Pricing equity-linked life insurance contracts with multiple risk factors by neural networks

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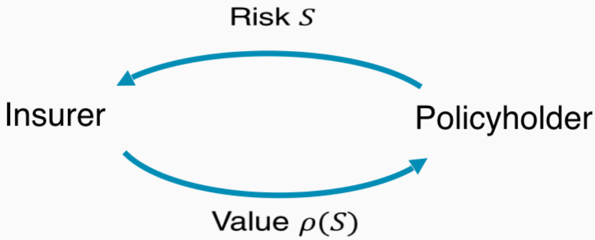
Introduction

Motivation



- What is the fair value of an insurance liability S?
  - Depends on financial risks: Interest rate, equity, volatility, etc.
  - Depends on biometric risks: diversifiable and undiversifiable mortality risk, etc.

Actuarial versus Financial valuation

Two types of valuations:

- Actuarial valuation:

$$\boxed{\rho[S] = e^{-r}\ \mathbb{E}^{\mathbb{P}}[S] + \mathsf{RM}[S]}$$

  - Based on principle of diversification (LLN).
  - Risk margin to cover non-diversified risk.
  - The valuation is performed under the real-world measure $\mathbb{P}$.

- Financial valuation:

$$\boxed{\rho[S] = \mathbb{E}^{\mathbb{Q}}\left[e^{-r}S\right]}$$

  - Based on principles of no-arbitrage and replication under a risk-neutral measure $\mathbb{Q}$.
  - Set of feasible $\mathbb{Q}$'s follows from observed market prices.
  - In incomplete markets: Infinite choice of measures $\mathbb{Q}$.

Liability decomposition: Two-step approach

Insurance liabilities are only partially hedgeable and therefore a two-step approach is considered:

- Two-step hedge-based approach (Dhaene et al. 2017):

$$\rho[S] = \underbrace{V^{\theta}(0)}_{\text{Hedgeable part}} + \underbrace{\pi[S - V^{\theta}(T)]}_{\text{Actuarial principle on the residual part}}$$

  - First part is typically determined by quadratic hedging.
  - Second part is priced via a standard actuarial principle.

- Two-step conditional approach (Pelsser & Stadje 2014):

$$\rho[S] = \underbrace{\mathbb{E}^{\mathbb{Q}}\left[\underbrace{\pi[S \mid \mathbf{Y}]}_{\text{Inner step}}\right]}_{\text{Outer step}}$$

  - Inner step: Actuarial valuation conditional on traded asset prices.
  - Outer step: Financial valuation.

Fair valuation: Recent literature

Growing research interest on fair valuation. In particular:

- The hedge-based valuation of Dhaene et al. (2017) in a one-period setting was generalized in a discrete multi-period setting in Barigou et al. (2019) and in continuous-time in Delong et al. (2019).
- We consider a general financial and insurance model with hedegable and non-hedgeable financial risks and non-hedgeable insurance risk,
- We start with a one-period hedge-based valuation where an optimal dynamic hedging portfolio for the liability is set up with traded assets and the non-hedgeable part of the liability is valuated via a subjective actuarial valuation. We define a multi-period valuation operator by backward iterations of the one-period valuation operator.
- We investigate the continuous-time limit of the multi-period, discrete-time iterations and derive a system of partial differential equations for the continuous-time valuation operator which satisfies the limit.
- The dynamic hedging strategy associated with the continuous-time valuation operator is established.

Aim of this paper

The aim of this paper is threefold:

- Price equity-linked contracts in a general incomplete market with stochastic interest rate, equity, volatility and mortality.
- Solve numerically the pricing problem using neural networks.
- Provide a sensitivity analysis to model parameters and study the accuracy of our approach.

Model for financial and insurance risks

Financial market

- The standard Black-Scholes model with constant interest rate and volatility is too restrictive for long-term liabilities.
- We consider a general financial market with
  - Two-factor Hull-White interest rate model

$$\begin{aligned} r(t) &= \psi(t) + x(t) + y(t), \quad r(0) = r_0, \\ dx(t) &= \big(\delta_x \sigma_x - a x(t)\big)dt + \sigma_x dW_1(t), \quad x(0) = 0, \\ dy(t) &= \big(\delta_y \sigma_y - b y(t)\big)dt + \sigma_y dW_2(t), \quad y(0) = 0, \end{aligned}$$

  - Heston stochastic volatility model

$$dS(t) = S(t)\Big(\big(r(t, x(t), y(t)) + \gamma\sqrt{v(t)}\big)dt + \sqrt{v(t)}dW_3(t)\Big), \quad S(0) = 1,$$

$$dv(t) = k(\eta - v(t))dt + \sigma_v\sqrt{v(t)}dW_4(t), \quad v(0) = v_0.$$

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Account dynamics and insurance guarantees

- The policyholder's account dynamics is then given by[1]

$$
\begin{aligned}
dF(t) \quad = \quad & uF(t) \underbrace{\Big(\big(r(t,x(t),y(t)) + \zeta(t)\big)dt + A(t)dW_1(t) + B(t)dW_2(t)\Big)}_{\text{Bond dynamics}} \\
& +(1-u)F(t) \underbrace{\Big(\big(r(t,x(t),y(t)) + \gamma\sqrt{v(t)}\big)dt + \sqrt{v(t)}dW_3(t)\Big)}_{\text{Equity dynamics}} \\
& -cF(t)dt,
\end{aligned}
$$

  where
  - $F(0) := F_0$ is the initial premium.
  - u is a fixed percentage of the account invested in the bond.
  - c is a constant fee to cover insurance guarantees.

- An equity-linked life insurance contract includes death and survival benefits:

$$D(t,F(t)) = (D^* - F(t))_+, \quad S(F(T)) = (S^* - F(T))_+.$$

[1] The portfolio is homogeneous, i.e. F_0, u and c are equal and fixed for all policyholders.

Insurance model

- We consider a portfolio consisting of n identical equity-linked life insurance contracts,
- We assume that the lifetimes of policyholders are conditionally i.i.d. where the stochastic intensity is given by (Luciano et al. (2008))

$$d\lambda(t) = q\lambda(t)dt + \sigma_\lambda \sqrt{\lambda(t)} dW_5(t), \quad \lambda(0) = \lambda_0,$$

- Such process accounts for diversifiable and undiversifiable mortality risk.
- The counting death process:

$$N(t) = \sum_{k=1}^{n} 1 \{\tau_k \le t\}.$$

- The number of in-force policies:

$$J(t) = n - N(t)$$

Pricing equity-linked life insurance using neural networks

Hedging portfolio dynamics

- The dynamics of the insurer's hedging portfolio is given by

$$
\begin{aligned}
dV^{\theta}(t) &= \theta_1(t)\Big(\big(r(t,x(t),y(t))+\zeta(t)\big)dt + A(t)dW_1(t) + B(t)dW_2(t)\Big) \\
&\quad +\theta_2(t)\Big(\big(r(t,x(t),y(t))+\gamma\sqrt{v(t)}\big)dt + \sqrt{v(t)}dW_3(t)\Big) \\
&\quad +(V^{\theta}(t)-\theta_1(t)-\theta_2(t))r(t,x(t),y(t))dt \\
&\quad +J(t-)cF(t)dt - D(t,F(t))dN(t), \\
V^{\theta}(0) &= 0.
\end{aligned}
$$

- We search for the asset porfolio $\theta = (\theta_1(t), \theta_2(t))$ which optimally matches the liabilities.
- Survival benefits $J(T)S(F(T))$ are paid at the terminal time T and therefore, do not affect the insurer's hedging portfolio.

Hedging and pricing in two steps

Let $Z(t) = (x(t), y(t), F(t), v(t), \lambda(t))$ be the risk factors and $X(t) = V^{\theta}(t) - \varphi^{J(t)}(t, Z(t)) = NAV(t)$, the net asset value at time t.

We proceed in two steps:

1. We determine the hedging strategy $\theta = (\theta_1(t), \theta_2(t))$ which minimizes the local variance of $X(t)$.
   - In a complete market, the minimized local volatility is zero (e.g. Black-Scholes model)
   - In an incomplete market, there is a residual local variance.
2. We price the non-hedgeable risk via a standard deviation principle (Delong et al. (2019)):

$$\Pi[S] = \mathbb{E}[S] + \alpha\sqrt{\mathrm{Var}[S]}$$

The resulting system of non-linear PDEs

- The fair price φ solves the following system of non-linear PDEs:

$$\begin{cases} \varphi_t^k(t,z) + \nabla\varphi^k(t,z)\cdot\mu^*(t,z) + \dfrac{1}{2}\operatorname{Tr}(\sigma(t,z)\mathcal{Q}\sigma(t,z)^\intercal \operatorname{Hess}_z \varphi^k(t,z)) \\ -ckf + \left(\varphi^{k-1}(t,z) + D(t,f) - \varphi^k(t,z)\right) k\lambda(t) - \varphi^k(t,z)r(t,x,y) \\ +\alpha\Big\{(\nabla\varphi^k(t,z)\sigma^*(t,z))\mathcal{Q}(\nabla\varphi^k(t,z)\sigma^*(t,z))^\intercal \\ +(\varphi^{k-1}(t,z) + D(t,f) - \varphi^k(t,z))^2 k\lambda\Big\}^{1/2} = 0, \\ \varphi^k(T,z) = kS(f), \end{cases}$$

  for $k \in \{0,\ldots,n\}$.

- n number of policyholders, $D(t,f)$ death guarantee, $S(f)$ survival benefit, α standard deviation parameter.

Representation as Forward Stochastic Differential Equation

- The fair price φ is also solution of the following controlled FSDE:

$$
\begin{aligned}
\varphi^{J(t)}(t, Z(t)) &= \varphi^{n}(0, Z(0)) \\
&\quad - \int_0^t \Upsilon^{J(s-)}\Big(s, Z(s), \varphi^{J(s-)}(s, Z(s)), \varphi^{J(s-)-1}(s, Z(s))\Big) ds \\
&\quad - \int_0^t D(s, F(s)) J(s-) \lambda(s) ds \\
&\quad + \int_0^t \Big(cJ(s-)F(s) + \varphi^{J(s-)}(s, Z(s)) r(s, x(s), y(s))\Big) ds \\
&\quad + \int_0^t \nabla \varphi^{J(s-)}(s, Z(s)) \sigma(s, Z(s)) dW(s) \\
&\quad + \int_0^t \Big(\varphi^{J(s-)-1}(s, Z(s)) - \varphi^{J(s-)}(s, Z(s))\Big) d\tilde{N}(s).
\end{aligned}
$$

- Starting from an initial price of the liabilities: $\varphi^{n}(0, Z(0))$, the FSDE describes the price dynamics of the insurance liabilities.

Discretization of the FSDE

- For the FSDE, the Euler scheme has the following form:

$$
\begin{aligned}
&\varphi^{J(t_{n+1})}(t_{n+1}, Z(t_{n+1})) = \varphi^{J(t_n)}(t_n, Z(t_n)) \\
&- \alpha\Big\{\Big(\nabla\varphi^{J(t_n)}(t_n, Z(t_n))\sigma^*(t_n, Z(t_n))\Big)\mathcal{Q}\Big(\nabla\varphi^{J(t_n)}(t_n, Z(t_n))\sigma^*(t_n, Z(t_n))\Big)^{\intercal} \\
&+ \Big(\varphi^{J(t_n)-1}(t_n, Z(t_n)) + D(t_n, F(t_n)) - \varphi^{J(t_n)}(t_n, Z(t_n))\Big)^2 J(t_n)\lambda(t_n)\Big\}^{1/2}\Delta t_n \\
&- D(t_n, F(t_n))J(t_n)\lambda(t_n)\Delta t_n \\
&+ \Big(cJ(t_n)F(t_n) + \varphi^{J(t_n)}(t_n, Z(t_n))r(t_n, x(t_n), y(t_n))\Big)\Delta t_n \\
&+ \nabla\varphi^{J(t_n)}(t_n, Z(t_n))\sigma(t_n, Z(t_n))\Delta W(t_n) \\
&+ \Big(\varphi^{J(t_n)-1}(t_n, Z(t_n)) - \varphi^{J(t_n)}(t_n, Z(t_n))\Big)\Delta\tilde{N}(t_n)
\end{aligned}
$$

where

$$
\Delta t_n = t_{n+1} - t_n, \quad \Delta W(t_n) = W(t_{n+1}) - W(t_n), \quad \Delta\tilde{N}(t_n) = \tilde{N}(t_{n+1}) - \tilde{N}(t_n).
$$

- $\nabla\varphi^{J(t_n)}(t_n, Z(t_n))$ and $\varphi^{J(t_n)-1}(t_n, Z(t_n))$ are estimated by neural networks.

Neural networks in our model

We have three neural networks:

N1. The price at time 0:

$$\varphi^{J(0)}(0, Z(0)) = \mathcal{N}_1^{\phi}(Z(0), J(0)).$$

N2. The price gradient:

$$\nabla \varphi^{J(t_n)}(t_n, Z(t_n)) = \mathcal{N}_2^{\chi}(t_n, Z(t_n), J(t_n)).$$

N3. The price with one less policyholder:

$$\varphi^{J(t_n)-1}(t_n, Z(t_n)) = \mathcal{N}_3^{\psi}(t_n, Z(t_n), J(t_n) - 1).$$

These parameters are then estimated in order to minimize the quadratic loss function:

$$l(\phi, \chi, \psi) = \mathbb{E}\left[\left| J(t_N) S(F(t_N)) - \widehat{\varphi}^{J(t_N)}(t_N, Z(t_N)) \right|^2 \right].$$

Solving PDEs using deep learning

- Han et al. (2018) considered semilinear parabolic PDEs of the form

$$\frac{\partial u}{\partial t}(t,x) + \frac{1}{2}\operatorname{Tr}\left(\sigma\sigma^{\mathrm{T}}(t,x)\left(\operatorname{Hess}_x u\right)(t,x)\right) + \nabla u(t,x)\cdot\mu(t,x) + f\left(t,x,u(t,x),\sigma^{\mathrm{T}}(t,x)\nabla u(t,x)\right) = 0$$

  where deep learning is used to approximate $\nabla u(t,x)$.
- Chan-Wai-Nam et al. (2019) compared different types of architectures and parameterizations.
- In our paper, we extended this approach for a system of coupled PDEs where the gradient *and* the jump component need to be approximated.

Numerical results

Model parameters

1. We search for $\varphi^{J(0)}(0, Z(0))$ with $Z(0) = (x(0), y(0), F(0), v(0), \lambda(0))$.
2. Death and survival guarantees:

$$D(t, F(t)) = (D^* - F(t))_+, \quad S(F(T)) = (S^* - F(T))_+.$$

Parameter	Value	Parameter	Value
c	0.01	$x(0)$	0
$D^\star$	1.02	$y(0)$	0
$S^\star$	1.02	$F(0)$	1
T	1	$v(0)$	0.1
Δt	0.01	$\lambda(0)$	0.015
α	0.1	$J(0)$	100

Table 1: The base values of the initial parameters.

Convergence diagnostics

Convergence of one price:

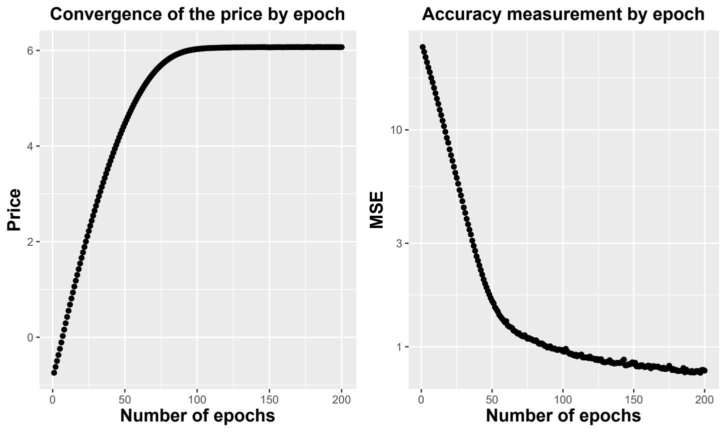

Figure 1: Diagnostics of the neural network training. On the left: Convergence of the price at time 0 per epoch. On the right: Mean Square Error per epoch.

Convergence diagnostics

We consider a surface of prices by changing $v(0), F(0), \lambda(0), J(0)$ by $+/-25\%$ and considering $x(0) \in [0, 0.01], y(0) \in [0, 0.01]$.

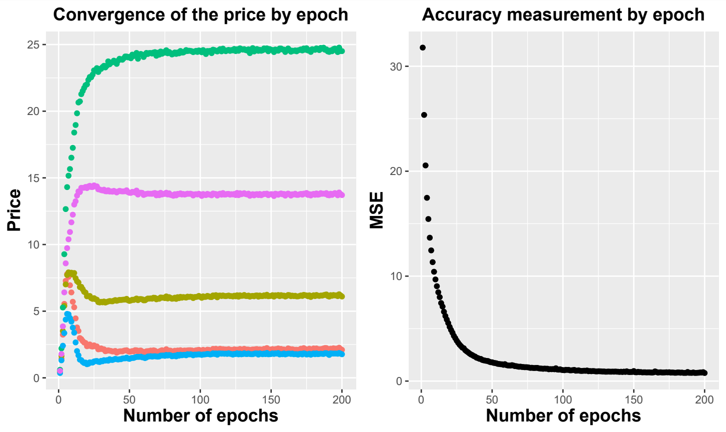

Figure 2: On the left: Convergence of the price at time 0 during the neural network training for five randomly chosen initial parameters. On the right: Mean Square Error per epoch during the neural network training of the surface of prices.

Sensitivity analysis

Sensitivity to interest rate:

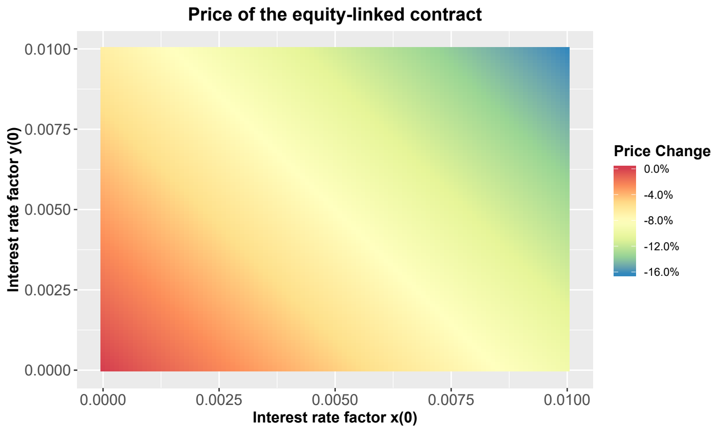

Figure 3: The price of the portfolio of equity-linked contracts for a range of the initial interest rate factors $x(0) \in [0; 0.01]$ and $y(0) \in [0; 0.01]$.

Sensitivity to the volatility and force of mortality:

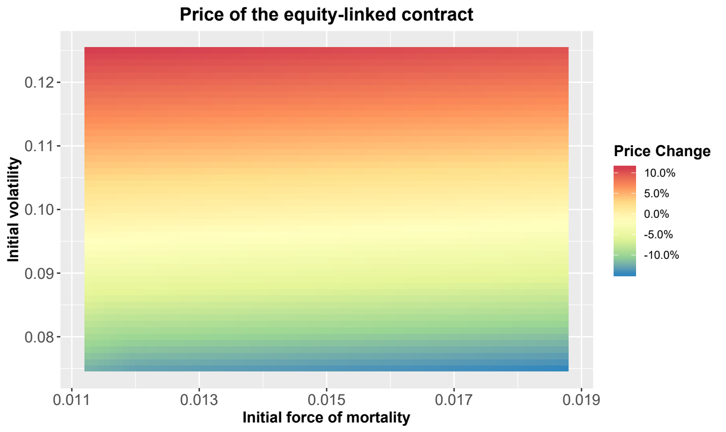

Figure 4: The price of the portfolio of equity-linked contracts for a range of the initial volatilities $v(0) \in [0.075; 0.125]$ and the forces of mortality $\lambda(0) \in [0.01125; 0.01875]$.

Pricing GMMB contracts in the Black-Scholes framework

Consider a survival benefit with payoff function $S(F(T))$ at maturity.

- Either we solve the following PDEs:

$$\begin{cases} \varphi_t^k(t,f) + \varphi_f^k(t,f)fr + \frac{1}{2}\varphi_{ff}^k(t,f)f^2\sigma_f^2 - \varphi^k(t,f)r \\ + \Big(\varphi^{k-1}(t,f) - \varphi^k(t,f)\Big)k\lambda\left(1 - \frac{\alpha}{\sqrt{k\lambda}}\right) = 0, \\ \varphi^k(T,f) = kS(f), \end{cases}$$

  for $k \in \{0,\ldots,n\}$

- or the price can be determined via

$$\varphi^n(0,F(0)) = \mathbb{E}^{\tilde{\mathbb{Q}}}\left[e^{-rT}J(T)S(F(T))\right],$$

  under a risk-adjusted measure $\tilde{\mathbb{Q}}$.

Pricing GMMB contracts in the Black-Scholes framework

Consider a survival benefit with payoff function $S(F(T))$ at maturity.

- Either we solve the following PDEs:

$$\begin{cases} \varphi_t^k(t,f) + \varphi_f^k(t,f) f r + \dfrac{1}{2}\varphi_{ff}^k(t,f) f^2 \sigma_f^2 - \varphi^k(t,f) r \\ \quad + \Big(\varphi^{k-1}(t,f) - \varphi^k(t,f)\Big) k\lambda \left(1 - \dfrac{\alpha}{\sqrt{k\lambda}}\right) = 0, \\ \varphi^k(T,f) = kS(f), \end{cases}$$

for $k \in \{0, \ldots, n\}$

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Pricing GMMB contracts in the Black-Scholes framework

A relative error of 0.2% after 100 epochs.

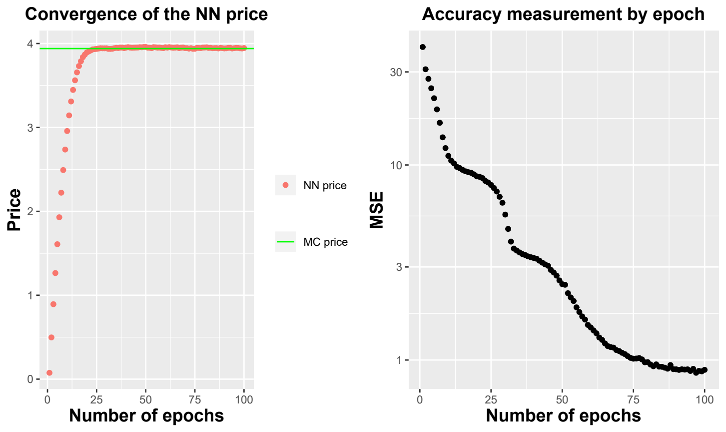

Figure 5: Neural Network price convergence to the Monte-Carlo price for the GMMB and Mean Square Error per epoch.

Conclusion

Conclusion

- We derived a system of non-linear PDEs for the fair pricing of equity-linked life insurance contracts in a general incomplete market.
- We used the connection with BSDEs with jumps and proposed an efficient neural network architecture to solve PDEs with jumps.
- The neural network algorithm provides a multi-dimensional price surface and a fast convergence.
- The paper is available on-line.

Thank you very much.

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References

Barigou, K., Chen, Z. & Dhaene, J. (2019), 'Fair dynamic valuation of insurance liabilities: Merging actuarial judgement with market-and time-consistency', *Insurance: Mathematics and Economics* **88**, 19–29.

Chan-Wai-Nam, Q., Mikael, J. & Warin, X. (2019), 'Machine learning for semi linear pdes', *Journal of Scientific Computing* **79**(3), 1667–1712.

Delong, Ł., Dhaene, J. & Barigou, K. (2019), 'Fair valuation of insurance liability cash-flow streams in continuous time: Theory', *Insurance: Mathematics and Economics* .

Dhaene, J., Stassen, B., Barigou, K., Linders, D. & Chen, Z. (2017), 'Fair valuation of insurance liabilities: Merging actuarial judgement and market-consistency', *Insurance: Mathematics and Economics* **76**, 14–27.

Han, J., Jentzen, A. & Weinan, E. (2018), 'Solving high-dimensional partial differential equations using deep learning', *Proceedings of the National Academy of Sciences* **115**(34), 8505–8510.

Luciano, E., Spreeuw, J. & Vigna, E. (2008), 'Modelling stochastic mortality for dependent lives', *Insurance: Mathematics and Economics* **43**(2), 234–244.

Pelsser, A. & Stadje, M. (2014), 'Time-consistent and market-consistent evaluations', *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics* **24**(1), 25–65.