v^2$ ensures that the variance process will remain strictly positive almost surely (see Cox, Ingersoll, Ross (1985)).

There is no analytical solution for S(·) available, thus numerical methods will be used in the simulation.

3.2 Valuation

3.2.1 Risk-Neutral Valuation

In order to determine the values (i.e. the risk-neutral expectations) of the assets in our model, we first have to transform the real-world measure P into its risk-neutral counterpart Q, i.e. into a measure under which the process of the discounted underlying's spot price is a (local) martingale. While the transformation to such a measure is unique under the Black-Scholes model, it is not under the Heston model.

If no dividends are paid on the underlying, the dynamics of the underlying's price with respect to the risk-neutral measure under the Black-Scholes model is given by the following equation (see for instance Bingham and Kiesel (2004)):

$$dS(t) = rS(t)dt + \sigma_{BS}S(t)dW^Q(t), \quad S(0) \geq 0, \tag{10}$$

where r denotes the risk-free rate of return and W^Q is a Wiener process under the risk-neutral measure Q.

In the Heston model, as there are two sources of risk, there are also two market-price-of-risk processes, denoted by γ_1 and γ_2 (corresponding to W_1 and W_2). Heston (1993) proposed the following restriction on the market price of volatility risk process, assuming it to be linear in volatility,

$$\gamma_1(t) = \lambda\sqrt{V(t)} \quad . \tag{11}$$

Provided both measures, P and Q, exist, the Q-dynamics of $S(t)$ and $V(t)$, again under the assumption that no dividends are paid, are given by

$$dS(t) = rS(t)dt + \sqrt{V(t)}S(t)\left(\rho dW_1^Q(t) + \sqrt{1-\rho^2}dW_2^Q(t)\right), \quad S(0) \geq 0 \tag{12}$$
$$dV(t) = \kappa^*\left(\theta^* - V(t)\right)dt + \sigma_v\sqrt{V(t)}dW_1^Q(t), \quad V(0) \geq 0 \tag{13}$$

where $W_1^Q(\cdot)$ and $W_2^Q(\cdot)$ are two independent Q-Wiener processes and where

$$\kappa^* = \left(\kappa + \lambda\sigma_v\right), \quad \theta^* = \frac{\kappa\theta}{\left(\kappa + \lambda\sigma_v\right)} \tag{14}$$

are the risk-neutral counterparts to κ and θ (see, for instance, Wong and Heyde (2006)).

Wong and Heyde (2006) also show that the equivalent local martingale measure that corresponds to the market price of volatility risk, $\lambda\sqrt{V(t)}$, exists if the inequality $-\kappa / \sigma_v \leq \lambda < \infty$ is fulfilled. They further show that, if an equivalent local martingale measure Q exists and $\kappa + \lambda\sigma_v \geq \sigma_v\rho$, the discounted stock price $\frac{S(t)}{B(t)}$ is a Q-martingale.

3.2.2 Valuation of the GMWB for Life Products

For both equity models, we use Monte Carlo Simulations to compute the value of the GMWB option value V defined in Section 2.3, i.e. the difference between expected future guarantee payments made by the insurer and expected future guarantee fees deducted from the policyholders' fund assets. We call the contract fair, if V_0=0.

3.2.3 Standard Option Valuation

In some of the hedging strategies considered in Section 5, European "plain vanilla" options are used. Under the Black-Scholes model, closed form solutions exist for the price of European call and put options. For strike price K and maturity T, the call option price at time t is given by the Black (1976) formula

$$Call^{BS}\left(S(t),t\right) = P(t,T)\left[FN(d_1) - KN(d_2)\right], \tag{15}$$

where

$$\begin{aligned} d_1 &:= \frac{\ln(F/K) + \left(\sigma_{BS}{}^2/2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &:= d_1 - \sigma_{BS}\sqrt{T-t} \\ F &:= S(t)e^{(r-q)(T-t)} \\ P(t,T) &:= e^{-r(T-t)} \quad , \end{aligned} \tag{16}$$

and N(·) denotes the cumulative distribution function of the standard normal distribution.

The price of a European put option is given by

$$Put^{BS}\left(S(t),t\right)=P(t,T)\left[KN(-d_2)-FN(-d_1)\right] \tag{17}$$

For the Heston stochastic volatility model, Heston (1993) found a semi-analytical solution for pricing European call and put options using Fourier inversion techniques. The formulas have the form

$$Call^{Heston}\left(S(t),V(t),t\right)=P(t,T)\left[F\cdot P_1-K\cdot P_2\right] \tag{18}$$

$$Put^{Heston}\left(S(t),V(t),t\right)=P(t,T)\left[F\cdot\left(P_1-1\right)-K\cdot\left(P_2-1\right)\right] \tag{19}$$

$$P_{1/2}:=\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty}f_{1/2}(u)du \tag{20}$$

$$f_1(u):=\mathrm{Re}\left(\frac{e^{-iu\ln K}\varphi(u-i)}{iuF}\right) \tag{21}$$

$$f_2(u):=\mathrm{Re}\left(\frac{e^{-iu\ln K}\varphi(u)}{iu}\right) \tag{22}$$

$$\varphi(u):=E_Q\left[e^{iu\ln S(T)}\middle|S(t)\right] \tag{23}$$

This means $\varphi(\cdot)$ is the log-characteristic function of the underlying's price under the risk-neutral measure Q. As Kahl and Jäckel (2006) point out, the computation of the terms $P_{1/2}$ includes the evaluation of the logarithm with complex arguments which may lead to numerical instabilities for certain sets of parameters and/or long-dated options. Therefore, we use the scheme proposed in their paper, which should allow for a robust computation of the fair values of European call and put options for (practically) arbitrary parameters. As in the proposed scheme, we use the adaptive Gauss-Lobatto quadrature method for the numerical integration of P_1 and P_2.

3.3 Computation of Sensitivities (Greeks)

Where no analytical solutions for the sensitivity of the option's or guarantee's value to changes in model parameters (the so-called Greeks, cf. e.g. Hull (2008)) exist, we use Monte Carlo methods to compute the respective sensitivities numerically. We use finite differences as approximations of the partial derivatives, where the direction of the shift is chosen accordingly to the direction of the risk, i.e. for the delta we shift the stock downwards in order to compute the backward finite difference, and shift the volatility upwards for the vega, this time to compute a forward finite difference.

4 Contract Analysis

4.1 Determination of the Fair Guaranteed Withdrawal Rate

In this section, we first calculate the guaranteed withdrawal rate x_{WL} that makes a contract fair, all other parameters given. In order to calculate x_{WL}, we perform a root search with x_{WL} as argument and the value of the option V_0 as function value. For all of the analyses we use the fee structure given in Table 1.

Acquisition fee	4.00 % of lump sum
Management fees	1.50 % p.a. of NAV
Guarantee fees	1.50 % p.a. of NAV
Withdrawal fees	0.00 % of withdrawal amount

Table 1: Assumed fee structure for all regarded contracts.

We further assume the policy holder to be a 65 years old male. For pricing purposes, we use best-estimate mortality probabilities given in the DAV 2004R table published by the German Actuarial Society (DAV).

4.1.1 Results for the Black-Scholes model

All results for the Black-Scholes model have been calculated assuming a risk-free rate of interest of $r = 4\%$.

Table 3 displays the fair guaranteed withdrawal rates for different ratchet mechanisms, different volatilities and different policyholder behavior assumptions: We assume that – as long as their contracts are still in force – policy holders every year withdraw exactly the maximum guaranteed annual withdrawal amount. Further, we look at the scenarios no surrender (no surr), surrender according to Table 2 (surr 1) and surrender with twice the probabilities given in Table 2 (surr 2).

Year	Surrender rate
1	6 %
2	5 %
3	4 %
4	3 %
5	2 %
≥ 6	1 %

Table 2: Assumed deterministic surrender rates.

Ratchet Mechanism / Volatility		I (No Ratchet)	II (Lookback Ratchet)	III (Remaining WBB Ratchet)	IV (Performance Bonus)
σ_{BS}=15 %	No surr	5.26 %	4.80 %	4.43 %	4.37 %
	Surr 1	5.45 %	5.00 %	4.62 %	4.57 %
	Surr 2	5.66 %	5.22 %	4.83 %	4.79 %
σ_{BS}=20 %	No surr	4.98 %	4.32 %	4.01 %	4.00 %
	Surr 1	5.16 %	4.50 %	4.18 %	4.19 %
	Surr 2	5.35 %	4.71 %	4.38 %	4.40 %
σ_{BS}=22 %	No surr	4.87 %	4.13 %	3.85 %	3.85 %
	Surr 1	5.04 %	4.30 %	4.01 %	4.03 %
	Surr 2	5.23 %	4.50 %	4.20 %	4.24 %
σ_{BS}=25 %	No surr	4.70 %	3.85 %	3.61 %	3.62 %
	Surr 1	4.86 %	4.01 %	3.76 %	3.81 %
	Surr 2	5.04 %	4.20 %	3.94 %	4.01 %

Table 3: Fair guaranteed withdrawal rates for different ratchet mechanisms, different policyholder behavior assumptions and under different volatilities.

A comparison of the different product designs shows that, obviously, the highest annual guarantee can be provided if no ratchet or performance bonus is provided at all. If no surrender is assumed and a volatility of 20% is assumed, the guarantee is similar to a “5 for life” product (4.98%). Including a Lookback Ratchet would need a reduction of the initial annual guarantee by 66 basis points to 4.32%. If a richer ratchet mechanism is provided such as the Remaining WBB Ratchet, the guarantee needs to be reduced to 3.61%. About the same annual guarantee (3.62%) can be provided if no ratchet is provided but a Performance Bonus is paid out annually.

Throughout our analyses, the Remaining WBB Ratchet and the Performance Bonus allow for about the same annual guarantee. However, for lower volatilities, the Remaining WBB Ratchet seems to be less valuable than the Performance Bonus and therefore allows for higher guarantees while for higher volatilities the Performance Bonus allows for higher guarantees. Thus, the relative impact of volatility on the price of a GLWB depends on the chosen product design and appears to be particularly high for ratchet type products (II and III). This can also observed comparing the No Ratchet case with the Lookback Ratchet. While – when volatility is increased from 15% to 25% – for the No Ratchet case, the fair guaranteed withdrawal decreases by just over half a percentage point from 5.26% to 4.7%, it decreases by almost a full percentage point from 4.8% to 3.85% in the Lookback Ratchet case (if no surrender is assumed). The reason for this is that for the products with ratchet, high volatility leads to a possible lock in of high positive returns in some years and thus is a rather valuable feature if volatilities are high.

If the insurance company assumes some deterministic surrender probability when pricing GLWBs, the guarantees increase for all model points observed. The increase of the annual guarantee is rather similar over all product types and volatilities. The annual guarantee increases by around 15-20 basis points if the surrender assumption from Table 2 is made and increases by another 20 basis points if this surrender assumption is doubled.

4.1.2 Results for the Heston model

We use the calibration given in Table 4, where the Heston parameters are those derived by Eraker (2004), and stated in annualized form for instance by Poulsen (2007).

Parameter	Numerical value
r	0.04
θ	0.220^2
κ	4.75
σ_v	0.55
ρ	-0.569
$V(0)$	θ

Table 4: Benchmark parameters for the Heston model.

One of the key parameters in the Heston model is the market price of volatility risk λ. Since absolute λ-values are hard to be interpreted, in the following table we show long-run local variance and speed of mean reversion for different parameter values of λ.

Market price of volatility risk	Speed of mean reversion κ	Long-run local variance θ
$\lambda = 3$	6.40	0.190^2
$\lambda = 2$	5.85	0.198^2
$\lambda = 1$	5.30	0.208^2
$\lambda = 0$	4.75	0.220^2
$\lambda = -1$	4.20	0.234^2
$\lambda = -2$	3.65	0.251^2
$\lambda = -3$	3.10	0.272^2

Table 5: Q-parameters for different choices of the market price of volatility risk factor.

Higher values of λ correspond to a lower volatility and a higher mean reversion speed while lower (e.g. negative) values of λ correspond to high volatilities and lower speed of mean reversion. $\lambda = 2$ implies a long-term volatility of 19.8% and $\lambda = -2$ implies a long-term volatility of 25.1%.

In the following table, we show the fair annual withdrawal guarantee under the Heston model for all different product designs and values of λ between -2 and 2.

Ratchet Mechanism / Market price of volatility risk	I (No Ratchet)	II (Lookback Ratchet)	III (Remaining WBB Ratchet)	IV (Performance Bonus)
λ = 2	4.99 %	4.36 %	4.03 %	4.00 %
λ = 1	4.93 %	4.27 %	3.95 %	3.93 %
λ = 0	4.87 %	4.17 %	3.86 %	3.84 %
λ = -1	4.79 %	4.05 %	3.75 %	3.74 %
λ = -2	4.70 %	3.90 %	3.62 %	3.62 %

Table 6: Fair guaranteed withdrawal rates for different ratchet mechanisms and volatilities when no surrender is assumed.

Under the Heston model, the fair annual guaranteed withdrawal appears to be the same as under the Black-Scholes model with a comparable constant volatility. E.g. for λ = 0, which corresponds with a long-term volatility of 22%, the fair annual guaranteed withdrawal rate for a contract without ratchet is given by 4.87%, exactly the same number as under the Black-Scholes model. In the Lookback Ratchet case, the Heston model leads to a fair guaranteed withdrawal rate of 4.17%, the Black-Scholes model of 4.13%. For the other two product designs, again, both asset models almost exactly lead to the same withdrawal rates.

Thus, for the pricing (as opposed to hedging, see Section 5) of GLWB benefits, the long-term volatility assumption is much more crucial than the question whether stochastic volatility should be modeled or not.

4.2 Distribution of Withdrawals

In this section, we compare the distributions of the guaranteed withdrawal benefits (given the policyholder is still alive) for each policy year and for all four different ratchet mechanisms that were presented in Section 2.

We use the Black-Scholes model for all simulations in this chapter and assume a risk-free rate of interest r = 4%, an underlying's drift μ = 7% and a constant volatility of σ_{BS} = 22 %. For all four ratchet types, we use the guaranteed withdrawal rates derived in section 4.1.1 (without surrender).

In the following figure, for each product design we show the development of arithmetic average (red line), median (yellow points), 10^{th} - 90^{th} percentile (light blue area), and 25^{th} - 75^{th} percentile (dark blue area) of the guaranteed annual withdrawal amount over time.

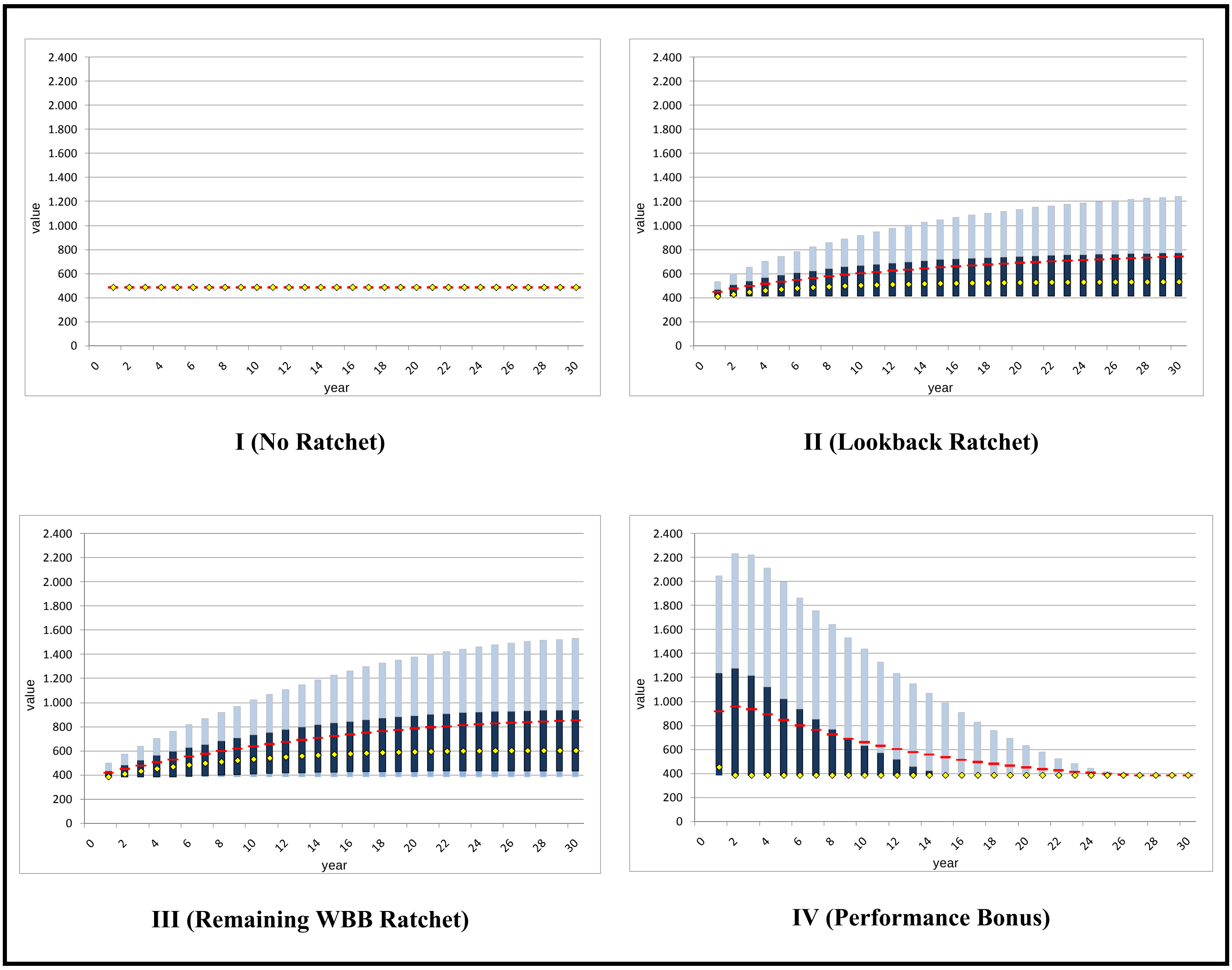

Figure 1: Development of percentiles, median and mean of the guaranteed withdrawal amount over policy years 0 to 30 for each ratchet type.

Obviously, the different considered product designs lead to significantly different risk/return-profiles for the policyholder. While the No Ratchet case provides deterministic cash flows over time, the other product designs differ quite considerably. Both ratchet products have potentially increasing benefits. For the Lookback Ratchet, however, the 25th percentile remains constant at the level of the first withdrawal amount. Thus, the probability that a ratchet never happens is higher than 25%. The median increases for the first 10 years and then reaches some constant level implying that with a probability of more than 50% no withdrawal increments will take place thereafter.

Product III (Remaining WBB Ratchet) provides more potential for increasing withdrawals: For this product, the 25th percentile increases over the first few years and the median is increasing for around 20 years. In the 90th percentile, the guaranteed annual withdrawal amount reaches 1,500 after slightly more than 25 years while the Lookback Ratchet hardly reaches 1,200. On average, the annual guaranteed withdrawal amount more than doubles over time while the Lookback Ratchet doesn't, of course this is only possible since the guaranteed withdrawal at t=0 is lower.

A completely different profile is achieved by the fourth product design, the product with Performance Bonus. Here, annual withdrawal amounts are rather high in the first years and are falling later. After 15 years, with a 75% probability no more performance bonus is paid, after 25 years, with a probability of 90% no more performance bonus is paid.

For all three product designs with some kind of bonus, the probability distribution of the annual withdrawal amount is rather skewed: the arithmetic average is significantly above the median. For the product with Performance Bonus, the median exceeds the guarantee only in the first year. Thus, the probability of receiving a performance bonus in later years is less than 50%. The expected value, however, is more than twice as high.

4.3 Distribution of Trigger Times

In the following figure, for each of the products, we show the probability distribution of trigger times, i.e. of the point in time where the account value drops to zero and the guarantee is triggered, (if the insured is still alive). Any probability mass at t=56 (i.e. age 121 which is the limiting age of the mortality table used), refers to scenarios where the guarantee is not triggered.

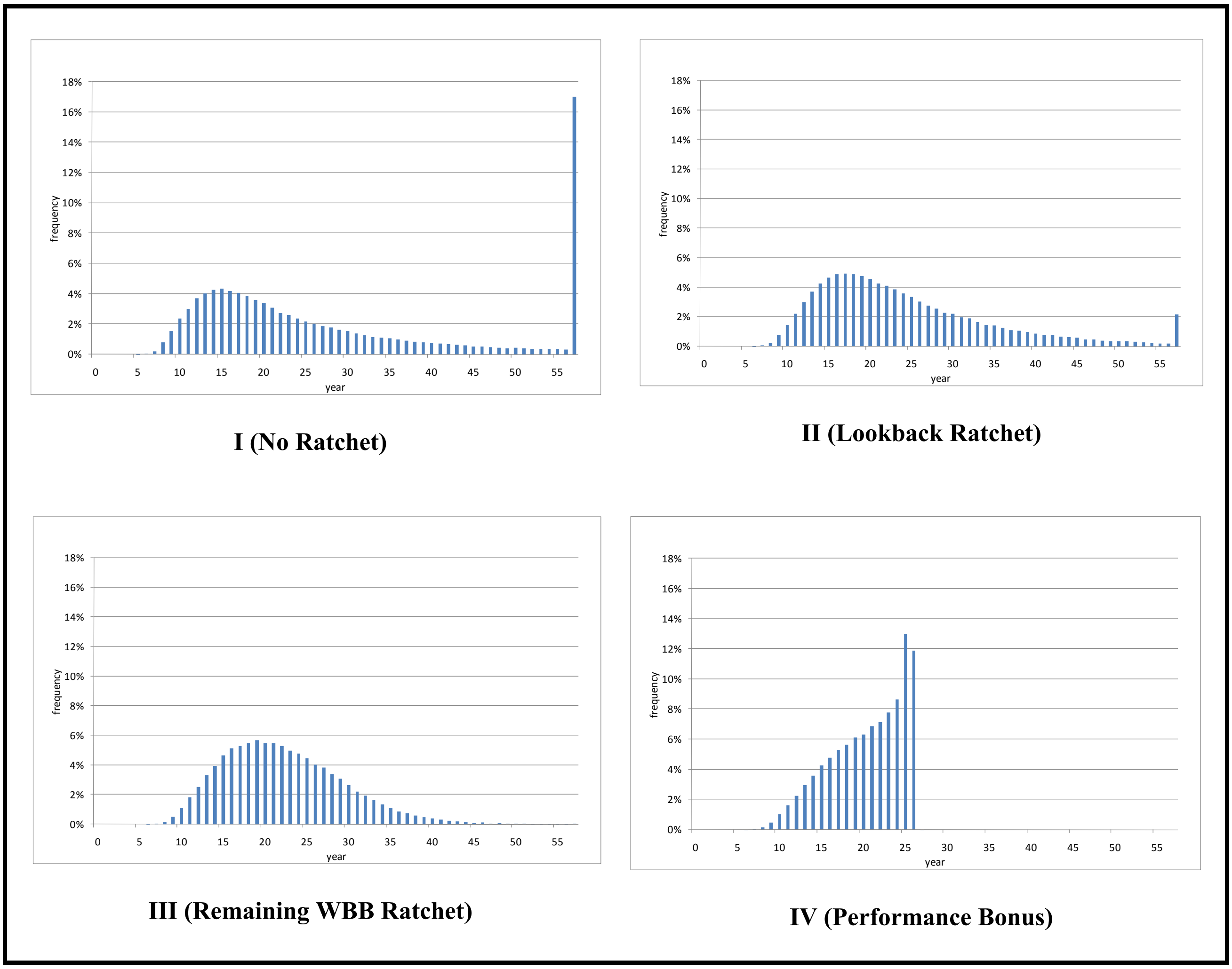

Figure 2: Distribution of trigger times for each of the product designs.

For the No Ratchet product, trigger times vary from 7 to over 55 years. With a probability of 17%, there is still some account value available at age 121. For this product, on the one hand, the insurance company's uncertainty with respect to if and when guarantee payments have to be paid is very high; on the other hand, there is a significant chance that the guarantee is not triggered at all, which reduces longevity (tail) risk[3] from the insurer's perspective.

For the products with ratchet features, very late or even no triggers appear to be less likely. The more valuable a ratchet mechanism is for the client, the earlier the guarantee tends to trigger. While for the Lookback Ratchet still 2% of the contracts do not trigger at all, the Remaining WBB Ratchet almost certainly triggers within the first 40 years. However, on average the guarantee is triggered rather late, after around 20 years.

The least uncertainty in the trigger time appears to be in the product with Performance Bonus. While the probability distribution looks very similar to that of the Remaining WBB Ratchet for the first 15 years, trigger probabilities then increase rapidly and reach a maximum at t=25 and 26 years. Later triggers did not occur at all within our simulation. The reason for this is quite obvious: The Performance Bonus is given by 50% of the difference between the current account value and the Remaining WBB. However, the Remaining WBB is annually reduced by the initially guaranteed withdrawal amount and therefore reaches 0 after 26 years (1 / 3.85%). Thus, after 20 years, almost half of the account value is paid out as bonus every year. This, of course, leads to a tremendously decreasing account value in later years. Therefore, there is not much uncertainty with respect to the trigger time on the insurance company's side. On the other hand, the complete longevity tail risk remains with the insurer.

Whenever the guarantee is triggered, the insurance company needs to pay an annual lifelong annuity equal to the last guaranteed annual withdrawal amount. This is the guarantee that needs to be hedged by the insurer. Thus, in the following section, we have a closer look on the Greeks of the guarantees of the different product designs.

4.4 Greeks

Within our Monte Carlo simulation, for each scenario we can calculate different sensitivities of the option value as defined in Section 2.3, the so called Greeks. All Greeks are calculated for a pool of identical policies with a total single premium volume of US$100m under assumptions of future mortality and future surrender. All the results shown in this section are calculated under standard mortality and no surrender assumptions.

In the following figure, we chose to show different percentiles as well as median and arithmetic average of the so called delta, i.e. the sensitivity of the option value as defined in with respect to changes in the price of the underlying:

[3] This risk is not modelled in our framework.

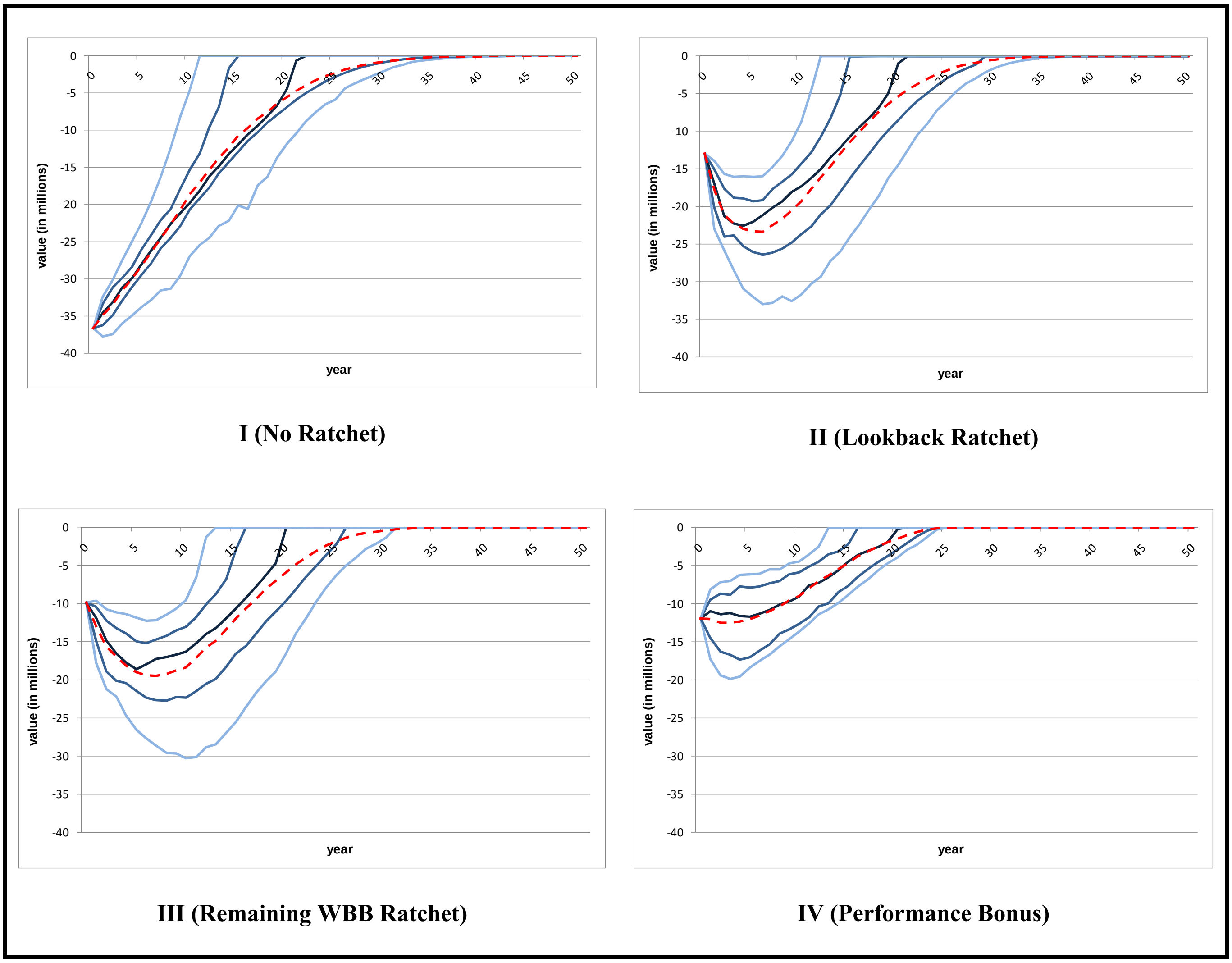

Figure 3: Development over time of the percentiles of the delta for a pool of policies multiplied by the current spot value.

First of all, it is rather clear that all products throughout do have negative deltas since the value of the guarantee increases with falling stock markets and vice versa. Once the guarantee is triggered, no more account value is available and thus, from this point on, the delta is zero. Thus, in what follows, we call delta to be "high" whenever its absolute amount is big.

At outset, the product without any ratchet or bonus does have the biggest delta and thus the highest sensitivity with respect to changes in the underlying's price. The reason for this mainly is that the guarantee is not adjusted when fund prices rise. In this case, the value of the guarantee decreases much stronger than with any product where either ratchet lead to an increasing guarantee or a performance bonus leads to a reduction of the account value. On the other hand, if fund prices decrease, the first product is deeper in the money since it does have the highest guarantee at outset. Over time, all percentiles of the delta in the No Ratchet case are decreasing.

For products II and III, the guarantee can never be far out of the money due to the ratchet feature. Thus delta increases in the first few years. All percentiles reach a maximum after ten years and tend to be decreasing from then on.

For the product with Performance Bonus, delta exposure is by far the lowest. This is consistent with our results of the previous section where we concluded that the uncertainty for the insurance company is the highest in the No Ratchet case and the lowest in the Performance Bonus case.

5 Analysis of Hedge Efficiency

In this Section, we analyze the performance of different (dynamic) hedging strategies, which can be applied by the insurer in order to reduce the financial risk of the guarantees (and thereby the required economic risk capital). We first describe the analyzed hedging strategies, before we define the risk measures that we use to compare the simulation results of the hedging strategies, which are presented in the last part of this Section.

5.1 Hedge Portfolio

We assume that the insurer has sold a pool of policies with GLWB guarantees. We denote by $\Psi(\cdot)$ the option value for that pool, i.e. the sum of the option values V defined in Section 2.3 of all policies. We assume that the insurer cannot influence the value of the guarantee $\Psi(\cdot)$ by changing the underlying fund (i.e. changing the fund's exposure to risky assets or forcing the insured to switch to a different, e.g. less volatile, fund). We further assume that the insurer invests the guarantee fees in some hedge portfolio $\Pi^{Hedge}(\cdot)$ and performs some hedging strategy within this hedging portfolio. In case a guarantee is triggered, guaranteed payments are made from that portfolio. Thus,

$$\Pi(t) = -\Psi(t, S(t),...) + \Pi^{Hedge}(t) \tag{24}$$

is the insurer’s cumulative profit/loss (in what follows sometimes just denoted as insurer’s profit) stemming from the guarantee and the corresponding hedging strategy.

The following hedging strategies aim at reducing the insurer's risk by implementing certain investment strategies within the hedge portfolio $\Pi^{Hedge}(\cdot)$. Note that the value $\Psi(\cdot)$ of the pool of policies at time t does not only depend on the number and size of contracts and the underlying fund's current level, but also on several retrospective factors, such as the historical prices of the fund at previous withdrawal dates, and on model and parameter assumptions.

The insurer’s choice of model and parameters can also have a significant impact on the hedging strategies. Therefore, we will differentiate in the following between the hedging model that is chosen and used by the insurer, and the data-generating model that we use to simulate the development of the underlying and the market prices of European call and put options. This allows us, e.g., to analyze the impact on the insurer’s risk situation if the insurer bases pricing and hedging on a simple Black-Scholes model (hedging model) with deterministic volatility whereas in reality (data-generating model) volatility is stochastic. We assume the value of the guarantee to be marked-to-model, where the same model is used for valuation as the insurer uses for hedging. All other assets in the insurer's portfolio are marked-to-market, i.e. their prices are determined by the (external) data-generating model.

We assume that, additional to the underlying $S(\cdot)$ and the money-market account $B(\cdot)$, a market for European “plain vanilla” options on the underlying exists. However, we assume that only options with limited time to maturity are liquidly traded. As well as the underlying and the money-market account, we assume the option prices (i.e. the implied volatilities) to be driven by the data-generating model, and presume risk-neutrality with respect to volatility risk, i.e. the market price of volatility is set to zero in case the Heston model is used as data-generating model. Additionally, we assume the spread between bid and ask prices/volatilities to be zero.

For all considered hedging strategies we assume the hedging portfolio to consist of three assets, whose quantities are rebalanced at the beginning of each hedging period: a position of quantity $\Delta_S(\cdot)$ in the underlying, a position of quantity $\Delta_B(\cdot)$ in the money-market account and a position of $\Delta_X(\cdot)$ in a 1-year ATMF straddle (i.e. an option consisting of one call and one put, both with one year maturity and at the money with respect to the maturity's forward, ATMF). We assume the insurer to hold the position in the straddle for one hedging period, then sell the options at then-current prices, and set up a new position in a then 1-year ATMF straddle. For each hedging period, the new straddle is denoted by $X(\cdot)$. We assume that the portion of the hedge portfolio that was not invested in either $S(\cdot)$ or $X(\cdot)$ is invested in (or borrowed from) the money market. Thus, the hedge portfolio at time t has the form

$$\Pi^{Hedge}(t) = \Delta_S(t)S(t) + \Delta_B(t)B(t) + \Delta_X(t)X(t) \ , \tag{25}$$

where

$$\Delta_B(t) := \frac{\Pi^{Hedge}(t) - \Delta_S(t)S(t) - \Delta_X(t)X(t)}{B(t)} \ . \tag{26}$$

5.2 *Dynamic Hedging Strategies*

For both considered hedging models, Black-Scholes and Heston, we analyze three different types of (dynamic) hedging strategies.

No Hedge (NH)

The first strategy simply invests all guarantee fees in the money-market account. The strategy is obviously identical for both models.

Delta Hedge (D)

The second type of hedging strategy uses a position in the underlying in order to immunize the portfolio against small changes in the underlying's level. In the Black-Scholes framework without transaction costs, such a position is sufficient to perform a perfect hedge. In reality however, time-discrete trading and transaction costs cause imperfections.

Using the Black-Scholes model as hedging model, in order to immunize the portfolio against small changes in the underlying's price (i.e. to attain delta-neutrality), Δ_S is chosen as the delta of $\Psi(\cdot)$, i.e. the partial derivative of $\Psi(\cdot)$ with respect to the underlying .

While delta hedging under the Black-Scholes model (given the typical assumptions), constitutes a theoretically perfect hedge, it does not under the Heston model. This leads to (locally) risk minimizing strategies that aim to minimize the variance of the instantaneous change of the portfolio. Under the Heston model[4], the problem

$$\operatorname{var}\big(d\Pi(t)\big) \to \min, \quad \Delta_S \in IR, \Delta_X = 0 \tag{27}$$

has the solution (see e.g. Ewald, Poulsen and Schenk-Hoppe, 2007)

$$\Delta_S(t) = \frac{\partial \Psi^{Heston}(t, S(t), V(t), \ldots)}{\partial S(t)} + \frac{\rho\sigma_v}{S(t)} \frac{\partial \Psi^{Heston}(t, S(t), V(t), \ldots)}{\partial V(t)}. \tag{28}$$

To keep notation simple, this (locally) risk minimizing strategy under the Heston model is also referred to as delta hedge.

Delta and Vega (DV)

The third type of hedging strategies incorporates the use of the straddle $X(\cdot)$, exploiting its sensitivity to changes in volatility for the sake of neutralizing the portfolio's exposure to changes in volatility.

Under the Black-Scholes model, volatility is assumed to be constant; therefore using it to hedge against a changing volatility appears rather counterintuitive. Nevertheless, following Taleb (1997), we analyze some kind of *ad-hoc* vega hedge in our simulations, that aims at compensating the deficiencies of the Black-Scholes model: For performing the vega hedge, we do not compute the Black-Scholes vega of the guarantee $\Psi(\cdot)$ and compare it to the corresponding Black-Scholes vega of the option $X(\cdot)$, but, instead, we will be using the so-called *modified Vega* of $\Psi(\cdot)$ for comparison. Since all maturities cannot be expected to react the same way to changes in today's volatility, the *modified Vega* applies a different weighting to the respective vega of each maturity. We use the inverse of square root of time as simple weighting method and use the maturity of the hedging instrument $X(\cdot)$, i.e. one year, as benchmark maturity. The modified vega of $\Psi(\cdot)$ at a policy calculation date τ then has the form

$$ModVega(\tau) = \sum_{t=\tau+1}^{T} \nu_t \frac{1}{\sqrt{t-\tau}} \tag{29}$$

where the ν_t denote the respective Black-Scholes vega of each discounted future cash flow of the pool of policies. This determines the option position (i.e. the quantity of straddles) required to achieve vega neutrality.

Under the Heston model, we compare the two derivatives of $\Psi(\cdot)$ and $X(\cdot)$ with respect to the current local variance $V(\cdot)$ and then analogously determine the option position required to achieve vega neutrality.

Of course, under both hedging models, the position in the underlying must be adjusted for the delta of the option position $\mathbf{\Delta}_X X(\cdot)$.

[4] Note that a (time-continuously) Delta-hedged portfolio under the Black-Scholes model is already risk-free. Therefore for the Black-Scholes model, the Delta-hedging strategy coincides with the locally risk minimizing strategy.

The hedge ratios for all three strategies used in our simulations are summarized in Table 7 for the Black-Scholes model, and in Table 8 for the Heston model.

	Δ_S	Δ_X
(NH)	0	0
(D-BS)	$\dfrac{\partial \Psi^{BS}(t,S(t),\sigma^{BS},...)}{\partial S(t)}$	0
(DV-BS)	$\dfrac{\partial \Psi^{BS}(t,S(t),\sigma^{BS},...)}{\partial S(t)} - \Delta_X \dfrac{\partial X^{BS}(t,S(t),\sigma^{BS},...)}{\partial S(t)}$	$ModVega(t) \Big/ \dfrac{\partial X^{BS}(t,S(t),\sigma_X^{BS},...)}{\partial V(t)}$

Table 7: Hedge ratios for the different strategies if the Black-Scholes model is used as hedging model.

	Δ_S	Δ_X
(NH)	0	0
(D-H)	$\dfrac{\partial \Psi^{Heston}(t,S(t),V(t),...)}{\partial S(t)} + \dfrac{\rho\sigma_v}{S(t)} \dfrac{\partial \Psi^{Heston}(t,S(t),V(t),...)}{\partial V(t)}$	0
(DV-H)	$\dfrac{\partial \Psi^{Heston}(t,S(t),V(t)...)}{\partial S(t)} - \Delta_X \dfrac{\partial X^{Heston}(t,S(t),V(t)...)}{\partial S(t)} + \dfrac{\rho\sigma_v}{S(t)} \left(\dfrac{\partial \Psi^{Heston}(t,S(t),V(t)...)}{\partial V(t)} - \Delta_X \dfrac{\partial X^{Heston}(t,S(t),V(t)...)}{\partial V(t)} \right)$	$\dfrac{\partial \Psi^{Heston}(t,S(t),V(t)...)}{\partial V(t)} \Big/ \dfrac{\partial X^{Heston}(t,S(t),V(t)...)}{\partial V(t)}$

Table 8: Hedge ratios for the different strategies if the Heston model is used as hedging model.

Additionally, for all dynamic hedging strategies (Delta and Delta-Vega), we assume that the hedger buys 1-year European put options at each policy anniversary such that the possible guarantee payments for the next policy anniversary are fully hedged by the put options (assuming surrender and mortality rates are deterministic and known). This strategy aims at avoiding having to hedge an option with short time to maturity and hence having to deal with a potentially rapidly alternating delta (high gamma) if the option is near the strike. This is possible for all four ratchet mechanisms, since the guaranteed withdrawal amount is known one year in advance.

For all considered hedging strategies we assume that the hedge portfolio is rebalanced on a monthly basis.

5.3 Simulation Results

We use the following three ratios to compare the different hedging strategies, all of which will be normalized as a percentage of the sum of the premiums paid to the insurer at t=0:

- $E_P\left[e^{-rT}\Pi_T\right]$, the discounted expectation of the final value of the insurer's final profit under the real-world measure *P*, where $T=\omega-x_0$. This is a measure for the insurer's expected profit and constitutes the "performance" ratio in our context. A value of 1 means that, in expectation, for a single premium of 100 paid by the client, the insurance company's expected profit is 1.

- $CTE_{1-\alpha}(\chi) = E_P\left[-\chi \middle| -\chi \geq VaR_\alpha(\chi)\right]$, the conditional tail expectation of the random variable χ, where χ is defined as the minimum of the discounted values of the insurer's profit/loss at all policy calculation dates, i.e. $\chi = \min\left\{e^{-rt}\Pi_t \middle| t = 0,...,T\right\}$, and VaR denotes the Value at Risk. This is a measure for the insurer's risk given a certain hedging strategy: It can be interpreted as the additional amount of money that would be necessary at outset such that the insurer's profit/loss would never become negative over the life of the contract, even if the world develops according to the average of the α (e.g. 10%) worst scenarios in the stochastic model. Thus a value of 1 in the above table means that, in expectation over the 10% worst scenarios, for a single premium of 100 paid by the client, the insurance company would need to hold 1 additional unit of capital upfront.

- $CTE_{1-\alpha}(\Pi_T) = E_P\left[-\Pi_T \middle| -\Pi_T \geq VaR_\alpha(\Pi_T)\right]$, the conditional tail expectation of the profit/loss' final value. This is also a risk measure which, however, focuses on the value of the profit/loss at time *T*, i.e. after all liabilities have been met, and does not care about negative portfolio values over time. Thus a value of 1 in the above table means that, in expectation over the α (e.g. 10%) worst scenarios, for a premium of 100 paid by the client, the insurance company's expected loss is 1. By definition, of course $CTE_{1-\alpha}(\chi) \geq CTE_{1-\alpha}(\Pi_T)$.

In the numerical analyses below, we set α=10% for both risk measures and assume a pool of identical policies with parameters as given in Section 4 assuming no surrender. Our analysis focuses on model risk rather than parameter risk. Therefore, we use the benchmark parameters for the capital market models presented in Section 4 for both, the hedging and the data-generating model.

The following Table gives the results for different hedging strategies and different data-generating models as a percentage of the single premium paid by the client.

		Data-Generating model							
		Black-Scholes				Heston			
		Product				Product			
		I	II	III	IV	I	II	III	IV
No hedge **(NH)**	$E_P\left[e^{-rT}\Pi_T\right]$	10.43	7.77	6.67	3.88	10.36	7.97	6.82	4.13
	$CTE_{1-\alpha}(\chi)$	25.29	20.07	17.54	15.12	25.76	20.97	18.54	15.97
	$CTE_{1-\alpha}(\Pi_T)$	23.41	18.27	15.90	13.35	22.93	18.55	16.25	13.51
Delta hedge Black-Scholes **(D-BS)**	$E_P\left[e^{-rT}\Pi_T\right]$	0.48	0.27	0.21	0.17	0.57	0.29	0.17	0.13
	$CTE_{1-\alpha}(\chi)$	1.71	3.25	3.12	2.02	2.77	4.76	4.51	3.35
	$CTE_{1-\alpha}(\Pi_T)$	1.44	2.74	2.71	1.78	2.44	4.14	3.99	3.02
Delta hedge Heston **(D-H)**	$E_P\left[e^{-rT}\Pi_T\right]$					0.52	0.42	0.34	0.21
	$CTE_{1-\alpha}(\chi)$					2.63	4.59	4.44	3.36
	$CTE_{1-\alpha}(\Pi_T)$					2.28	4.03	3.98	2.95
Delta-Vega hedge Black-Scholes **(DV-BS)**	$E_P\left[e^{-rT}\Pi_T\right]$					0.82	0.81	0.75	0.47
	$CTE_{1-\alpha}(\chi)$					1.75	2.41	3.01	1.88
	$CTE_{1-\alpha}(\Pi_T)$					1.35	1.80	2.40	1.53
Delta-Vega hedge Heston **(DV-H)**	$E_P\left[e^{-rT}\Pi_T\right]$					0.49	0.41	0.33	0.19
	$CTE_{1-\alpha}(\chi)$					1.40	1.99	1.95	1.49
	$CTE_{1-\alpha}(\Pi_T)$					1.15	1.58	1.60	1.21

Table 9: Results for different hedging strategies and different data-generating models as a percentage of the single premium paid by the client

If no hedging is in place, obviously, the insurance company has a long position in the underlying and thus faces a rather high expected return combined with high risk. No hedging effectively means that the insurance company, on average over the worst 10% scenarios, would need additional capital between 15% and 25% of the premium volume paid by the clients in order to avoid a loss over time. The $CTE_{1-\alpha}(\Pi_T)$ are around 23 for product I (No Ratchet) and around 13 for product IV (Performance Bonus) in both data-generating models. The corresponding values for the products with ratchet are in between. The difference in risk and expected return between the two data-generating models is rather small.

If the insurance company sets up a delta hedging strategy based on the Black-Scholes model, risk is significantly reduced for all products and both data-generating models. If the data-generating model is also the Black-Scholes model, risk is reduced to less than 10% of its unhedged value for product I (No Ratchet). This of course goes hand in hand with a reduction of the expected profit of the insurer. While without hedging, the No Ratchet product appeared to be the riskiest, after delta hedging, the products with a ratchet (Lookback Ratchet and Remaining WBB Ratchet) now are the riskiest. The reason for this is that delta is rather "volatile" for the products with ratchet, cf. Figure 3 in Section 4.4. Since fast changes in the delta lead to potential losses, this increases the risk for the ratchet type products. This basically shows the effect of a high gamma (second order derivative of the option value with respect to the underlying price). The higher the gamma, the higher discretization errors and thus the higher the risk of a delta-only hedge.

We now look how the results of the Black-Scholes delta hedge change, if the data-generating model is the Heston model. By solely introducing stochastic volatility into the capital market, the risk of the hedging strategy, throughout all product types, is increased by roughly 50%. This demonstrates the effect model risk can have on hedge efficiency. At the same time, the insurance company's expected return hardly changes.

If the calculation of the hedge position is also performed within the Heston model (D-H), risk is only reduced by a small amount. However, for both products with a ratchet mechanism and the Performance Bonus product, the insurer's expected profit is significantly increased. Thus, by adopting the hedging model to the data-generating model, the insurance company's profit increases while risk is slightly reduced.

We now analyze the two hedging strategies where volatility risk is also tackled. The DV-BS hedge further reduces risk significantly compared to the two delta-only hedges, even though the hedge is set up under a model with deterministic volatility. Risk is further reduced by almost 50% and the results are even better than a D-BS hedge under the Black-Scholes model, which is not surprising, as the hedge instrument used for vega hedging (a straddle) also introduces a partial hedge against the gamma of the insurer's liability. If the vega hedge is set up within the Heston model, results improve even further. Market risk within our model now is below 2% of the initial single premium paid by the client and thus, e.g. below solvency capital requirements for traditional with-profits business within the European Union.

We would like to close this section with some more comments about vega hedging: First, we would like to stress that - since on the one side there are different types of volatility (e.g. actual vs. implied) that can change with respect to their level, skew, slope, convexity, etc., and on the other side there is a great variety of hedging instruments in the market that exhibit some kind of sensitivity to changes in volatility - a unique vega hedging strategy does not exist. Second, we would like to point out the shortcomings of a somewhat intuitive and straightforward (but unfortunately ill-advised) way of setting up a vega hedge portfolio within the Black-Scholes model: One could simply calculate the 1st order derivate of the option value with respect to the unmodified volatility parameter and use this number to set up a vega hedge portfolio. This would, however, result in a rather bad hedge performance due to the following reasons: A change in current asset volatility under the Heston model would mean a change in short term volatility and a much smaller change in long term volatility. Since volatility in the Black-Scholes model is assumed to be constant over time, volatility risk would be significantly overestimated. Thus, the resulting hedge portfolio would lead to increasing risk, foiling the very idea of hedging. To illustrate this effect, we calculated above risk measures for this unmodified vega hedge using the Heston model for data generation:

	Product			
	I	II	III	IV
$E_P\left[e^{-rT}\Pi_T\right]$	1.38	2.05	2.08	1.12
$CTE_{1-\alpha}(\chi)$	7.8	16.36	18.01	9.66
$CTE_{1-\alpha}(\Pi_T)$	6.48	13.73	15.23	8.45

Table 10: Results of the unmodified Vega hedge

6 Summary

In the present paper, we have analyzed different types of guaranteed withdrawal benefits for life, the latest guarantee feature within Variable Annuities, both, from a client's perspective and from an insurer's perspective. We found that different ratchet and bonus features can lead to significantly different cash-flows to the insured. Similarly, the probability that guaranteed payments become payable and their amount varies significantly for the different products, even if they all come at the same fair value.

The development of the Greeks – i.e. the sensitivities of the value of the guarantees with respect to certain market parameters – over time is also significantly different, depending on the selected product features. Thus both, the constitution of a hedging portfolio (following a certain hedging strategy) and the insurer's risk after hedging differ significantly for the different products.

We analyzed different hedging strategies (no hedging, delta only, delta and vega) and analyzed the distribution of the insurer's cumulative profit/loss and certain risk measures thereof. We found that the insurer's risk can be reduced significantly by suitable hedging strategies.

We then quantified the model risk by using different capital market models for data generation and calculation of the hedge positions. This is an indication for the model risk, i.e. the risk an insurer takes by assuming a certain model whilst in the real world, capital markets display different properties. In this paper, we focussed on the risk an insurer takes by assuming constant volatilities in the hedging model whilst in the real world volatilities are stochastic and showed that this risk can be substantial.

We were also able to show that whereas a hedging strategy based on modified vega can lead to a significant reduction of volatility risk even if a model with deterministic volatilities is being used as a hedging model. On the other hand, a somewhat more intuitive and straightforward attempt to hedge volatilities based on an unmodified vega can lead to results inferior to the case with no vega hedging at all.

Our results – in particular with respect to model risk – should be of interest to both, insurers and regulators. The latter appear to systematically neglect model risk if analyzing hedge efficiency in the same model that it used by the insurer as a hedging model.

Further research could aim at extending our findings to other products or other capital market models (e.g. with equity jumps, stochastic interest rates and/or other approaches to the stochasticity of actual and implied equity volatility). Also, a systematic analysis of parameter risk and robustness of the hedging strategies against policyholder behaviour appears worthwhile.

Finally, it would be interesting to analyze how the insurer can reduce risk by product design, e.g. by offering funds as an underlying that are managed to meet some volatility target or by reserving the right to switch the insured's assets to less risky funds (e.g. bond or money market funds) if market volatilities increase. Such product features can already be observed in some insurance markets.

7 References

[1] Bauer, D., Kling A., and Ruß, J. (2008), *A Universal Pricing Framework for Guaranteed Minimum Benefits in Variable Annuities*. ASTIN Bulletin, Volume: 38 (2), 621 - 651, November 2008.

[2] Bingham, N.H., Kiesel, R. (2004), *Risk-Neutral Valuation: Pricing and Hedging of Financial Derivatives*, Springer Verlag, Berlin.

[3] Coleman, T.F., Kim, Y., Li, Y., and Patron, M. (2005), *Hedging Guarantees in Variable Annuities Under Both Equity and Interest Rate Risks,* Cornell University, New York.

[4] Coleman, T.F., Kim, Y., Li, Y., and Patron, M. (2007), *Robustly Hedging Variable Annuities With Guarantees Under Jump and Volatility Risks,* The Journal of Risk and Insurance, 2007, Vol. 74, No.2, 347-376

[5] Cox, J.C., Ingersoll, J.E., & Ross, S.A. (1985), *A Theory of the Term Structure of Interest Rates*. Econometrica 53, 385–407.

[6] Eraker, B. (2004), *Do stock prices and volatility jump? Reconciling evidence from spot and option prices*, Journal of Finance 59, 1367–1403.

[7] Ewald, C.-O., Poulsen, R., Schenk-Hoppe, K.R. (2007), *Risk Minimization in Stochastic Volatility Models: Model Risk and Empirical Performance*, Swiss Finance Institute Research Paper Series N°07 – 10.

[8] Heston, S. (1993), *A closed form solution for options with stochastic volatility with applications to bond and currency options*, Review of Financial Studies 6, no. 2, 327–344.

[9] Holz, D., Kling, A., and Ruß, J. (2007), *GMWB For Life – An Analysis of Lifelong Withdrawal Guarantees*. Working Paper, Ulm University, 2007.

[10] Hull, J.C. (2008), *Options, Futures and Other Derivatives.* Prentice Hall

[11] Kahl, C. & Jäckel, P. (2006), Not-so-comp*lex logarithms in the Heston model*.

[12] Mikhailov, S. & Nögel, U. (2003), *Heston's Stochastic Volatility Model Implementation,Calibration and Some Extensions*, Wilmott Magazine (2003)

[13] Milevsky, M. and Posner, S.E. (2001), *The Titanic Option: Valuation of the Guaranteed Minimum Death Benefit in Variable Annuities and Mutual Funds*. The Journal of Risk and Insurance, Vol. 68, No. 1, 91 – 126, 2001.

[14] Milevsky, M. and Salisbury, T.S. (2006), *Financial Valuation of Guaranteed Minimum Withdrawal Benefits*. Insurance: Mathematics and Economics: 38, 21 – 38, 2006.

[15] Poulsen, R., Schenk-Hoppe, K.R. & Ewald, C.-O. (2007*), Risk Minimization in Stochastic Volatility Models: Model Risk and Empirical Performance*

[16] Taleb, N. (1997), *Dynamic Hedging: Managing Vanilla and Exotic Options*, Wiley

[17] Wong, B., Heyde, C.C. (2006), *On changes of measure in stochastic volatility models*,

Journal of Applied Mathematics and Stochastic Analysis.