Ordered Risk Aggregation under Dependence Uncertainty

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Risk measure

Risk measure: Let $\mathcal{M}$ be the set of cdfs on $\mathbb{R}$. A risk measure is defined as

$$\rho : \mathcal{M} \to \mathbb{R}.$$

For $F \in \mathcal{M}$, if $X \sim F$, we also write $\rho(X) = \rho(F)$.

Examples: For $F \in \mathcal{M}$,

- Left Value-at-Risk:

$$\mathrm{VaR}^L_q(F) = F^{-1}(q) = \inf\{t \in \mathbb{R} : F(t) \geqslant q\}, q \in (0, 1].$$

- Right Value-at-Risk:

$$\mathrm{VaR}^R_p(F) = F^{-1}(p+) = \inf\{t \in \mathbb{R} : F(t) > p\}, p \in [0, 1).$$

- Expected Shortfall:

$$\mathrm{ES}_p(F) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}^R_u(F)\mathrm{d}u, p \in (0, 1).$$

- Range-VaR:

$$\mathrm{RVaR}_{p,q}(F) = \frac{1}{q-p}\int_p^q \mathrm{VaR}^R_u(F)\mathrm{d}u, 0 \leqslant p < q < 1.$$

Dependence uncertainty

Suppose that we have a portfolio of two risks X and Y. We are interested in

$$\rho(X+Y).$$

We assume:

- Known mariginal distributions;
- Uncertain dependence structure;
- Order constraint: $X \leqslant Y$ almost surely.

For $F, G \in \mathcal{M}$ such that F is stochastically smaller than G i.e., $F \geqslant G$, define the set

$$\mathcal{F}_2^o(F,G) = \{(X,Y) : X \sim F,\ \ Y \sim G,\ \ X \leqslant Y\}.$$

Risk bounds with order constraint

The worst-case and best-case values of $\rho(X+Y)$ over the set $\mathcal{F}_2^o(F,G)$ denoted by

$$\overline{\rho}(\mathcal{F}_2^o(F,G)) := \sup\{\rho(X+Y) : (X,Y) \in \mathcal{F}_2^o(F,G)\},$$

and

$$\underline{\rho}(\mathcal{F}_2^o(F,G)) := \inf\{\rho(X+Y) : (X,Y) \in \mathcal{F}_2^o(F,G)\}.$$

Concave order

A distribution F is called smaller than a distribution G in *concave order*, denoted by $F \leqslant_{\mathrm{cv}} G$, if $\int \phi \, \mathrm{d}F \leqslant \int \phi \, \mathrm{d}G$ for all concave $\phi : \mathbb{R} \to \mathbb{R}$, provided that both integrals exist.

For a risk measure $\rho : \mathcal{M} \to \mathbb{R}$, we define three commonly used properties:

- A risk measure ρ is *monotone* if $\rho(F) \leqslant \rho(G)$ whenever $F \leqslant_{\mathrm{st}} G$;
- A risk measure ρ is $\leqslant_{\mathrm{cv}}$*-consistent* if $\rho(F) \leqslant \rho(G)$ whenever $F \leqslant_{\mathrm{cv}} G$;
- A risk measure ρ is $\leqslant_{\mathrm{cx}}$*-consistent* if $\rho(F) \leqslant \rho(G)$ whenever $G \leqslant_{\mathrm{cv}} F$.

The concave ordering bounds of $X + Y$?

Concave ordering in unconstrained case

Concave ordering bounds of $X + Y$:

- A random vector (X, Y) is *comonotonic* if there exists a random variable U and two increasing functions f and g such that $X = f(U)$ and $Y = g(U)$ almost surely.
- A random vector (X, Y) is *countermonotonic* if $(X, -Y)$ is comonotonic.
- It is well known that

$$X^c + Y^c \leqslant_{\mathrm{cv}} X + Y \leqslant_{\mathrm{cv}} X^{co} + Y^{co}.$$

Countermonotonicity may violate the order constraint!

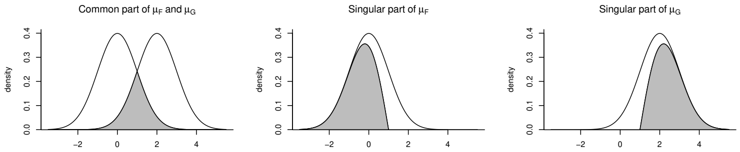
DL coupling

Directional Lower (DL) coupling: Arnold et al. (2020) and Nutz and Wang (2020)

Denote by μ_F and μ_G the Borel probability measures generated by continuous distributions F and G, respectively.

- The common part $\mu_F \wedge \mu_G$ of F and G is the maximal measure θ such that $\theta \leqslant \mu_F$ and $\theta \leqslant \mu_G$.
- The singular parts of F and G are defined as $\mu'_F = \mu_F - \mu_F \wedge \mu_G$ and $\mu'_G = \mu_G - \mu_F \wedge \mu_G$.

Figure: $F = N(0,1)$ and $G = N(2,1)$



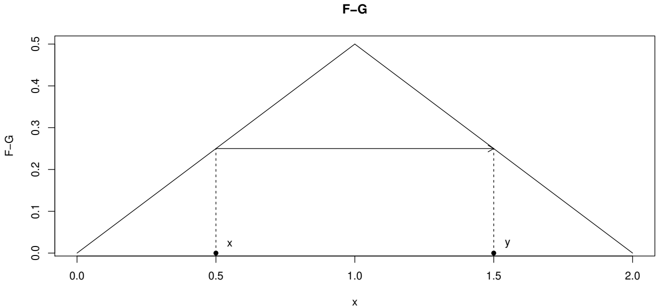
DL coupling

The DL coupling between F and G has two parts.

- The common part of F and G couples identically with each other.
- The transport from the singular part of F to the singular part of G, denoted by $T^{F,G}$, is defined as

$$T^{F,G}(x) = \inf\left\{z \geqslant x : F(z) - G(z) < F(x) - G(x)\right\}.$$

Figure: $F = \mathrm{Unif}[0,2]$ and $G = \mathrm{Unif}[1,2]$



Concave orderings in constrained case

Lemma

For $(X, Y), (X^c, Y^c), (X', Y') \in \mathcal{F}_2^o(F, G)$ such that (X^c, Y^c) is comonotonic and (X', Y') is DL-coupled, we have

$$X^c + Y^c \leqslant_{\mathrm{cv}} X + Y \leqslant_{\mathrm{cv}} X' + Y'.$$

Corollary

Suppose that $(X, Y), (X^c, Y^c), (X', Y') \in \mathcal{F}_2^o(F, G)$ such that (X^c, Y^c) is comonotonic and (X', Y') is DL-coupled. If ρ is $\leqslant_{\mathrm{cv}}$-consistent, then

$$\underline{\rho}(\mathcal{F}_2^o(F, G)) = \rho(X^c + Y^c) \leqslant \rho(X + Y) \leqslant \rho(X' + Y') = \overline{\rho}(\mathcal{F}_2^o(F, G)).$$

If ρ is $\leqslant_{\mathrm{cx}}$-consistent, then

$$\underline{\rho}(\mathcal{F}_2^o(F, G)) = \rho(X' + Y') \leqslant \rho(X + Y) \leqslant \rho(X^c + Y^c) = \overline{\rho}(\mathcal{F}_2^o(F, G)).$$

Tail risk measures

Let $F^{[p,1]}$ be the upper p-tail distribution of $F \in \mathcal{M}$, namely

$$F^{[p,1]}(x) = \frac{(F(x)-p)_+}{1-p}, \quad x \in \mathbb{R}.$$

Definition (Liu and Wang (2020))

For $p \in (0,1)$, a risk measure ρ is a *p-tail risk measure* if $\rho(F) = \rho(G)$ for all $F, G \in \mathcal{M}$ such that $F^{[p,1]} = G^{[p,1]}$.

For a p-tail risk measure ρ, there always exists another risk measure ρ^*, called the generator, such that $\rho(F) = \rho^*\left(F^{[p,1]}\right)$. We call (ρ, ρ^*) a *p-tail* pair of risk measures.

p-concentration

p-concentration:

- A *p-tail event* of a random variable X is an event $A \in \mathcal{A}$ with $0 < \mathbb{P}(A) = 1 - p < 1$ such that $X(\omega) \geqslant X(\omega')$ holds for all $\omega \in A$ and $\omega' \in A^c$.
- A random vector (X, Y) is *p-concentrated* if X and Y shares a common p-tail event of probability $1 - p$.

Bounds on tail risk measures

DL coupling: $(X, Y) \sim D_*^{F,G}$.

Theorem

Suppose that $F \leqslant_{\mathrm{st}} G$*,* $p \in (0,1)$*,* (ρ, ρ^*) *is a p-tail pair of risk measure, and* ρ^* *is monotone and* $\leqslant_{\mathrm{cv}}$*-consistent. We have*

$$\overline{\rho}(\mathcal{F}_2^o(F,G)) = \overline{\rho^*}\left(\mathcal{F}_2^o\left(F^{[p,1]}, G^{[p,1]}\right)\right) = \rho^*(X+Y),$$

where $(X, Y) \sim D_*^{F^{[p,1]}, G^{[p,1]}}$.

The class of $\leqslant_{\mathrm{cv}}$-consistent generators ρ^*:

- $\rho^* = \text{ess-inf}$, corresponding to $\rho = \mathrm{VaR}_p^R$;
- $\rho^* = \mathbb{E}$, corresponding to $\rho = \mathrm{ES}_p$;
- $\rho^* : X \mapsto -\mathrm{ES}_t(-X)$, corresponding $\rho = \mathrm{RVaR}_{p,q}$, where $t = (1-q)/(1-p)$.

VaR bounds

Proposition

For continuous distributions F and G such that $F \leqslant_{\mathrm{st}} G$ and $p \in (0,1)$, we have

$$\overline{\mathrm{VaR}}_p^R(\mathcal{F}_2^o(F,G)) = \min\left\{ \inf_{x\in[F^{-1}(p+),G^{-1}(p+)]} \left\{ T^{F^{[p,1]},G^{[p,1]}}(x) + x \right\}, 2G^{-1}(p+) \right\},$$

and

$$\underline{\mathrm{VaR}}_p^L(\mathcal{F}_2^o(F,G)) = \max\left\{ \sup_{x\in[F^{-1}(p),G^{-1}(p)]} \left\{ x + \hat{T}^{F^{[0,p]},G^{[0,p]}}(x) \right\}, 2F^{-1}(p) \right\},$$

where $\hat{T}^{F^{[0,p]},G^{[0,p]}}(x) = \sup\left\{ t \leqslant x : F^{[0,p]}(t) - G^{[0,p]}(t) < F^{[0,p]}(x) - G^{[0,p]}(x) \right\}$.

Unconstrained problem: Rüschendorf (1982)

Proposition

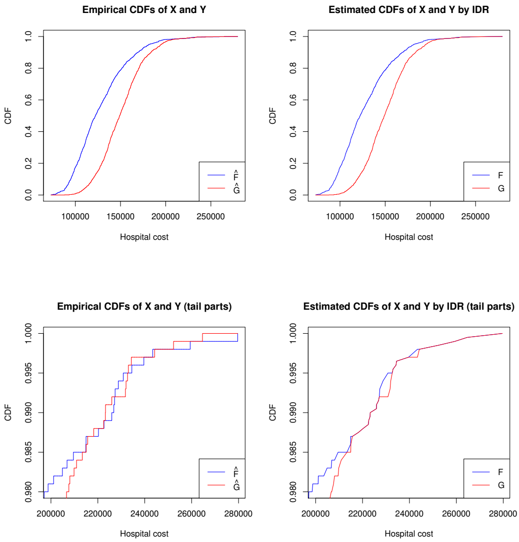
Suppose that F and G are strictly increasing continuous distribution functions such that $F \leqslant_{\mathrm{st}} G$. For $p \in (0,1)$, we have

$$\overline{\mathrm{VaR}}_p^L(\mathcal{F}_2^o(F,G)) = \overline{\mathrm{VaR}}_p^R(\mathcal{F}_2^o(F,G)) \quad \textit{and} \quad \underline{\mathrm{VaR}}_p^L(\mathcal{F}_2^o(F,G)) = \underline{\mathrm{VaR}}_p^R(\mathcal{F}_2^o(F,G)).$$

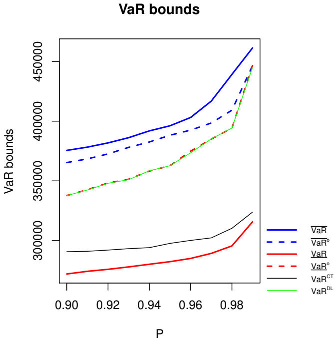
A case study

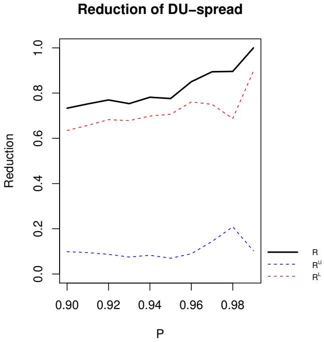
- The aggregate loss $S = X + Y$ where $X \sim F$ and $Y \sim G$ represent the losses caused by females and males, respectively, from a portfolio of 50 males and 50 females
- $X \leqslant Y$ is reasonable due to many common risk factors
- Cannot reject the hypothesis $\hat{F} \leqslant_{\mathrm{st}} \hat{G}$
- Estimate F and G such that $F \leqslant_{\mathrm{st}} G$ using the isotonic distributional regression (Henzi et al. (2019))

Numerical results



Numerical results





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Thank You!