STOCHASTIC MORTALITY: THE IMPACT ON TARGET CAPITAL

BY

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ABSTRACT

In this paper, we take the point of view of an insurer dealing with life annuities, which aims at building up a (partial) internal model in order to quantify the impact of mortality risks, namely process and longevity risk, in view of taking appropriate risk management actions. We assume that a life table, providing a best-estimate assessment of annuitants' future mortality is available to the insurer; conversely, the insurer has no access to data sets and the methodology underlying the construction of the life table. Nonetheless, the insurer is aware that, in the presence of mortality risks, a stochastic approach is required. The (projected) life table, which provides a deterministic description of future mortality, should then be used as the basic input of a stochastic model.

The model we propose focuses on the annual number of deaths in a given cohort, which we represent allowing for a random mortality rate. To this purpose, we adopt the widely used Poisson model, first assuming a Gamma-distributed random parameter, and second introducing time-dependence in the parameter itself. Further, we define a Bayesian-inferential procedure for updating the parameters to experience in some situations. The setting we define does not demand advanced analytical tools, while allowing for process and longevity risk in a rigorous way.

The model is then implemented for capital allocation purposes. We investigate the amount of the required capital for a given life annuity portfolio, based on solvency targets which could be adopted within internal models. The outcomes of such an investigation are compared with the capital required according to some standard rules, in particular those proposed within the Solvency 2 project.

KEYWORDS

Life annuities, Random fluctuations, Systematic deviations, Process risk, Longevity risk, Solvency, Insurance risk management, Internal models.

1. INTRODUCTION

Advantages provided by "large" portfolio sizes in respect of the risk of random fluctuations in mortality justify, to some extent, the traditional deterministic

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approach to mortality in life insurance and pension calculations. However, the uncertainty in the level of future mortality, mainly due to unanticipated trends at adult and old ages, originates risks which may heavily affect portfolio results. Special attention is required when dealing with long-term living benefits, for example life annuities.

Examples of a stochastic approach in respect of the uncertain mortality trend may be found in the recent literature, with applications to life annuities and longevity-linked securities. Some proposals design a simplified but practicable setting. For example, CMI (2002, 2006) recommend to adopt scenarios which represent alternatives to the best-estimate one. Olivieri (2001) considers a similar structure, suggesting also to weight the alternative assumptions, so to come to unconditional valuations of portfolio results. Olivieri and Pitacco (2002) then define a Bayesian-inferential procedure for updating the weighting distribution. Due to the high number of items relying on experts' opinion (namely number and types of alternative scenarios, initial level of their weight), such an approach can hardly lead to a standard valuation setting; indeed, it has been proposed just for internal valuations, such as capital allocation (see, for example, Olivieri and Pitacco (2003, 2002)).

Moving from assumed similarities between the force of mortality and interest rates or between mortality and default occurrence, some authors have tested stochastic models developed originally for financial purposes. See, for example, Biffis (2005), Biffis and Millossovich (2006), Dahl (2004), Dahl and Møller (2006), Cairns et al. (2006). Proposals in this respect mainly aim at solving pricing problems for life annuities and longevity-linked securities. Some underlying assumptions still need to be validated against experience and a standard valuation code has not yet been defined. Further, we note that the relevant implementation would require an advanced technical knowledge of stochastic processes, which may be currently lacking in practical work.

In this paper, we take the point of view of an insurer dealing with life annuities, which aims at building up a (partial) internal model in order to quantify the impact of mortality risks, longevity risk in particular, in view of taking appropriate risk management actions (for example, capital allocation). We assume that a life table, providing a best-estimate assessment of annuitants' future mortality is available to the insurer; conversely, the insurer has no access to data sets and the methodology underlying the construction of the life table. Nonetheless, the insurer is aware that, in the presence of mortality risks, a stochastic approach is required. The (projected) life table, which provides a deterministic description of future mortality, should then be used as the basic input of a stochastic model. The impact of the process risk (namely, random fluctuations around the expected mortality given by the life table) can be quantified by using the traditional actuarial tool-kit (that is, binomial models, Poisson models, and so on). Conversely, quantifying the impact of the uncertainty risk (leading to possible short or long-term systematic deviations from the expected mortality) constitutes a crucial step in the risk assessment process, mainly because of the lack of information about alternative mortality scenarios.

To achieve a reasonable assessment of the impact of the various components of the mortality risk in such a situation, we focus on the annual number of deaths in a given cohort, which we model allowing for a random mortality rate. To this purpose, we adopt a doubly stochastic process which, in general terms, consists in describing a two-step randomization procedure (see, for example, Brémaud (1981)). In our application, the first step consists in choosing (for each age and time) the mortality rate, whereas the second step generates the number of deaths. In particular, we adopt the widely used Poisson model, first assuming a Gamma-distributed random parameter, and second introducing time-dependence in the parameter itself. Thus, we use a non-homogeneous doubly stochastic (or "conditional") Poisson process.

Further, we define a Bayesian-inferential procedure for updating the parameters to experience in some situations. Then we investigate the amount of the required capital for a given life annuity portfolio, based on solvency targets which could be adopted within internal models. The outcomes of such an investigation are compared with the capital required according to some standard rules, in particular those proposed within the Solvency 2 project.

The paper is organized as follows. In Section 2 we define the mortality model. In Section 3 we revise the standard formula proposed in Solvency 2 for the longevity risk and we describe internal models to be implemented adopting the mortality model previously defined; some comparisons are discussed through numerical investigations. Then, in Section 4 we conclude with some final remarks.

2. The mortality model

2.1. Modelling the annual number of deaths: basic structure

We refer to a portfolio of immediate life annuities, with fixed benefits. Risks other than mortality are disregarded and the same annual amount is considered for each annuitant. Thus, the only random variable affecting portfolio results is the number of survivors, or equivalently the number of deaths.

We let t_0 be the starting time of the portfolio and x_0 the age required at entry. We further let t denote the portfolio duration since time t_0, so that $t = 0, 1, 2, \ldots$. The random number of deaths in year $(t-1, t)$ is denoted as D_t. In more detail, if more than one cohort is present in $(t-1, t)$ and ages range between x_0 and ω (the latter denoting the maximum attainable age), then $D_t = \sum_{x=x_0}^{\omega} D_{x,t}$, with $D_{x,t}$ expressing the number of deaths for those aged x at the beginning of the year (i.e. at time $t-1$). In the following, reference is mainly to one cohort only. In this case, D_t is simply given by $D_{x,t}$, with x the only current age of annuitants. Similarly, we denote by N_t the number of annuitants at time t, understanding that when more than one cohort is present, then $N_t = \sum_{x=x_0}^{\omega} N_{x,t}$, with $N_{x,t}$ representing the number of annuitants aged x at time t. Possible (and observed) outcomes of the random variables D_t, $D_{x,t}$, N_t and $N_{x,t}$ will be denoted with small letters, i.e. d_t, $d_{x,t}$, n_t and $n_{x,t}$, respectively.

The random annual number of deaths is affected by random fluctuations and systematic deviations. Random fluctuations represent a traditional risk in life insurance (and, more generally, in insurance), known as process risk or simply as insurance risk, which can be offset through pooling. Indeed, conditional on a given best-estimate mortality rate, say $q^*_{x,t}$ for those aged x at the beginning of year t, whenever insured risks are independent and similar one to the other and the size of the portfolio is large enough, then the ratio $\frac{D_{x,t}}{n_{x,t-1}}$ will be close to $q^*_{x,t}$ with high probability. Once the number of annuitants at time $t-1$ is known, the probability distribution of $D_{x,t}$ conditional on $q^*_{x,t}$ allows us to assess the impact of random fluctuations. In particular, under our assumptions (homogeneous and independent lives), the (conditional) probability distribution of $D_{x,t}$ is Binomial, so that

$$[D_{x,t} \mid q^*_{x,t};\, n_{x,t-1}] \sim \mathrm{Bin}(n_{x,t-1}, q^*_{x,t}) \tag{1}$$

Provided that $n_{x,t-1}$ is large enough, $q^*_{x,t}$ is low and the product $n_{x,t-1}q^*_{x,t}$ is stable, we could accept the Poisson approximation, namely

$$[D_{x,t} \mid q^*_{x,t};\, n_{x,t-1}] \sim \mathrm{Poi}(n_{x,t-1}, q^*_{x,t}) \tag{2}$$

widely adopted in the actuarial literature (see, for example, Gerber (1995), Panjer and Willmot (1992)). We note that the approximation could be unsatisfactory at very high ages, when $n_{x,t-1}$ is low and $q^*_{x,t}$ is high. On the other hand, the case of more than one cohort can be easily developed under the Poisson setting. Take (2) for any age x which can be in-force at time $t-1$, and assume that, conditional on the life table $\{q^*_{x,t}\}$, the numbers of deaths $D_{x,t}$ are independent, i.e. the individual lifetimes are independent not only between each cohort (as required by (2)), but also among cohorts. Then

$$[D_{x,t} \mid \{q^*_{x,t}\};\{n_{x,t-1}\}] \sim \mathrm{Poi}\left(\sum_{x=x_0}^{\omega} n_{x,t-1}\, q^*_{x,t}\right) \tag{3}$$

Clearly, if some age x is not in-force at time $t-1$, in (3) the relevant number of survivors $n_{x,t-1}$ is set to 0. We note that, thanks to considering multiple cohorts, the Poisson assumption may be satisfactory also when the oldest ages are included, as the weight of this age-class in respect of the overall population should be low. In the following, one cohort is mainly focused, and assumption (2) is accepted henceforth.

Whenever the probability that $\frac{D_{x,t}}{n_{x,t-1}}$ is not close to $q^*_{x,t}$ keeps high also in large portfolios, a risk of systematic deviations, also referred to as deviations in aggregate mortality, is present. The representation of deviations in aggregate mortality requires the adoption of a stochastic mortality rate.

Deviations in aggregate mortality can be temporary or permanent. In the former case, the deviation typically consists of an upward shock in mortality, due to critical living conditions, such as influenza epidemics, severe climatic conditions (e.g. hot summers), natural disasters and so on; the relevant risk is usually named the risk of catastrophe mortality. It is reasonable to assume that the shock in a year is independent of previous ones; conversely, the impact of each shock could be age-dependent. Permanent deviations occur when the underlying trend, for the whole population or for some cohorts, is other than the one described by the best-estimate mortality rates. In particular, the term longevity risk is used to denote a situation of experienced mortality rates permanently below the best-estimate rate, $q^*_{x,t}$. Reasonably, deviations are positively correlated in time. In the recent literature (see, for example, Willets (2004)), deviations specific of a cohort have been referred to as the "cohort effect", whilst those common to the whole population as the "period effect". In order to detect a cohort effect, cohort mortality data must be available, and this occurs just in some countries (for example, in UK).

In order to represent systematic deviations, we allow for a random mortality rate, $Q_{x,t}$. We make the following assumption

$$Q_{x,t} = q^*_{x,t}\, Z_{x,t} \tag{4}$$

where $Z_{x,t}$ is a (positive) random coefficient, and such that

$$0 \leq Q_{x,t} \leq 1 \tag{5}$$

Note that $Z_{x,t}$ expresses a deviation of the mortality rate in respect of the best-estimate one. As such, it represents a deviation in aggregate mortality (above or below $q^*_{x,t}$). The multiplicative model (4) has already been used in actuarial mathematics, to represent heterogeneity in the pool of lives, due to observable or unobservable risk factors. In the former case, the multiplicative coefficient is assumed to be deterministic, typically expressing extra-mortality for substandard lives (and hence with a value higher than 1); see Haberman and Olivieri (2008) and Olivieri (2006) for details and references. In the case of unobservable risk factors, model (4) has been introduced by Vaupel et al. (1979) with reference to the force of mortality (instead of the mortality rate), with the random coefficient representing the (unknown) frailty of one individual in respect of the standard mortality level (assumed to be deterministic). Again, see Olivieri (2006) for details and references. The difference between the frailty model by Vaupel et al. (1979) and the use we are making in this paper of assumption (4) stands, besides the underlying quantity (force of mortality vs mortality rate), on the fact that we are referring to a situation of dynamic mortality for a homogeneous group of lives.

Expressing a deviation in aggregate mortality, the coefficient $Z_{x,t}$ should account for both temporary and permanent deviations. To this purpose, several assumptions could be made about its behaviour. In particular, the $Z_{x,t}$'s could be assumed to be independent in time, with either fixed or age- and time-dependent parameters; or they could be assumed to be correlated in time, and hence with age- and time-dependent parameters.

We make the following assumption about the probability distribution of $Z_{x,t}$:

$$Z_{x,t} \sim \text{Gamma}(\alpha_{x,t}, \beta_{x,t}) \tag{6}$$

due to the versatility of the Gamma distribution. A similar assumption has been adopted by Olivier and Jeffrey (2004) and Smith (2005), working in the framework of a forward-rate approach to stochastic mortality modelling.

Thanks to (4), assuming (6) it follows that

$$Q_{x,t} \sim \text{Gamma}\left(\alpha_{x,t}, \frac{\beta_{x,t}}{q^*_{x,t}}\right) \tag{7}$$

Further, assuming (2), we can easily show (see, for example, Panjer and Willmot (1992)) that the unconditional distribution (in respect of $Q_{x,t}$) of the number of deaths in a year is Negative Binomial, namely

$$[D_{x,t} | n_{x,t-1}] \sim \text{NBin}\left(\alpha_{x,t}, \frac{\theta_{x,t}}{\theta_{x,t}+1}\right) \tag{8}$$

where $\theta_{x,t} = \dfrac{\beta_{x,t}}{n_{x,t-1}\, q^*_{x,t}}$.

Note that (8) introduces time-dependence in the parameters of the Poisson-Gamma model, well-known in non-life insurance for describing the number of claims for a heterogeneous pool of risks in a static setting (see, for example, Bühlmann (1970)). A similar generalization is discussed in Pitacco (1992), with reference to the number of claims in sickness insurance; in that case, similarly to the traditional framework, a heterogenous pool of risks is considered, but with a claim frequency depending on age (conversely, no stochastic dynamics is addressed).

Before moving on with the discussion, we comment on the approximations implied by (8). Such approximations are originated by the Poisson assumption (2) and by the fact that (6) does not necessarily fulfil constraint (5), especially at the oldest ages (when $q^*_{x,t}$ is high). We think that the disadvantages of such approximations are more than balanced by the advantages of model (8). First, model (8) extends some findings which are well-known both in the actuarial literature and practice, and is thus quite practicable for implementation purposes. Second, a Bayesian-inferential procedure can be naturally defined (see Section 2.2), which allows to account for correlations among the $Z_{x,t}$'s (without having to define explicitly such correlations), as well as to update parameters to experience (which is useful as long as data on the stochastic evolution of mortality are scanty). In this regard, we note that in Bayesian inference the Gamma distribution is the conjugate prior of the Poisson distribution, so that posterior distributions can be easily derived, as we will see in Section 2.2.

It is interesting to note that whilst according to (2), or also to (1), we have

$$\mathbb{E}[D_{x,t} \mid q^*_{x,t}; n_{x,t-1}] = n_{x,t-1}\, q^*_{x,t} \tag{9}$$

according to (8) we have

$$\mathbb{E}[D_{x,t} \mid n_{x,t-1}] = \frac{\alpha_{x,t}}{\theta_{x,t}} = \frac{\alpha_{x,t}}{\beta_{x,t}}\, n_{x,t-1}\, q^*_{x,t} \tag{10}$$

So, when accounting for the uncertainty of the mortality rate, it turns out $\mathbb{E}[D_{x,t} \mid n_{x,t-1}] \gtreqless \mathbb{E}[D_{x,t} \mid q^*_{x,t}; n_{x,t-1}]$, depending on the ratio $\frac{\alpha_{x,t}}{\beta_{x,t}}$.

As far as the choice of the parameters of (6), and hence of (8), is concerned, statistical tests will dictate the nature of the $Z_{x,t}$'s in practical applications, namely whether they are independent or correlated in time. This latter case is discussed in detail in Section 2.2, where we further assume that the mortality experience from the portfolio is reliable as an evidence of the trend of the cohort.

Before moving to that case, we note that a different modelling choice for the annual number of deaths, alternative to the Poisson-Gamma model, is as follows. First, assume for the annual number of deaths the (natural) Binomial distribution (see (1)), however with a random mortality rate. Second, assume a Beta distribution for the random mortality rate. These assumptions lead to the Pólya-Eggenberger model (also called Binomial-Beta, or Negative Hypergeometric; see, for example, Panjer and Willmot (1992)). This model has been adopted by Marocco and Pitacco (1998) to represent the annual number of deaths in a portfolio of life annuities.

The Poisson-Gamma choice, however, has several advantages. In particular, the Poisson assumption can be extended to more general cases (for example, more than one cohort or, via compounding, allowing for heterogeneity in the amounts of benefits) and offers a better manageability for computation purposes. We just sketch the extension to more than one cohort.

For any age x, we let

$$Q_{x,t} = q^*_{x,t}\, Z_t$$

where Z_t is a (positive) random coefficient, expressing a systematic deviation in mortality. Note that, as expressed by notation, the deviation in aggregate mortality is here defined univocally over the several cohorts. Assumption (3) can be extended as follows. Conditional on $Z_t = z$, we have

$$[D_t \mid z; \{n_{x,t-1}\}] \sim \mathrm{Poi}\left(z \sum_{x=x_0}^{\omega} n_{x,t-1}\, q^*_{x,t} \right)$$

If we accept

$$Z_t \sim \text{Gamma}(\alpha_t, \beta_t) \tag{11}$$

then we find

$$[D_t | \{n_{x,t-1}\}] \sim \text{NBin}\left(\alpha_t, \frac{\theta_t}{\theta_t + 1}\right)$$

where $\theta_t = \dfrac{\beta_t}{\sum_{x=x_0}^{\omega} n_{x,t-1}\, q^*_{x,t}}$.

Clearly, under (11) not necessarily constraint (5) is fulfilled (again: especially at the oldest ages). However, what is now modeled is the number of deaths for the population, where an offsetting effect emerges among the young and the old ages. Clearly, appropriate statistical tests should be used to understand whether the relevant approximation is negligible.

2.2. Updating the parameters to experience

We now refer to one cohort only (which enters the portfolio at time t_0), and we assume that the $Z_{x,t}$'s are correlated in time and further that the mortality experienced in the portfolio is informative in respect of the actual trend of the cohort. It is then possible to design an inferential procedure for updating the parameters of the probability distribution of $Z_{x,t}$ to experience.

The inferential procedure we are adopting is the classical one, relying on the property of conjugacy (see, for example, Carlin and Louis (2000)). To build-up more general Bayesian inferential procedures we could resort, in particular, to Markov Chains Monte Carlo (MCMC) methods, which can be used to approximate posterior distributions. See, for example, Carlin and Louis (2000) and Robert and Casella (2004)).

At time 0, when no experience is available, we adopt (6) with a value for the parameters which is the same for all times t, $t = 0, 1, \ldots$ and ages x, $x = x_0 + t$. We denote this initial level of parameters respectively as $\bar{\alpha}$ and $\bar{\beta}$. Thus

$$Z_{x,t} \sim \text{Gamma}(\bar{\alpha}, \bar{\beta}) \tag{12}$$

Then it follows

$$[D_{x_0,1} | n_{x_0,0}] \sim \text{NBin}\left(\bar{\alpha}, \frac{\theta_{x_0,1}}{\theta_{x_0,1} + 1}\right)$$

where $\theta_{x_0,1} = \dfrac{\bar{\beta}}{n_{x_0,0}\, q^*_{x_0,1}}$ (compare with (8)).

At time 1, the number of deaths observed in year (0,1) is available. Let $d_{x_0,1}$ be such number. Then, $n_{x_0+1,1} = n_{x_0,0} - d_{x_0,1}$. More important, it is possible to calculate the posterior probability distribution of $Q_{x_0,1}$ conditional on the information $D_{x_0,1} = d_{x_0,1}$. We find

$$[Q_{x_0,1} | d_{x_0,1}] \sim \text{Gamma}\left(\bar{\alpha} + d_{x_0,1}, \frac{\bar{\beta}}{q^*_{x_0,1}} + n_{x_0,0}\right)$$

Thanks to (4), we have also

$$[Z_{x,t} | d_{x_0,1}] \sim \text{Gamma}\left(\bar{\alpha} + d_{x_0,1}, \bar{\beta} + n_{x_0,0}\, q^*_{x_0,1}\right) \tag{13}$$

i.e., the posterior pdf of $Z_{x,t}$ is still Gamma-distributed.

Going back to (13), we note that, thanks to experience, parameters are updated. In particular, when comparing (13) to (12), it turns out that the first parameter of the pdf of $Z_{x,t}$ is increased by the observed number of deaths ($d_{x_0,1}$), whilst the second by the expected number of deaths for that year ($n_{x_0,0}\, q^*_{x_0,1}$). Thus, whilst the prior expected value of $Z_{x,t}$ is

$$\mathbb{E}[Z_{x,t}] = \frac{\bar{\alpha}}{\bar{\beta}}$$

the posterior expected value at time 1 is

$$\mathbb{E}[Z_{x,t} | d_{x_0,1}] = \frac{\bar{\alpha} + d_{x_0,1}}{\bar{\beta} + n_{x_0,0}\, q^*_{x_0,1}}$$

so that $\mathbb{E}[Z_{x,t} | d_{x_0,1}] \gtreqless \mathbb{E}[Z_{x,t}]$ depending on the comparison between $d_{x_0,1}$ and the relevant expected value $n_{x_0,0}\, q^*_{x_0,1}$.

Let us now move to the valuations performed at time 1 involving the next year. From (13), it follows

$$[Q_{x_0+1,2} | d_{x_0,1}] \sim \text{Gamma}\left(\bar{\alpha} + d_{x_0,1}, \frac{\bar{\beta} + n_{x_0,0}\, q^*_{x_0,1}}{q^*_{x_0+1,2}}\right)$$

so that

$$[D_{x_0+1,2} | n_{x_0,0}, d_{x_0,1}] \sim \text{NBin}\left(\bar{\alpha} + d_{x_0,1}, \frac{\theta_{x_0+1,2}}{\theta_{x_0+1,2} + 1}\right)$$

with $\theta_{x_0+1,2} = \dfrac{\bar{\beta} + n_{x_0,0}\, q^*_{x_0,1}}{n_{x_0+1,1}\, q^*_{x_0+1,2}}$.

Following similar steps, we can generalize as follows. At time $t-1$, having observed the annual numbers of deaths $D_{x_0,1} = d_{x_0,1}, D_{x_0+1,2} = d_{x_0+1,2}, \ldots, D_{x_0+t-2,t-1} = d_{x_0+t-2,t-1}$ and therefore the number of survivors $n_{x_0+h,h} = n_{x_0+h-1,h-1} - d_{x_0+h-1,h}$ at time h, $h = 1, 2, \ldots, t-1$, it turns out that

$$[D_{x_0+t-1,t} \mid n_{x_0,0}, d_{x_0,1}, d_{x_0+1,2}, \ldots, d_{x_0+t-2,t-1}] \\ \sim \text{NBin}\left(\alpha_{x_0+t-1,t}, \frac{\theta_{x_0+t-1,t}}{\theta_{x_0+t-1,t}+1}\right) \tag{14}$$

where:

$$\alpha_{x_0+t-1,t} = \bar{\alpha} + \sum_{h=1}^{t-1} d_{x_0+h-1,h} \text{ and } \theta_{x_0+t-1,t} = \frac{\bar{\beta} + \sum_{h=1}^{t-1} n_{x_0+h-1,h-1}\, q^*_{x_0+h-1,h}}{n_{x_0+t-1,t-1}\, q^*_{x_0+t-1,t}}$$

(the summations in the expressions of $\alpha_{x_0+t-1,t}$ and $\theta_{x_0+t-1,t}$ must be meant equal to 0 when $t = 1$).

It is interesting to comment on the expected number of deaths in each year. At time $t-1$, $t = 1, 2, \ldots$, we have

$$\mathbb{E}\left[D_{x_0+t-1,t} \middle| n_{x_0,0}, d_{x_0,1}, d_{x_0+1,2}, \ldots, d_{x_0+t-2,t-1}\right] \\ = \frac{\bar{\alpha} + \sum_{h=1}^{t-1} d_{x_0+h-1,h}}{\bar{\beta} + \sum_{h=1}^{t-1} n_{x_0+h-1,h-1}\, q^*_{x_0+h-1,h}}\, n_{x_0+t-1,t-1}\, q^*_{x_0+t-1,t}$$

We note that, similarly to Section 2.1 (see (9) and (10)), the unconditional expected number of deaths in a year is given by the expected value conditional on the best-estimate mortality rate, namely $n_{x_0+t-1,t-1}\, q^*_{x_0+t-1,t}$, adjusted by a coefficient. Such a coefficient is now updated to the observed number of deaths in respect of those expected at the beginning of each year. Thus, if experience is consistent with what expected, such coefficient will remain stable in time; conversely, if experience is worse than expected (that is, the number of deaths is lower than anticipated), then such coefficient will decrease in time. This may have an impact on the assessment of the capital to allocate to face risks, as we are going to investigate in Section 3.

We mention that an inferential procedure similar to what described above was developed by Pitacco (1992) for sickness insurance and then applied for pricing purposes, within an experience-rating scheme. However, in that case only age-dependence is allowed for, whilst a stochastic dynamics of the underlying claim frequency is not considered.

The inferential procedure can easily be extended also to the case of multiple cohorts, taking advantage of a larger data set on which recording experience. For the sake of brevity, we do not give details in this respect, as the following implementation is developed for one cohort only.

3. Solvency investigations

3.1. A regulatory requirement: the Solvency 2 proposal

Insurers and supervisors have become aware of the impact of longevity risk just in recent years. In respect of mortality, most solvency standards (see

Sandström (2006) for a comprehensive review of regulatory solvency systems) allow for the risk of random fluctuations and for the catastrophe mortality risk. Conversely, just few examples are available in respect of longevity risk. A request of capital for longevity risk is present in the developing Solvency 2 system. Due to the importance of such a system, we discuss the relevant proposed requirement. We focus on mortality risks only, whilst other risk sources are disregarded, and we refer just to the case of a life annuity portfolio. In particular, we consider a cohort of immediate conventional life annuities, with fixed benefits (that is, without any participation in financial or other profits).

In defining the new regulatory capital requirement, the Solvency 2 project sets the parameters of the relevant formulae so that the amount of the required capital is consistent with a Value-at-Risk assessment at a 99.5% confidence level with a one year time-horizon. Details may be found in CEIOPS (2007, 2008). Adoption of internal models is permitted, provided that they are validated by the supervisor.

In the proposed standard formula, only systematic deviations are considered with reference to mortality. However, the terminology used to name risks is slightly different from what is common in the literature, and unfortunately this may originate some misunderstanding. In Solvency 2, the possible permanent situation of extra-mortality for insurance covers with a positive sum-at-risk (i.e. life insurance covers) is named mortality risk. In contrast, longevity risk is the term used to define a permanent situation of under-mortality for insurance products with a negative sum at risk (e.g. life annuities). Catastrophe mortality refers (as usual) to a sudden (positive) jump in mortality rates, and the relevant risk is considered in relation to those insurance products bearing a positive sum-at-risk.

Given that we are considering life annuities, and disregarding risks other than those linked to the lifetime as well, we address only the charge for longevity risk (with the meaning just specified).

The capital charge for longevity risk at time t, $\text{Life}_{\text{long},t}$, is defined as follows

$$\text{Life}_{\text{long},t} = \Delta\text{NAV} \mid \text{longevity shock} \tag{15}$$

where, with regard to the in-force portfolio, ΔNAV is the change in the Net Asset Value, namely the value of assets minus liabilities, in face of a longevity shock defined as a (permanent) 25% decrease (in respect of the best-estimate assumption) in the mortality rate at each age. An explicit reduction of the capital charge is admitted in case the insurer has the possibility to reduce profit participation in the adverse scenario; we will not consider such possibility, as we are referring to fixed benefits. We note that in defining $\text{Life}_{\text{long},t}$ a deterministic setting is considered in respect of systematic deviations, as only one level for the possible shock is considered.

In order to avoid a double charge for risks, the value of liabilities in (15) is not risk-adjusted. If we denote by A_t the amount of portfolio assets at time t and by $V_t^{(\Pi)[BE]}$ the expected value of liabilities of a given portfolio Π, based on best-estimate assumptions about the future scenario, in (15) we can define the NAV as follows

$$\mathrm{NAV}_t = A_t - V_t^{(\Pi)[BE]}$$

Since we are addressing immediate life annuities, ΔNAV at time t and then the capital charge for longevity risk reduce to

$$\Delta\mathrm{NAV} \mid \text{longevity shock} = \mathrm{Life}_{\mathrm{long},t} = V_t^{(\Pi)[-25\%]} - V_t^{(\Pi)[BE]} \tag{16}$$

where $V_t^{(\Pi)[-25\%]}$ is the expected value of future payments, calculated with a life table whose mortality rates are 25% lower than those in the best-estimate table. Indeed, current assets are not affected by the shock scenario, whilst the value of liabilities must be updated.

The (total) portfolio reserve, which we denote by $V_t^{(\Pi)}$, consists of the best-estimate assessment of liabilities, $V_t^{(\Pi)[BE]}$, and a risk margin facing adverse future scenarios, RM_t; thus

$$V_t^{(\Pi)} = V_t^{(\Pi)[BE]} + RM_t$$

In the Solvency 2 framework, the risk margin is calculated according to a Cost-of-Capital (CoC) logic. More specifically, the risk margin at time t, RM_t, is the amount of money rewarding the capital to be allocated to the business of the insurance company in the current and in future years, until exhaustion. The CoC factor is set to 6% above the risk-free interest rate, r_f, and the capital to be allocated is stated in terms of the so-called Solvency Capital Required (SCR), which under our assumptions reduces to the capital charge for longevity risk. The risk margin is then calculated as follows

$$RM_t = \sum_{h=0}^{m} 0.06 \cdot SCR_{t+h}(1+r_f)^{-h}$$

where m is the time to exhaustion of the in-force portfolio and $SCR_{t+h} = \mathrm{Life}_{\mathrm{long},t+h}$ in our case. The (future) quantities SCR_{t+h} are estimated through the current best-estimate assumptions. It is interesting to note that the size of the risk margin is consistent with the risks accounted for into the SCR, i.e. the longevity risk in our case. However, RM_t clearly provides assets for facing all the residual risks in respect of those met by the SCR.

In order to compare results obtained with the internal models described in Section 3.2 to those obtained with the Solvency 2 proposed standard formula, we define

$$M_t^{[\mathrm{Solv2}]} = \mathrm{Life}_{\mathrm{long},t} + RM_t$$

as the total amount of assets required, under Solvency 2, to face mortality risks.

3.2. Solvency rules within internal models

In this section we describe possible solvency rules to be adopted within an internal model. The advantage of an internal model in respect of a standard formula stands in the possible improved consistency with the value of the obligations of the insurer, which in particular may lead to a lower required amount of capital. Because of that, validation by the supervisor is required before the standard formula can be replaced by the internal model, and therefore the rationale of the internal solvency rule must be rigourously described and the underlying assumptions accurately disclosed.

As in Section 3.1, we refer to conventional immediate life annuities, with fixed benefits, and we disregard risks other than those linked to the lifetime. Further, expenses and related expense loadings are disregarded. We refer to a portfolio consisting of one cohort, with all the annuitants entitled to receive the annual benefit b. As in Section 2, we denote by x_0 the initial age and by t_0 the initial time; time t, $t = 0, 1, \ldots$, then denotes the duration of the portfolio.

Annual outflows for the portfolio are defined, for $t = 1, 2, \ldots, \omega - x_0$, as

$$B_t^{(\Pi)} = bN_{x_0+t,t}$$

We calculate the present value of future portfolio outflows as

$$Y_t^{(\Pi)} = \sum_{h=t+1}^{\omega - x_0} B_h^{(\Pi)}(1+i)^{-(h-t)} = \sum_{h=t+1}^{\omega - x_0} bN_{x_0+h,h}(1+i)^{-(h-t)}$$

where the discount rate i is set as the risk-free rate, assumed to be known. Given that we are assuming there is no investment risk (as we focus just on mortality issues), we take a basic setting for assets backing liabilities. Assets consist of risk-free bonds with appropriate maturities, so that the annual investment yield is i. Thus, given the amount A_z of portfolio assets at time z, $z = 0, 1, \ldots$, the random path of portfolio assets is simply defined by the recursion

$$A_t = A_{t-1}(1+i) - bN_{x_0+t,t} \tag{17}$$

for $t = z+1, z+2, \ldots$ We stress that A_t is random just because of the stochastic nature of the number of survivors. The quantity

$$M_t = A_t - V_t^{(\Pi)[BE]} \tag{18}$$

represents the assets available to meet risks, given the meaning of $V_t^{(\Pi)[BE]}$ (see Section 3.1). Thus M_t should be dedicated to meeting the needs of both the risk margin, to be then included into the portfolio reserve, and the required capital. We note that if the risk margin is funded by premiums paid by policyholders (as it should be), then M_t is partially built up with profit to the insurer. Indeed, if we replace (17) in (18), after adding/subtracting the quantity $V_{t-1}^{(\Pi)[BE]}(1+i)$ and rearranging, we may write

$$M_t = M_{t-1}(1+i) + U_t \tag{19}$$

where

$$U_t = V_{t-1}^{(\Pi)[BE]}(1+i) - bN_{x_0+t,t} - V_t^{(\Pi)[BE]} \tag{20}$$

represents the (industrial) profit originated by the portfolio in year $(t-1, t)$ (of course, a loss may be incurred, i.e. it may turns out $U_t < 0$).

Solvency rules suitable for internal models assume that the insurer can be considered solvent in respect of portfolio obligations if, with an assigned (high) probability, assets meet liabilities within a chosen time-horizon, according to a realistic probability distribution for the future scenario. Several details need to be specified for a practical implementation of such a definition.

– Assets can be considered in terms of those available to meet risks, M_t, which are required to be non-negative, or in terms of the total amount of assets, A_t, which is required to be non-lower than the present value of future payments, $Y_t^{(\Pi)}$.
– Generally speaking, the portfolio needs to be defined in a run-off or a going concern perspective. Standard solvency rules typically refer to the in-force portfolio, so that this choice must be adopted also in the internal model.
– The time-horizon, T, may range from a short-medium term (1 to 5 years, say), to the time to exhaustion of the in-force portfolio, that is $m = \omega + 1 - x_0 - t$ at time t ($t = 0, 1, \ldots$).
– When a time-horizon longer than one year is adopted, the overall trajectory of assets within the time-horizon can be addressed, or just their final value at time T.

We let z be the time at which solvency is ascertained ($z = 0, 1, \ldots$). According to the possible choices listed above, the following rules can be adopted for assessing the capital required at time t

$$[R1]:\ \mathbb{P}[(M_{z+1} \geq 0) \cap (M_{z+2} \geq 0) \cap \ldots \cap (M_{z+T} \geq 0) \mid n_{x_0+z,z}] = 1 - \varepsilon_1 \tag{21}$$

$$[R2]:\ \mathbb{P}[M_{z+T} \geq 0 \mid n_{x_0+z,z}] = 1 - \varepsilon_2 \tag{22}$$

$$[R3]:\ \mathbb{P}[(A_{z+1} \geq Y_{z+1}^{(\Pi)}) \cap (A_{z+2} \geq Y_{z+2}^{(\Pi)}) \cap \ldots \cap (A_{z+T} \geq Y_{z+T}^{(\Pi)}) \mid n_{x_0+z,z}] = 1 - \varepsilon_3 \tag{23}$$

where ε_i, $i = 1, 2, 3$, is the accepted default probability under the chosen rule. Clearly, in all the solvency models above, the relevant probability is assessed conditional on the information available at time z.

The difference between rules [R1] and [R2] is easy to understand. We first recall that the quantity M_t originates from the initial capital allocation at time z, whose amount must indeed be assessed through a solvency rule, and from annual profits (possibly negative) emerging year by year (see (19)), which gradually accumulate a surplus (possibly negative). Under rule [$R2$], only the total surplus in the time-interval $[z, z+T]$ is addressed, whilst under rule [$R1$] its yearly emergence is also checked. Compensation of current losses with future profits is admitted under rule [$R2$], whilst under rule [$R1$] a loss is recorded as soon as it occurs. Thus, the supervisor should give preference to adoption of [$R1$] instead of [$R2$].

With reference to rule [$R3$], through a little algebra it is possible to rewrite it as follows (see Appendix)

$$[R3]:\ \mathbb{P}[A_{\omega+1-x_0} \geq 0 \mid n_{x_0+z,z}] = 1 - \varepsilon_3 \tag{24}$$

If we further note that $A_{\omega+1-x_0} = M_{\omega+1-x_0}$ (as all the obligations have expired at time $\omega + 1 - x_0$), we can also write

$$[R3]:\ \mathbb{P}[M_{\omega+1-x_0} \geq 0 \mid n_{x_0+z,z}] = 1 - \varepsilon_3 \tag{25}$$

Comparing rules [$R1$] and [$R3$], the apparent difference stands in the way liabilities are represented, that is in terms of their best-estimate value in [$R1$] or in terms of their random present value in [$R3$]. A deeper comparison emerges when we consider (25). Assume that in (21) the time-horizon $T = \omega + 1 - x_0 - z$ is chosen. In both cases, the total amount of the surplus would then be considered. However, whilst under rule [$R1$] its emergence in time is accounted for, rule [$R3$] simply involves the total amount of the surplus, as it is originated by the initial amount of assets and the annual outflows (as is witnessed by (24)). Recursion (19) for M_t makes clear that [$R1$] first focus on annual profits/losses and then, through their accumulation, on the total surplus. According to a valuation terminology (see, for example, Sandström (2006) and Abbink and Saker (2002)), this follows a "Deferral and Matching" (D/M) approach. We point out that the annual profits U_t are affected by the reserve (see (20)), which may thus have a smoothing effect on the emergence of the total surplus. Rule [$R3$], as already noted, addresses directly the total surplus in terms of the accumulation of annual cashflows (rather than profits/losses), and as such it follows an "Asset and Liability" (A/L) approach. While the D/M approach is common in standard actuarial valuation models, the A/L logic is usual in a market-consistent setting. We further point out that rule [$R1$] allows for a preferred time-horizon, whilst under [$R3$] the maximum possible time-horizon is necessarily adopted; these are features, respectively, of the D/M and A/L approach.

We do not comment on the comparison between rules [$R2$] and [$R3$], as it should now be straightforward.

Solving (21) (or (22)) in respect of M_z, through stochastic simulation, one finds the amount of assets to meet risks required at time z according to rule [$R1$] (or [$R2$]); we will denote such amount by $M_z^{[R1]}(T)$ ($M_z^{[R2]}(T)$). Then, $A_z^{[R1]}(T) = V_z^{(\Pi)[BE]} + M_z^{[R1]}(T)$ ($A_z^{[R2]}(T) = V_z^{(\Pi)[BE]} + M_z^{[R2]}(T)$) is the total amount of assets required at time z. Solving (25), again through stochastic simulation, one finds the total amount of assets required at time z, denoted as $A_z^{[R3]}$; the required amount of assets to meet risks at time z is then: $M_z^{[R3]} = A_z^{[R3]} - V_z^{(\Pi)[BE]}$.

The implementation is based on nested simulations which, in fact, reflect the doubly stochastic nature of the processes involved. Clearly, simulations of real-world portfolios require a significant computation time, and hence feasible short-cut formulae should be found to approximate the relevant results. As is well-known, a short-cut formula expresses the required capital, e.g. $M_z^{[R1]}(T)$, as a function of some known quantities (e.g. the total amount of insured benefits, the portfolio reserve, etc.) and a set of parameters which should reflect the risk profile of the portfolio (or the insurance company). However, serious difficulties arise in choosing the quantities mentioned above, as well as in quantifying the parameters, in order to obtain a formula applicable to a wide range of real-world portfolios. So, we now only focus on some numerical results, obtained via simulation, aiming in particular at a comparison between the results themselves, which constitute the potential output of an internal model, and the requirements emerging from standard formulae. The numerical investigation is presented and discussed in Section 3.3.

3.3. Numerical investigations

In this section we compare the capital required under the standard formula described in Section 3.1 with the amount resulting from the solvency rules discussed in Section 3.2. To this purpose, some parameters in such rules will be set consistently with choices made in Solvency 2. Thus, considering that the standard formula is calibrated on a VaR assessment at a 99.5% confidence interval, we set the default probabilities $\varepsilon_i = 0.005$, $i = 1,2,3$. As far as the solvency time-horizon is concerned, we note that although Solvency 2 explicitly refers to one year for the VaR assessment, equation (16) makes clear that in the capital charge for longevity risk the time-to-exhaustion of the portfolio is implicitly involved; indeed, both $V_t^{(\Pi)[BE]}$ and $V_t^{(\Pi)[-25\%]}$ express the expected present value of all future payments to current annuitants, calculated under different assumptions about the mortality scenario. According to this remark, at any valuation time z ($z = 0,1,\dots$) we set $T = m = \omega + 1 - x_0 - z$. In this case, rule [$R2$] reduces to [$R3$], so that the former will not be considered.

Similarly to Section 3.2, we refer to a portfolio of immediate and conventional life annuities, and we disregard risks other than mortality. The entry age is $x_0 = 65$, the risk-free annual interest rate $i = 0.03$ and the annual amount $b = 1$. The best-estimate life table is taken from IPS55, which is a projected life table for Italian males, cohort 1955. The maximum attainable age in such a table is $\omega = 119$.

Rules [$R1$] and [$R3$] are implemented adopting the stochastic mortality model described in Section 2, assuming either independence or correlation in time among the $Z_{x,t}$'s. In the former case, in (6) and then in (7) we set $\alpha_{x,t} = 0.75\beta_{x,t}$, so that $\mathbb{E}[Q_{x,t}] = 0.75\,q^*_{x,t}$, consistently with the assumption of Solvency 2 for the shock scenario. Similarly, under the assumption of correlation, in (12) we set $\bar{\alpha} = 0.75\bar{\beta}$, so that at age x_0 (= 65) we have $\mathbb{E}[Q_{x_0,1}] = 0.75 q^*_{x_0,1}$.

In both cases, one parameter has still to be assigned, either $\alpha_{x,t}$ ($\bar{\alpha}$) or $\beta_{x,t}$ ($\bar{\beta}$); a calibration to the observed volatility of the mortality rate would be appropriate. However, available data on stochastic mortality are still too scanty to provide a fully reliable estimate. On the other hand, if parameters are updated to the mortality experience (as described in Section 2.2), then the importance of their initial value reduces in time. The adoption of the setting described in Section 2 within an internal model is in particular interesting in this perspective (indeed, the case of independence is discussed here mainly for comparison). In this paper, we thus do not address estimation issues (which, of course, deserve attention and may be the subject of future work). We just make a tentative choice, suggested by recent projections for Italian males (which underly the life table IPS55). Under the assumption of independence, we simplify by setting $\beta_{x,t} = 100$ for any age/time, so that the coefficient of variation of the mortality rate (which expresses its volatility in terms of a unit-free measure) under (7) takes the value: $\mathbb{CV}[Q_{x,t}] = \frac{\sqrt{\mathbb{V}\mathrm{ar}[Q_{x,t}]}}{\mathbb{E}[Q_{x,t}]} = \frac{1}{\sqrt{\alpha_{x,t}}} = \frac{1}{\sqrt{0.75\,\beta_{x,t}}} = 11.52\%$ at any age/time. Under the assumption of correlation among the $Z_{x,t}$'s, we set $\bar{\beta} = 100$, so that at age x_0 we have $\mathbb{CV}[Q_{x_0,1}] = \frac{1}{\sqrt{\bar{\alpha}}} = 11.52\%$ (we stress that, in this case, for the following ages the value of parameters and then the value of $\mathbb{CV}[Q_{x,t}]$ change depending on experience).

With regard to the required capital calculated with an internal model (either based on rule [R1] or [R3]), we find useful to make some comparisons involving deterministic assumptions about systematic deviations. In particular, we adopt model (2); this way, only random fluctuations are addressed. Then, we adopt again model (2), but multiplying $q^*_{x,t}$ by 0.75; this way, systematic deviations in respect of the best-estimate life table are also addressed but, similarly to Solvency 2, through a deterministic representation.

The outcomes of our numerical investigations are reported in terms of the ratios $\frac{M_z^{[\mathrm{Solv2}]}}{V_z^{(\Pi)\,[BE]}}$, $\frac{M_z^{[R1]}(T)}{V_z^{(\Pi)\,[BE]}}$ and $\frac{M_z^{[R3]}}{V_z^{(\Pi)\,[BE]}}$. First we compare the proposed regulatory requirement with the solvency rule [R1]. Figures 1 and 2 plot the following cases:

- case (a): standard formula proposed in Solvency 2;
- case (b): rule [R1], where the number of deaths is generated through (8);
- case (c): rule [R1], where the number of deaths is generated through (14), assuming that the experienced number of deaths is as expected under the best-estimate life table (which means a better experience than what depicted by the Solvency 2 shock scenario);
- case (d): rule [R1], where the number of deaths is generated through (14), assuming that the experienced number of deaths is as expected under the Solvency 2 shock scenario;
- case (e): rule [R1], where the number of deaths is generated through (2);

– case (f): rule $[R1]$, where the number of deaths is generated through (2), but assuming a mortality rate given by $0.75\, q^*_{x,t}$ (instead of $q^*_{x,t}$).

It emerges, in particular, that the ratio $\frac{M_z^{[\text{Solv2}]}}{V_z^{(\Pi)\,[BE]}}$ is constant in respect of the portfolio size, since all the quantities contributing to $M_z^{[\text{Solv2}]}$ are linear in respect of n_z. On the contrary, $\frac{M_z^{[R1]}(\omega+1-x_0-z)}{V_z^{(\Pi)\,[BE]}}$ decreases in respect of the portfolio, size, due to the decreasing importance of random fluctuations (which are not considered by Solvency 2). Cases (b), (c) and (f) generate similar amounts of the required capital. Indeed, due to the choice of parameters in case (b), and in case (c) due also to the assumed mortality experience, the magnitude of the systematic deviations is similar in these three cases. However, under cases (b) and (c) also the uncertainty of such deviations is considered, and this originates

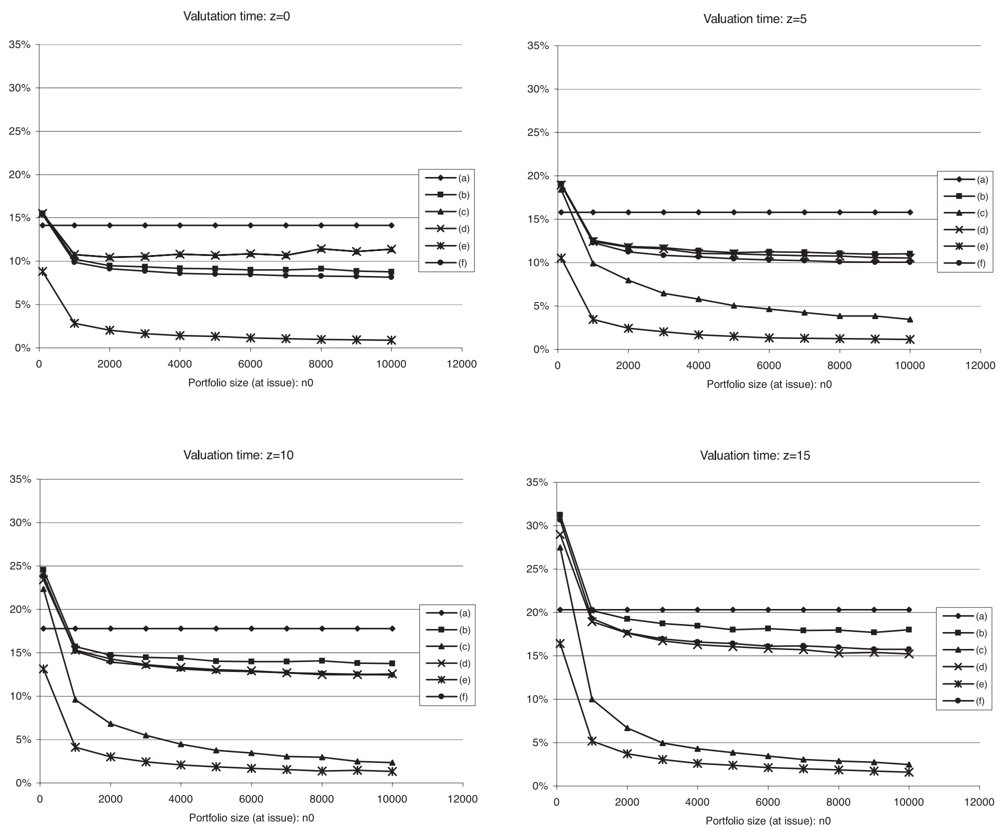

FIGURE 1: Ratios $\frac{M_z^{[\text{Solv2}]}}{V_z^{(\Pi)\,[BE]}}$ and $\frac{M_z^{[R1]}\left(\omega+1-x_0-z\right)}{V_z^{(\Pi)\,[BE]}}$ at various valuation times, for several portfolio sizes.

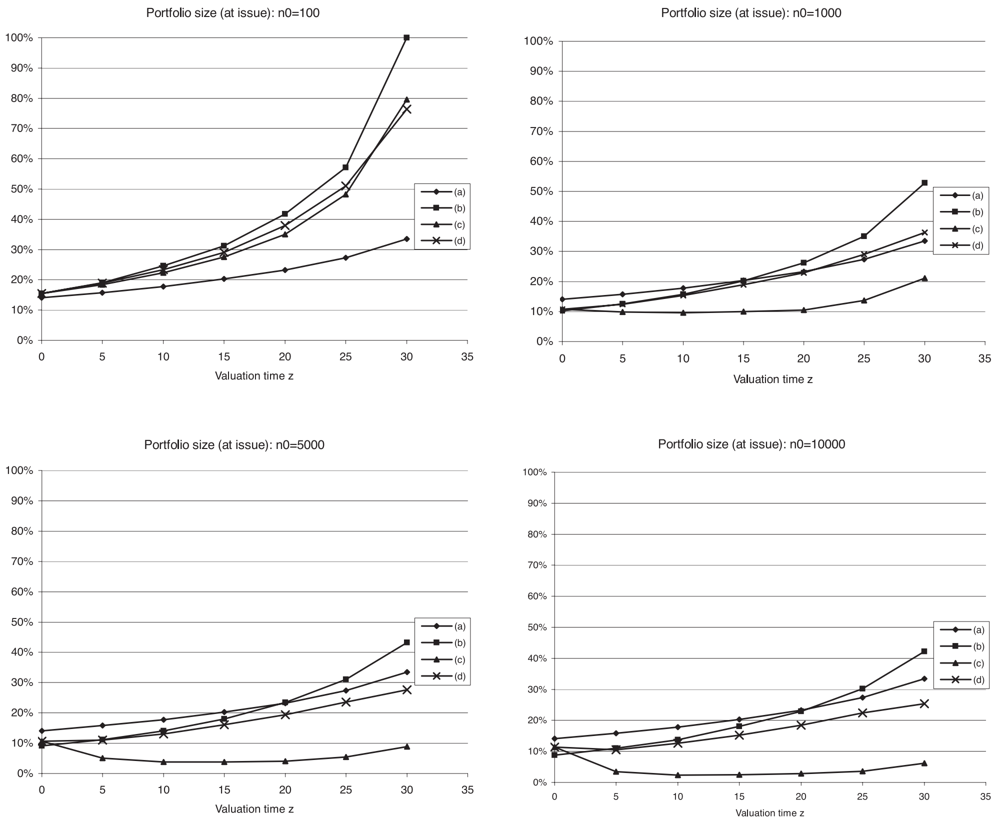

FIGURE 2: Ratios $\dfrac{M_z^{[\mathrm{Solv2}]}}{V_z^{(\Pi)}[BE]}$ and $\dfrac{M_z^{[R1]}(\omega+1-x_0-z)}{V_z^{(\Pi)}[BE]}$ for various portfolio sizes, at several valuation times.

a required capital which is slightly higher than in case (f). Case (c) witnesses that, adopting an accurately designed internal model, it is possible to gain efficiency in capital allocation. We note that the shock scenario referred to by the standard formula can be far away from the actual experience of the insurer, and thus may lead to a biased allocation of capital. On the other hand, a reduction of the required capital needs to be justified in a rigorous way. Case (e) provides us with some information about the relevance of random fluctuations also in large portfolios; disregarding them when allocating capital may not seem a prudent decision.

In Figure 3 we compare rules $[R1]$ and $[R3]$, addressing cases (b)-(f). We note that case by case the two rules lead to similar results. This is due to the fact that the same time-horizon is involved (i.e. $T = \omega + 1 - x_0 - z$), and to the long-term nature of mortality risks in general, and of longevity risk in particular.

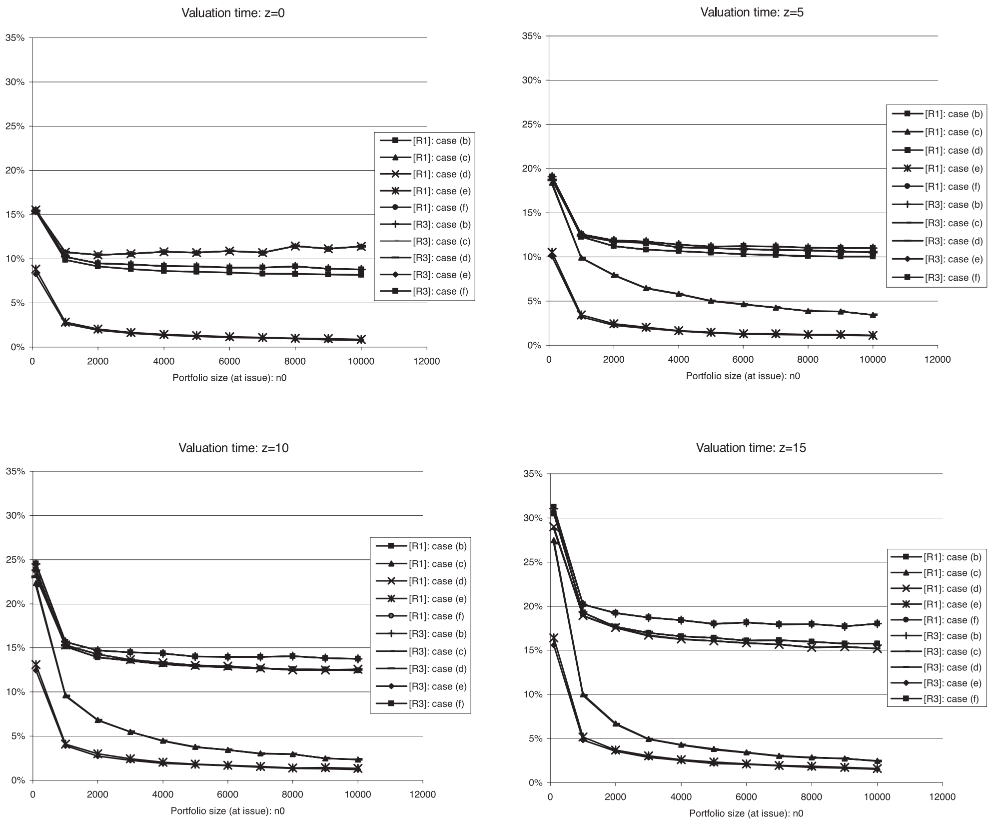

FIGURE 3: Ratios $\dfrac{M_z^{[R1]}\left(\omega+1-x_0-z\right)}{V_z^{(\Pi)}\left[BE\right]}$ and $\dfrac{M_z^{[R3]}}{V_z^{(\Pi)}\left[BE\right]}$ at various valuation times, for several portfolio sizes.

Indeed, if mortality risks originate an annual loss, then typically a global loss at maturity is also recorded. Thus, there is no ambiguity in the choice of the internal solvency rule when addressing longevity risk.

Having in mind a consistent assessment of the amount of required capital, we finally suggest an alternative use of the mortality model described in Section 2.2. Due to its simplicity, the insurer could prefer the adoption of the standard rule, instead of an internal model. An update of the parameters defining the shock scenario (now set to –25% of the best-estimate mortality rates), provided this is permitted by the supervisor, could be obtained through the inferential procedure described in Section 2.2, thus avoiding an unnecessary overestimate, or even an undesirable underestimate, of the liabilities of the insurer.

4. Concluding remarks

In this paper we have proposed a stochastic mortality model which is rigorously defined, but practicable to handle, and can be useful when working on capital allocation. Indeed, even though the insurer does not have the expertise to deal with the methodologies underlying the best-estimate life table, a rational assessment of the impact of mortality risks, longevity risk in particular, can be obtained. The numerical investigations we have performed about the required capital have shown the advantages which can be gained because of an improved consistency, when compared to a standard rule, between the amount to be allocated and the risks dealt with. It is worthwhile to note, in particular, that irrespective of the stronger simplicity of a standard rule, a biased assessment of the impact of the possible cost of the prevailing risks could lead to a capital amount which is unnecessary high or undesirably low.

The procedure we have proposed is based on a stochastic simulation approach; however, the simulation of real-world portfolios may be time-consuming. Then, future research should also focus on the construction of short-cut formulae to determine capital requirements, which are consistent with the main findings of the simulation-based assessments. Anyhow, we mention that a less time-consuming implementation of the model could be as commented at the end of Section 3.3; the model described in Section 2.2 could be used to update the parameters of the standard formula (and clearly this can be performed without stochastic simulations).

In the paper, the mortality model has been described in a simple setting (the case of just one cohort has been mainly addressed, in particular when defining the inferential procedure). We think it is interesting to go deeper on possible extensions to more than one cohort, as well as to investigate on the calibration of parameters to existing data (after noting, however, that the inferential procedure naturally leads to an update of their value which is consistent with experience). Further, business other than life annuities could be addressed.

As far as the investigation of the required capital is concerned, we are well aware of the importance of the Solvency 2 project, in particular in respect of the overall attention addressed to the risk management process. However, some strong simplifications have been introduced in the standard formula, and adoption of internal models or an adjustment of the relevant parameters (based on experience) could lead to a more rigorous capital allocation policy.

APPENDIX

Proof of (24)

Recursion (17) leads to

$$A_t = A_z(1+i)^{t-z} - \sum_{h=z+1}^{t} bN_{x_0+h,h}(1+i)^{t-h}$$

so that the probability in (23) can be rewritten as

$$\mathbb{P}\left[\bigcap_{t=z+1}^{z+T} A_z(1+i)^{t-z} - \sum_{h=z+1}^{\omega-x_0} bN_{x_0+h,h}(1+i)^{t-h} \geq 0 \middle| n_{x_0+z,z}\right] \tag{26}$$

$$= \mathbb{P}\left[\bigcap_{t=z+1}^{z+T} (1+i)^{t-(\omega+1-x_0)} \left(A_z(1+i)^{\omega+1-x_0-z} - \sum_{h=z+1}^{\omega-x_0} bN_{x_0+h,h}(1+i)^{\omega+1-x_0-h}\right) \geq 0 \middle| n_{x_0+z,z}\right]$$

We can note that

$$A_z(1+i)^{\omega+1-x_0-z} - \sum_{h=z+1}^{\omega-x_0} bN_{x_0+h,h}(1+i)^{\omega+1-x_0-h} = A_{\omega+1-x_0}$$

so that we can write (26) as

$$\mathbb{P}\left[\bigcap_{t=z+1}^{z+T} (1+i)^{t-(\omega+1-x_0)} A_{\omega+1-x_0} \geq 0 \middle| n_{x_0+z,z}\right]$$

$$= \mathbb{P}\left[\bigcap_{t=z+1}^{z+T} A_{\omega+1-x_0} \geq 0 \middle| n_{x_0+z,z}\right] = \mathbb{P}\left[A_{\omega+1-x_0} \geq 0 \middle| n_{x_0+z,z}\right]$$

and hence statement in (24).

It is useful to note that such results hold in particular because: (a) the portfolio is closed to new entrants; (b) the probability in (23) (as well as in (21) and (22)) is assessed according to the real-world probability distribution of assets and liabilities (so that no risk-adjustment is applied, for example in a risk-neutral sense); (c) such probability is conditional on the information available at time z on the relevant variables (current number of survivors, investment yields – when random, and so on). A similar simplification could however be obtained when multiple cohorts are considered and the term structure of interest rates is not flat.

Acknowledgments

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