Pandemic Mortality Rates

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P. Andersson, M. Lindholm (2021) *A Note on Pandemic Mortality Rates*
Scandinavian Actuarial Journal, to appear, available online

Aim for today

- Give a brief introduction to frailty models
- Illustrate how these can be used to describe pandemic mortality stresses
- Describe how implied post-pandemic mortality rates can be obtained

The notion of a "pandemic" will be used even though the results apply to general stress / scenario testing

Outline of the presentation

- Background and motivation
- Frailty models and some useful observations
- Pandemic and post-pandemic mortality rates
- The Gompertz-Makeham law as a frailty model
- Numerical illustration

Background

Let

- $\mu(x)$ denote the unstressed population mortality rate for an x yr old ("pre-Covid" mortality rate)
- $\mu^*(x)$ denote the corresponding stressed population mortality rate ("Covid" mortality rate)

Observation by Spiegelhalter (2020) and Milevsky (2020):

$$\mu^*(x) \approx \mu(x+k)$$

for some $k \geq 0$, during the pandemic

Background

Question:

How will the temporarily stressed mortality rates change the post-Covid-19 mortality rates?

Observation: If you stress $\mu(x)$ directly this is not obvious(!?)

Suggestion: Make use of selection effects $\rightsquigarrow$ **frailty models!**

Proportional frailty models

Construction:

- At birth each individual is assigned an individual random frailty $Z \geq 0$
- All individuals follow the same common baseline mortality rate $\alpha(x)$

$\Rightarrow$ more frail individuals (i.e. those having higher Z) will tend to die earlier

Idea in the present paper:

- The frailty assigned at birth will be unaffected by the pandemic
- The baseline mortality rate will be temporarily affected by the pandemic

$\Rightarrow$ changes in selection effects due to changes in $\alpha(x)$

Proportional frailty models

Let

- T denote the age at death of an individual
- $A(x) = \int_0^x \alpha(t)\mathrm{d}t$ the cumulative baseline mortality rate

The survival function is then given by

$$S(x) := \mathrm{P}(T > x) := \mathrm{E}[S(x \mid Z)] = \underbrace{\mathrm{E}[\mathrm{e}^{-ZA(x)}]}_{\text{Prop. frailty}} = \mathcal{L}(A(x)), \quad (1)$$

where $\mathcal{L}(\cdot)$ is the Laplace transform of Z

$$\Rightarrow \mu(x) := -\frac{\mathrm{d}}{\mathrm{d}x} \log S(x) = \underline{\text{pop. mortality rate}} \neq \alpha(x) \qquad (2)$$

Proportional frailty models

Basic relations:

- The population mortality rate can be expressed as

$$\mu(x) = -\alpha(x)\frac{\mathcal{L}'(A(x))}{\mathcal{L}(A(x))} = \alpha(x)h(A(x)), \tag{3}$$

  where $h(x) := -\mathcal{L}'(x)/\mathcal{L}(x)$

- Expected frailty conditional on survival:

$$\bar{z}(x) := \mathrm{E}[Z \mid T > x] = -\frac{\mathcal{L}'(A(x))}{\mathcal{L}(A(x))} = h(A(x)) \tag{4}$$

$\Rightarrow \mu(x) = \alpha(x)\bar{z}(x)$

Pandemic stress

Introduce the stressed

- baseline mortality rate

$$\beta(x) \geq \alpha(x) \tag{5}$$

during the pandemic, which is active for $x \in [\tau_0, \tau_1)$

- cumulative baseline mortality rate

$$A^*(x) = \int_0^x (1_{\{t\in[\tau_0,\tau_1)\}}\beta(t) + 1_{\{t\notin[\tau_0,\tau_1)\}}\alpha(t))\mathrm{d}t \tag{6}$$

$$\Rightarrow \begin{cases} S^*(x) & = \mathcal{L}(A^*(x)) \\ \mu^*(x) & = -\frac{\frac{\mathrm{d}}{\mathrm{d}x}S^*(x)}{S^*(x)} = \alpha^*(x)h(A^*(x)) = \alpha^*(x)\bar{z}^*(x) \end{cases}$$

Pandemic stress

Observations / preliminary results:

- $\beta(x) \geq \alpha(x) \Rightarrow A^*(x) \geq A(x)$
- **Lemma:** $h(\cdot)$ is non-increasing
  (see e.g. (Andersson & Lindholm 2021, Lemma 2))

$\Rightarrow$ $h(A^*(x)) \leq h(A(x)) \Leftrightarrow \bar{z}^*(x) \leq \bar{z}(x)$

$\Rightarrow$ $\mu^*(x) \leq \mu(x)$ for $x > \tau_1$, since $\alpha^*(x) = \alpha(x)$ for $x > \tau_1$

Q: what happens for $x \in (\tau_0, \tau_1]$!? ...not obvious, since

$$\mu^*(x) = \beta(x)\bar{z}^*(x), \text{ where } \beta(x) \geq \alpha(x), \text{ while } \bar{z}^*(x) \leq \bar{z}(x)$$

Pandemic stress

Lemma 1 (Andersson & Lindholm (2021))

Let $\beta(x) = \gamma(x)\alpha(x), x \geq 0$, where $\gamma(x) \geq 1, \gamma'(x) \geq 0$, and let $B(x) = \int_0^x \beta(t)\mathrm{d}t$. If $u(x) := 1/h(x)$ is positively sub-homogeneous for constants $\rho \geq 1$, that is

$$u(\rho x) \leq \rho u(x),\ \ \rho \geq 1,$$

it holds that

$$\mu_\beta(x) := \beta(x)h(B(x)) \geq \alpha(x)h(A(x)) = \mu(x)$$

Pandemic stress

Theorem 2 (Andersson & Lindholm (2021))

Let $A^*(x)$ be given by (6) and $\beta(x)$, $\gamma(x)$ and $u(x)$ be defined as in Lemma 1. It then holds that

$$\mu^*(x) = \mu(x), \quad x < \tau_0 \tag{i}$$

$$\gamma(x)\mu(x) \geq \mu^*(x) \geq \mu(x), \quad \tau_0 \leq x < \tau_1 \tag{ii}$$

$$\mu(x) \geq \mu^*(x) = \frac{\bar{z}^*(x)}{\bar{z}(x)}\mu(x), \quad x \geq \tau_1. \tag{iii}$$

Pandemic stress: Gompertz - Makeham

Special case: The Gompertz-Makeham law (GML)

$$\mu(x) = a + be^{cx}, \;\; x \geq 0, \tag{7}$$

with parameters $a, b, c \geq 0$

Proposition : The GML can be constructed using a proportional frailty model where

$$\mathcal{L}(w) = \frac{1}{(1 + \frac{c}{\delta} w)^{a/c}} \exp\{-\frac{b}{\delta} w\}, \tag{8}$$

and $\alpha(x) = \delta e^{cx}, \delta > 0$ (see Lindholm (2017))

N.B. $\mathcal{L}(w)$ above corresponds to a certain translated gamma-distribution

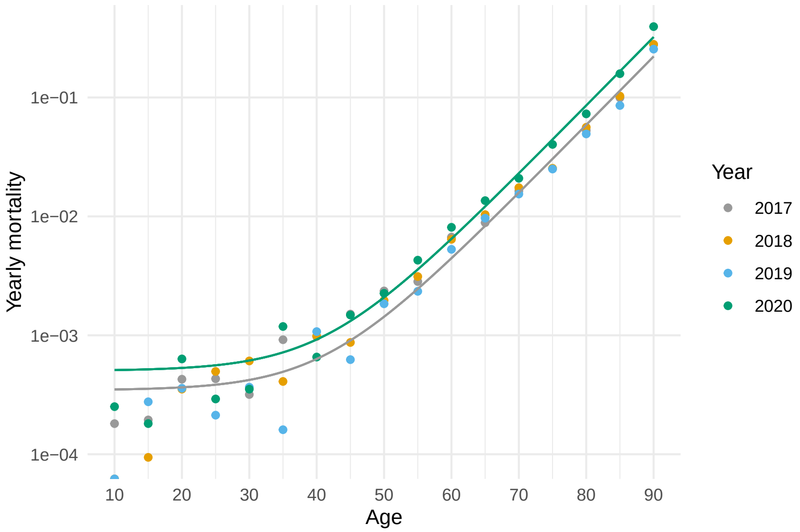
Pandemic stress: Gompertz - Makeham

Pandemic stress for GML: $\beta(x) = \alpha(x+k) \Leftrightarrow \beta(x) = \gamma\alpha(x)$, with $\gamma = e^{ck} \geq 1$

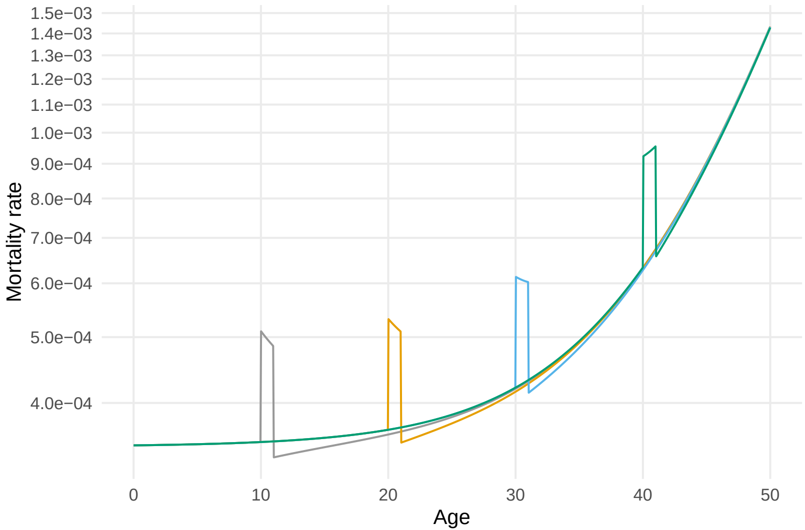
$\Rightarrow$ Theorem 1 applies (results can also be verified directly)

$\Rightarrow \mu^*(x) \nearrow \mu(x)$ as $x \to \infty$ (post-pandemic)

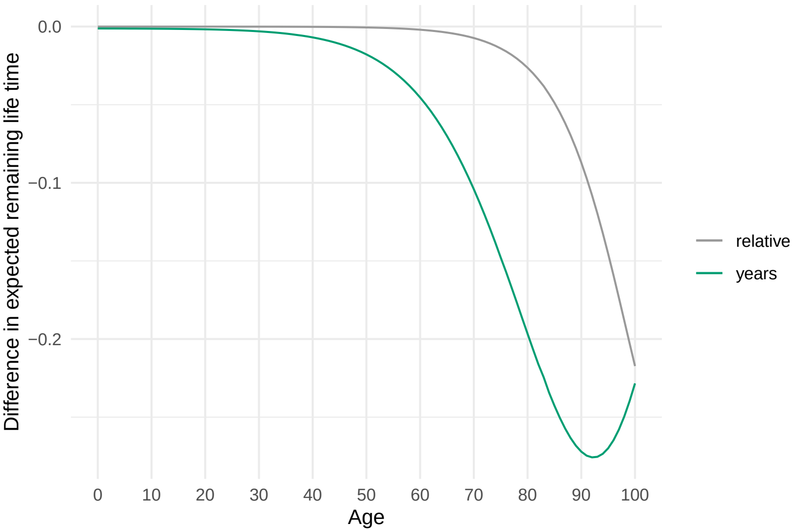
Next: illustration based on Swedish Covid-19 data



Yearly mortality
1e−01
1e−02
1e−03
1e−04
10
20
30
40
50
60
70
80
90
Age
Year
2017
2018
2019
2020



1.5e−03
1.4e−03
1.3e−03
1.2e−03
1.1e−03
1.0e−03
9.0e−04
8.0e−04
7.0e−04
6.0e−04
5.0e−04
4.0e−04
Mortality rate
0
10
20
30
40
50
Age


0.0
−0.1
−0.2
Difference in expected remaining life time
0
10
20
30
40
50
60
70
80
90
100
Age
relative
years

Conclusions

- We suggest to apply temporal mortality stresses to the common baseline hazard rate in proportional frailty models
- This allows us to exploit selection effects
  - ⇒ Post-stress population mortality rates
  - **Result:** Post-stress rates will always be lower!
- The results hold for any proportional frailty model
  - Particularly easy for the Gompertz-Makeham law, which can be constructed using a proportional frailty model
  - **Result:** Post-stress rates for the GML will converge to the unstressed ones (from below)
- The results are valid for any temporal stress on the common baseline mortality rate

References

Andersson, P. & Lindholm, M. (2021), 'A note on pandemic mortality rates', *Scandinavian Actuarial Journal (available online)* .

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