Bootstrap Consistency for the Mack Bootstrap

Actuarial Colloquia
06/23/2022

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Introduction

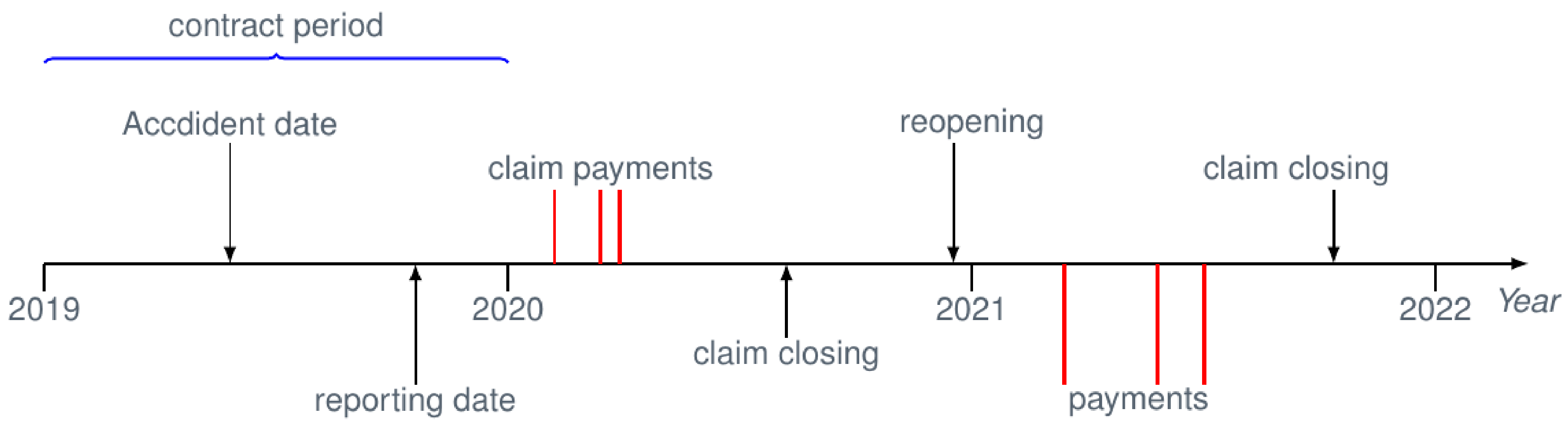

Figure: Typical process of loss adjustment for a non-life insurance

Aim: Prediction of total amount of claims for known and unknown claims that are already happened so far.

Loss Triangle – an Example

	Development Year j				
Accident Year i	0	1	2	3	4
2017 (i.e. $i = 0$)	$C_{0,0}$	$C_{0,1}$	$C_{0,2}$	$C_{0,3}$	$C_{0,4}$
2018	$C_{1,0}$	$C_{1,1}$	$C_{1,2}$	$C_{1,3}$	
2019	$C_{2,0}$	$C_{2,1}$	$C_{2,2}$		
2020	$C_{3,0}$	$C_{3,1}$			
2021	$C_{4,0}$				

Notation:

- $C_{i,j}$ cumulative claim for accident year i and development year j
- $\mathcal{D}_I = \{C_{i,j} \,|\, 0 \leq i + j \leq I\}$ set for all observed claims at time I for $i = 0, \ldots, I$ and $j = 0, \ldots, I$
- $\mathcal{Q}_I = \{C_{i,I-i} \,|\, i = 0, \ldots, I\}$ set of the last observed cumulative claims at time I

Mack's Model (1993)

Let $(C_{i,j}, i, j = 0, \ldots, I)$ denote random variables on some probability space and suppose the following holds:

i. There exist so-called development factors $f_0, \ldots, f_{I-1}$ such that

$$E\left(C_{i,j+1} \middle| C_{i,j}\right) = f_j C_{i,j}$$

for all $i = 0, \ldots, I$ and $j = 0, \ldots, I-1$.

ii. There exist variance parameters $\sigma_0^2, \ldots, \sigma_{I-1}^2$ such that

$$Var\left(C_{i,j+1} \middle| C_{i,j}\right) = \sigma_j^2 C_{i,j}$$

for all $i = 0, \ldots, I$ and $j = 0, \ldots, I-1$.

iii. The cumulative payments are stochastically independent over the accident years $i,\ i = 0, \ldots, I$.

Best Estimate and MSE of Prediction

	Development Year j				
Acc. Year i	0	1	2	3	4
2017	$C_{0,0}$	$C_{0,1}$	$C_{0,2}$	$C_{0,3}$	$C_{0,4}$
2018	$C_{1,0}$	$C_{1,1}$	$C_{1,2}$	$C_{1,3}$	
2019	$C_{2,0}$	$C_{2,1}$	$C_{2,2}$		
2020	$C_{3,0}$	$C_{3,1}$			
2021	$C_{4,0}$				

development factor estimator:

$$\hat{f}_j = \frac{\sum_{i=0}^{I-j-1} C_{i,j+1}}{\sum_{i=0}^{I-j-1} C_{i,j}}$$

variance parameter estimator:

$$\hat{\sigma}_j^2 = \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} C_{i,j} \left(\frac{C_{i,j+1}}{C_{i,j}} - \hat{f}_j \right)^2$$

Best estimate of total reserve

$$\hat{R}_I = \sum_{i=0}^{I} C_{i,I-i} \left(\prod_{j=I-i}^{I-1} \hat{f}_j - 1 \right)$$

Mean squared error of prediction

$$MSEP(\hat{R}_I | D_I) = \sum_{i=0}^{I} \left(\hat{C}_{i,I}^2 \sum_{j=I-i}^{I-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2} \frac{1}{\hat{C}_{i,j}} + \hat{C}_{i,I}^2 \sum_{j=I-i}^{I-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2} \frac{1}{\sum_{k=0}^{I-j-1} C_{k,j}} \right) + 2 \sum_{i,l=0, i<l}^{I} \left(\hat{C}_{i,I} \hat{C}_{l,I} \sum_{j=I-i}^{I-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2} \frac{1}{\sum_{k=0}^{I-j-1} C_{k,j}} \right)$$

Note: Mack's Model allows to estimate the variance of the reserve but we are interested in the whole distribution of the reserve.

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Goal: Asymptotic Theory for the Reserve

We have to impose further assumptions about Mack's Model:

- $C_{i,0}$ for $i = 0, \dots, I$ are i.i.d. random variables with finite first and second moment.
- The individual development factors $F_{i,j}$, $i = 0, \dots, I$ and $j = 0, \dots, I-1$, are positive random variables conditional on $C_{i,j}$, such that $F_{i,j}$ and $F_{k,l}$ are independent given $(C_{i,j}, C_{k,l})$ for all $(i,j) \neq (k,l)$. The conditional moments of $F_{i,j}$ on $C_{i,j}$ satisfy

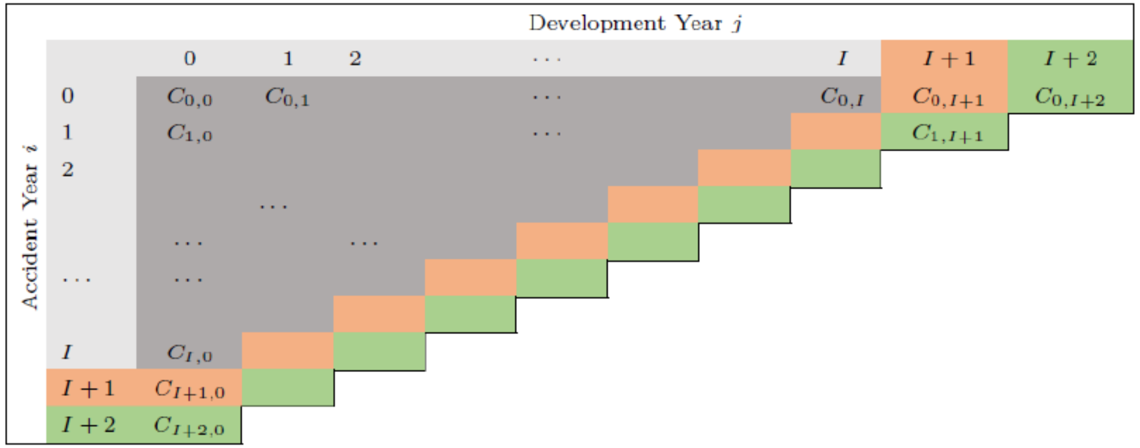
$$E(F_{i,j}|C_{i,j}) = f_j \quad \text{and} \quad \mathrm{E}\left(F_{i,j}^2 \middle| C_{i,j}\right) = f_j^2 + \frac{\sigma_j^2}{C_{i,j}}.$$

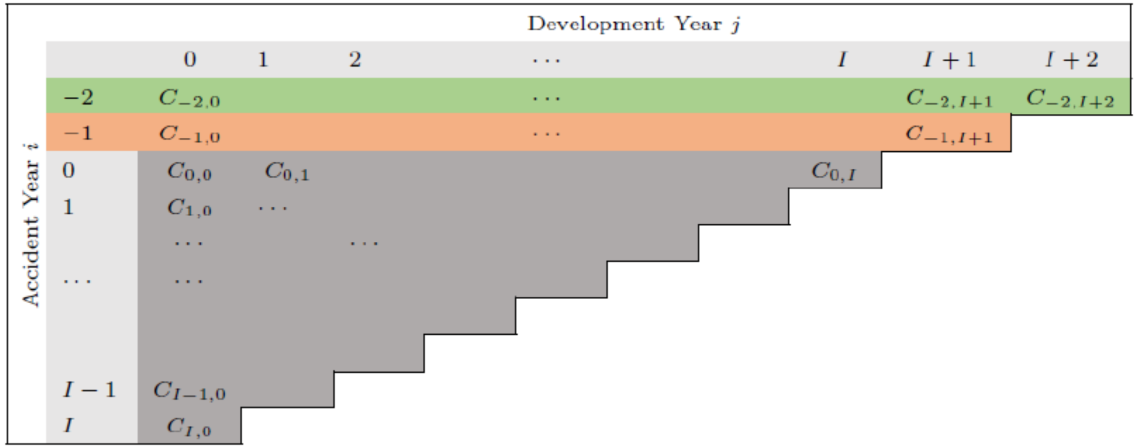
- Recursive Model:

$$C_{i,j} = C_{i,j-1} F_{i,j-1} = C_{i,0} \prod_{k=0}^{j-1} F_{i,k}$$

- $f_j \to 1$ for $j \to \infty$ such that $\prod_{j=0}^{\infty} f_j < \infty$.
- $\sigma_j^2 \to 0$ for $j \to \infty$ such that $\sum_{j=0}^{\infty} (j+1)^2 \sigma_j^2 < \infty$.

Asymptotic Framework of Loss Triangle





Hence, in the following we keep I fix and let $n \to \infty$.

Predictive Root of the Reserve

The predictive root of the reserve conditional on $\mathcal{Q}_{I,n}$ computes to

$$R_{I,n} - \hat{R}_{I,n} = \sum_{i=0}^{I+n} C_{I-i,i}\left(\prod_{j=i}^{I+n-1} F_{I-i,j} - \prod_{j=i}^{I+n-1} \hat{f}_j\right) = \sum_{i=0}^{I+n} C_{I-i,i}\left(\prod_{j=i}^{I+n-1} F_{I-i,j} - \prod_{j=i}^{I+n-1} f_j\right) + \sum_{i=0}^{I+n} C_{I-i,i}\left(\prod_{j=i}^{I+n-1} f_j - \prod_{j=i}^{I+n-1} \hat{f}_j\right)$$

$$= \underbrace{(R_{I,n} - \hat{R}_{I,n})_1}_{process\ uncertainty} + \underbrace{(R_{I,n} - \hat{R}_{I,n})_2,}_{parameter\ uncertainty}$$

where $F_{I-i,j} | C_{I-i,j} \sim \left(f_j, \frac{\sigma_j^2}{C_{I-i,j}}\right)$ for $i + j > I$.

Analogously, we define a bootstrap predictive root of the reserve conditional on $\mathcal{D}_{I,n}$.

$$R_{I,n}^* - \hat{R}_{I,n} = \sum_{i=0}^{I+n} C_{I-i,i}\left(\prod_{j=i}^{I+n-1} F_{I-i,j}^* - \prod_{j=i}^{I+n-1} \hat{f}_j\right) = \sum_{i=0}^{I+n} C_{I-i,i}\left(\prod_{j=i}^{I+n-1} F_{I-i,j}^* - \prod_{j=i}^{I+n-1} \hat{f}_j^*\right) + \sum_{i=0}^{I+n} C_{I-i,i}\left(\prod_{j=i}^{I+n-1} \hat{f}_j^* - \prod_{j=i}^{I+n-1} \hat{f}_j\right)$$

$$= (R_{I,n}^* - \hat{R}_{I,n}^*)_1 + (R_{I,n}^* - \hat{R}_{I,n}^*)_2,$$

where $F_{I-i,j}^* | C_{I-i,j}^* \sim \left(\hat{f}_j^*, \frac{\hat{\sigma}_j^2}{C_{I-i,j}^*}\right)$ for $i + j > I$ and $\hat{f}_j^*$ denotes a suitable bootstrap estimator.

Bootstrap Consistency

- A bootstrap approach is valid if the limiting bootstrap distribution converges to the "true" limiting distribution in some sense.

- Therefore, we need to derive at first the limiting distributions of the predictive root of the reserve $(R_{I,n}-\hat{R}_{I,n})_1$ and $(R_{I,n}-\hat{R}_{I,n})_2$ conditional on $\mathcal{Q}_{I,n}$ separately and then jointly.

- First, we recap the limiting distribution of $(R_{I,n}-\hat{R}_{I,n})_2$ and $(R_{I,n}-\hat{R}_{I,n})_1$ conditional on $\mathcal{Q}_{I,n}$ and then we examine bootstrap consistency for $(R^*_{I,n}-\hat{R}^*_{I,n})_2$ and $(R^*_{I,n}-\hat{R}^*_{I,n})_1$, respectively.

Limiting Distribution – Parameter Uncertainty

Theorem (Steinmetz and Jentsch (2022a)):

Under some regularity conditions, and $(R_{I,n} - \hat{R}_{I,n})_2$ inflated with $\sqrt{I+n+1}$, as $n \to \infty$, we have

$$\sqrt{I+n+1}(R_{I,n} - \hat{R}_{I,n})_2 = \sqrt{I+n+1} \sum_{i=0}^{I+n} C_{I-i,i} \left(\prod_{j=i}^{I+n-1} f_j - \prod_{j=i}^{I+n-1} \hat{f}_j \right) \overset{d}{\to} < Q_{I,\infty}, \boldsymbol{Y}_\infty > \sim \mathcal{G}_2,$$

where $\boldsymbol{Y}_\infty = (Y_i, i \in \mathbb{N}_0)$ denotes a centred Gaussian process with covariances for $i_1, i_2 \in \mathbb{N}_0$

$$Cov(Y_{i_1}, Y_{i_2}) = \sum_{j=\max(i_1,i_2)}^{\infty} \frac{\sigma_j^2}{\mu_j} \prod_{l=\max(i_1,i_2)}^{\infty} f_l^2 \prod_{k=\min(i_1,i_2)}^{\max(i_1,i_2)-1} f_k < \infty,$$

where $\mu_j = E\left(C_{i,j}\right)$.

Here, the two random sequences $Q_{I,\infty}$ and $\boldsymbol{Y}_\infty$ are stochastically independent, and the limiting distribution $\mathcal{G}_2$ has mean zero and (finite) variance.

Two Possible Bootstrap Approaches

In general, for $(R^*_{I,n} - \hat{R}^*_{I,n})_2$ we consider two natural bootstrap approaches leading to estimator $\hat{f}^*_j$:

- parametric
- non-parametric

Parametric Approach:

The parametric bootstrap approach requires a parametric assumption for the family of the distribution of $F^*_{i,j}$ conditional on $C_{i,j}$.

Then the bootstrap estimator $\hat{f}^*_j$ can be computed as

$$\hat{f}^*_j = \frac{\sum_{i=-n}^{I-j-1} C_{i,j} F^*_{i,j}}{\sum_{i=-n}^{I-j-1} C_{i,j}},$$

where $F^*_{i,j}|C_{i,j} \sim \left(\hat{f}_j, \frac{\hat{\sigma}^2_j}{C_{i,j}}\right)$ has a certain parametric distribution.

Possible Bootstrap Estimators

Non-parametric Approach:

The non-parametric bootstrap relies on a residual representation of $F_{i,j}$ conditional on $C_{i,j}$:

$$F_{i,j} = f_j + \frac{\sigma_j}{\sqrt{C_{i,j}}} r_{i,j},$$

where $r_{i,j}$ i.i.d. random variables and $E\left(r_{i,j}\right) = 0$, $\mathrm{Var}\left(r_{i,j}\right) = 1$ and $P(F_{i,j} > 0|C_{i,j}) = 1$ a.s..

Estimator for $r_{i,j}$:

$$\hat{r}_{i,j} = \frac{\sqrt{C_{i,j}}\,(F_{i,j} - \hat{f}_j)}{\hat{\sigma}_j}, \text{ if } \quad \hat{\sigma}_j > 0.$$

By i.i.d. re-sampling $\hat{r}_{i,j}$, we get $r^*_{i,j}$. Then the bootstrap estimator can be computed as

$$\hat{f}^*_j = \frac{\sum_{i=-n}^{I-j-1} C_{i,j} F^*_{i,j}}{\sum_{i=-n}^{I-j-1} C_{i,j}},$$

where $F^*_{i,j}|C_{i,j} = \hat{f}_j + \frac{\hat{\sigma}_j}{\sqrt{C_{i,j}}} r^*_{i,j}$.

Predictive Root – Parameter Uncertainty Bootstrap

Recall:

$$\sqrt{I+n+1}(R_{I,n}-\hat{R}_{I,n})_2 = \sqrt{I+n+1}\sum_{i=0}^{I+n} C_{I-i,i}\left(\prod_{j=i}^{I+n-1} f_j - \prod_{j=i}^{I+n-1} \hat{f}_j\right)$$

$$\sqrt{I+n+1}(R^{*}_{I,n}-\hat{R}^{*}_{I,n})_2 = \sqrt{I+n+1}\sum_{i=0}^{I+n} C_{I-i,i}\left(\prod_{j=i}^{I+n-1} \hat{f}^{*}_j - \prod_{j=i}^{I+n-1} \hat{f}_j\right),$$

where $\hat{f}^{*}_j$ is a introduced bootstrap estimator.

Theorem (Steinmetz and Jentsch (2022b)):

Under some regularity conditions, and as $n \to \infty$, we have

$$d_2\left(\mathcal{L}\left(\sqrt{I+n+1}\sum_{i=0}^{I+n}\left(\prod_{j=i}^{I+n-1} f_j - \prod_{j=i}^{I+n-1} \hat{f}_j\right)\right), \mathcal{L}^{*}\left(\sqrt{I+n+1}\sum_{i=0}^{I+n}\left(\prod_{j=i}^{I+n-1} \hat{f}_j - \prod_{j=i}^{I+n-1} \hat{f}^{*}_j\right)\right)\right) \to 0,$$

where d_2 denotes the Mallows metric.

Limiting Distribution – Process Uncertainty

Theorem (Steinmetz and Jentsch (2022a)):

Under some regularity conditions, as $n \to \infty$ and conditional on $\mathcal{Q}_{I,\infty}$, $(R_{I,n} - \hat{R}_{I,n})_1$ converges in L_2 −sense to a non-degenerate random distribution $(R_{I,\infty} - \hat{R}_{I,\infty})_1$,

$$\left(R_{I,\infty} - \hat{R}_{I,\infty}\right)_1 = \sum_{i=0}^{\infty} C_{I-i,i} \left(\prod_{j=i}^{\infty} F_{I-i,j} - \prod_{j=i}^{\infty} f_j \right) \sim \mathcal{G}_1 .$$

The limiting conditional distribution $\mathcal{G}_1$ conditional on $\mathcal{Q}_{I,\infty}$ has mean 0 and non-degenerate variance

$$\sum_{i=0}^{\infty} C_{I-i,i} \sum_{j=i}^{\infty} \left(\prod_{k=i}^{j-1} f_k \right) \sigma_j^2 \left(\prod_{l=j+1}^{\infty} f_l^2 \right).$$

Remark: The limiting distribution $\mathcal{G}_1 | \mathcal{Q}_{I,\infty}$ will be in general non-Gaussian.

Process Uncertainty – Parametric Bootstrap

Recall:

$$(R_{I,n} - \hat{R}_{I,n})_1 = \sum_{i=0}^{I+n} C_{I-i,i} \left(\prod_{j=i}^{I+n-1} F_{I-i,j} - \prod_{j=i}^{I+n-1} f_j \right)$$

and

$$(R^*_{I,n} - \hat{R}^*_{I,n})_1 = \sum_{i=0}^{I+n} C_{I-i,i} \left(\prod_{j=i}^{I+n-1} F^*_{I-i,j} - \prod_{j=i}^{I+n-1} \hat{f}^*_j \right),$$

where $F^*_{I-i,j} | C^*_{I-i,j} \sim \left(\hat{f}^*_j, \frac{\hat{\sigma}^2_j}{C^*_{I-i,j}} \right)$ has a certain parametric distribution.

Theorem (Steinmetz and Jentsch (2022b)):

Under some regularity conditions and if $F^*_{i,j}$ is chosen to have the true conditional parametric distribution of $F_{i,j}$, as $n \to \infty$, we have

$$d_2(\mathcal{L}((R_{I,n} - \hat{R}_{I,n})_1 | Q_{I,\infty}), \mathcal{L}^*((R^*_{I,n} - \hat{R}^*_{I,n})_1)) \to 0,$$

where d_2 denotes the Mallows metric.

Limiting distribution of the whole reserve

It turns out, that $(R_{I,n} - \hat{R}_{I,n})_1$ and $(R^*_{I,n} - \hat{R}^*_{I,n})_1$ dominate asymptotically.

Hence,

$$(R_{I,n} - \hat{R}_{I,n})|Q_{I,\infty} = (R_{I,n} - \hat{R}_{I,n})_1 + (R_{I,n} - \hat{R}_{I,n})_2|Q_{I,\infty} \overset{d}{\rightarrow} \mathcal{G}_1|Q_{I,\infty},$$

$$(R^*_{I,n} - \hat{R}^*_{I,n})|D_{I,\infty} = (R^*_{I,n} - \hat{R}^*_{I,n})_1 + (R^*_{I,n} - \hat{R}^*_{I,n})_2|D_{I,\infty} \overset{d^*}{\rightarrow} \mathcal{G}_1|Q_{I,\infty},$$

where $\mathcal{G}_1|Q_{I,\infty}$ is the limiting distribution of $(R_{I,n} - \hat{R}_{I,n})_1$ conditional on $Q_{I,\infty}$.

Theorem (Steinmetz and Jentsch (2022b)):

Under some regularity conditions and if $F^*_{i,j}$ is chosen to have the <u>true</u> conditional parametric distribution of $F_{i,j}$, as $n \to \infty$, we have

$$d_2(\mathcal{L}((R_{I,n} - \hat{R}_{I,n})|Q_{I,\infty}), \mathcal{L}^*(R^*_{I,n} - \hat{R}^*_{I,n})) \to 0,$$

where d_2 denotes the Mallows metric.

Is a Normal Distribution a valid Approximation?

In practice:

- For bootstrapping $(R^*_{I,n} - \hat{R}^*_{I,n})_2$ often a (non-parametric) residual based bootstrap is proposed.
- By contrast, for bootstrapping $(R^*_{I,n} - \hat{R}^*_{I,n})_1$, a (parametric) log-normal or gamma distribution is commonly used to ensure $F^*_{i,j} > 0$.

Limiting distribution:

- However, although the residuals are not normal, if $\sigma_j^2 / C_{i,j}$ is small, the distribution of the residuals become closer to a normal distribution such that the deviation from a normal approximation becomes small in practice.
- Hence, in this case a normal approximation for $F_{i,j}$, will be approximately correct and a normal approximation for $(R_{I,n} - \hat{R}_{I,n})_1$ conditional on $Q_{I,n}$ will be approximately valid.

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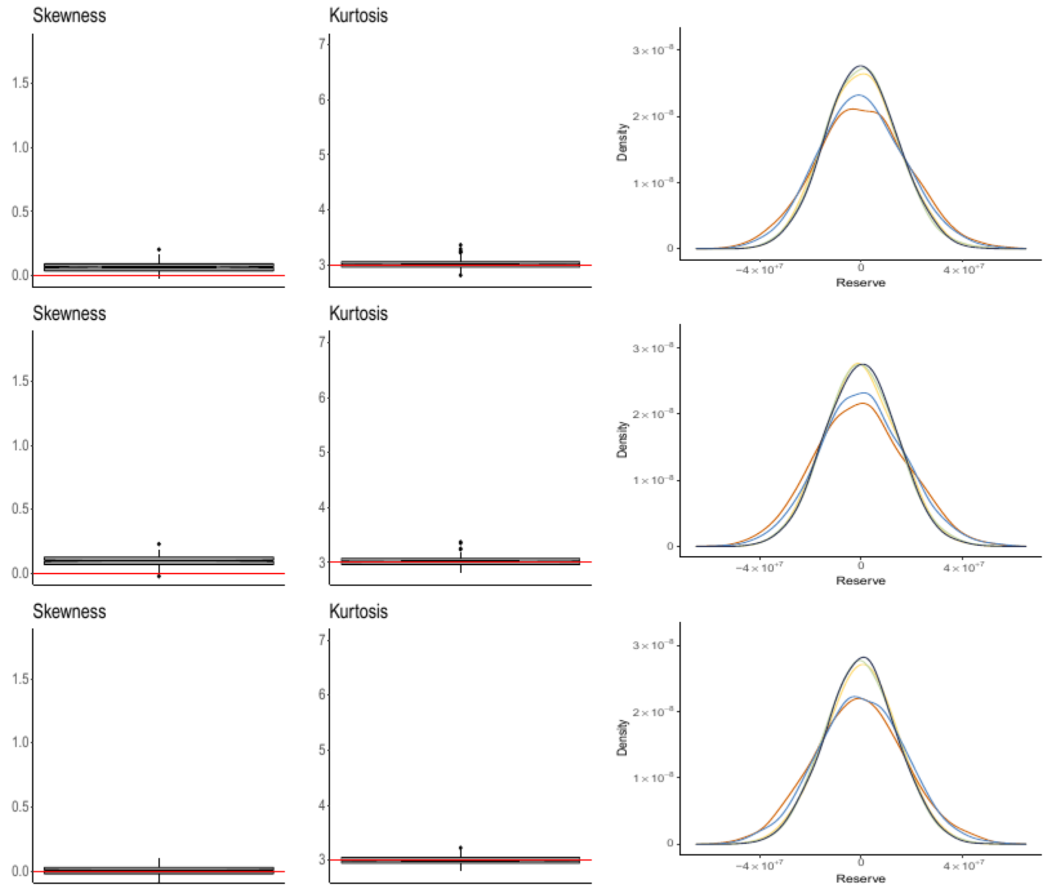
Simulation Study

- Simulate 500 upper loss triangles $D_{I,n}$.
- $I = 11$ and $n = 20$ (i.e. 31x31 loss triangle)
- f_j and σ_j^2 for all $j = 0, \dots, I + n - 1$, f_j and σ_j^2 decrease exponentially to 1 and 0, respectively.

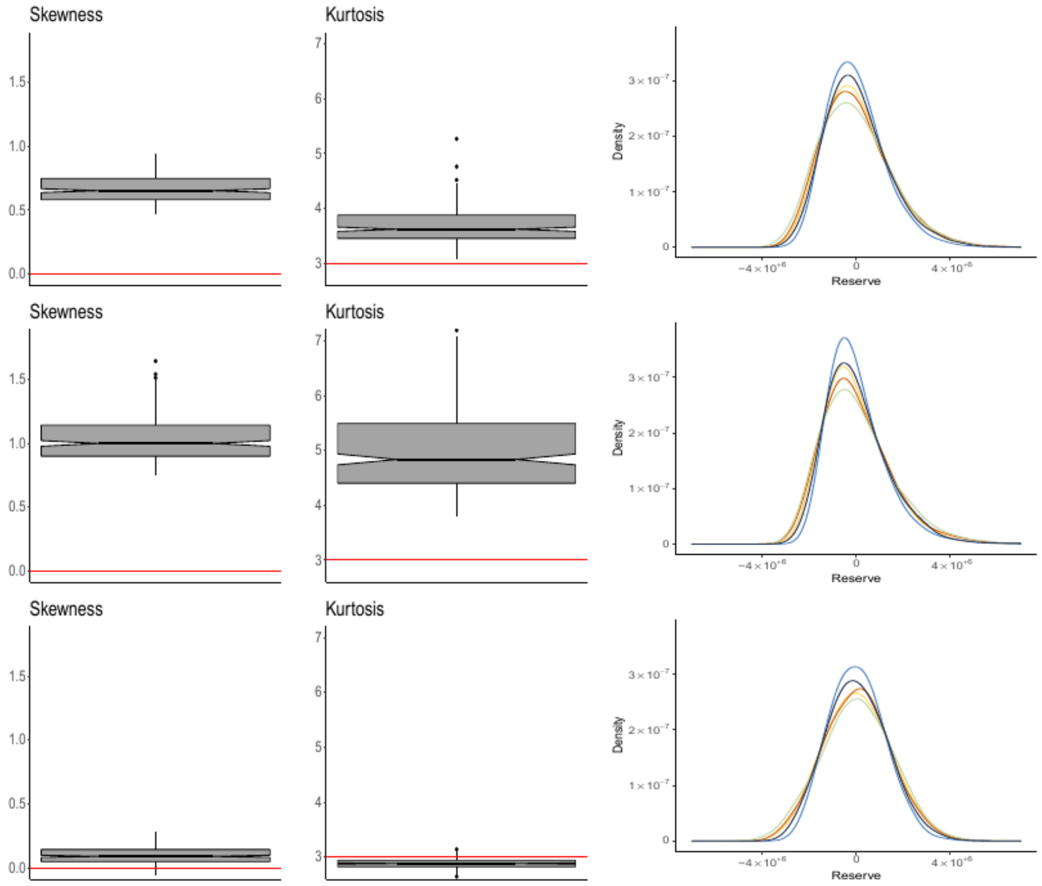
- Setup a) first column $C_{\bullet,0} = (C_{-20,0}, \dots, C_{10,0})$ of the (upper) loss triangle is (independently) uniformly generated from $[120 \cdot 10^6, 350 \cdot 10^6]$.
- Setup b) first column $C_{\bullet,0} = (C_{-20,0}, \dots, C_{10,0})$ of the (upper) loss triangle is (independently) uniformly generated from $[120 \cdot 10^4, 350 \cdot 10^4]$.

- Apply Mack's Bootstrap for each of the triangle using either a conditional (i) gamma, (ii) log-normal or (iii) left-tail truncated normal distribution (truncated at 0.1) and derive $(R_{I,n}^* - \hat{R}_{I,n}^*)_1$.

Setup a)



Setup b) (scaled by 100)



Boxplots of skewness, kurtosis and representative density plots for simulated bootstrap distributions of $(R^*_{I,n} - \hat{R}^*_{I,n})_1$ given the upper loss triangles in the setup of a) and b) for $I = 11, n = 20$ for (conditional) gamma (top), log-normal (center) and truncated normal distribution (bottom).

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Summary

- We derived asymptotic conditional and unconditional asymptotic theory for the predictive root of the reserve in Mack's model. For this purpose, the predictive root of the reserve is split in two additive parts that carry the process uncertainty and the estimation uncertainty.
- The process uncertainty part conditional on $\mathcal{Q}_{I,\infty}$ is in general asymptotically non-Gaussian and dominates asymptotically.
- The parameter uncertainty is inflated with $\sqrt{I+n+1}$ and converges conditional on $\mathcal{Q}_{I,\infty}$ asymptotically to normal distribution.
- The derived asymptotic theory allows to study Mack's bootstrap and enables the derivation of bootstrap consistency results.
- We prove bootstrap consistency for Mack's bootstrap for the predictive root of the reserve.

References

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- Mack, T. (1993). Distribution-free calculation of the standard error of chain ladder reserve estimates. ASTIN Bulletin: The Journal of the IAA, 23(2), 213-225.

- Steinmetz, J., Jentsch C. (2022a). Asymptotic Theory for Mack's Model (currently under review)

- Steinmetz, J., Jentsch C. (2022b). Bootstrap Consistency for the Mack Bootstrap (working paper)

Appendix

Central Limit Theorems for $\hat{f}_j$ and $\widehat{\sigma}_j^2$

Theorem (Steinmetz and Jentsch (2022a)):

Under certain assumptions, for each fixed j, and $n \to \infty$, we have

$$\sqrt{I+n+1}\left(\hat{f}_j - f_j\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma_j^2}{\mu_j}\right),$$

where $E\left(C_{i,j}\right) = \mu_j < \infty$.

Theorem (Steinmetz and Jentsch (2022a)):

Under additional moment assumptions, for fixed j, and $n \to \infty$, we have

$$\sqrt{I+n+1}(\widehat{\sigma}_j^2 - \sigma_j^2) \xrightarrow{d} \mathcal{N}(0, \kappa_j^4 - \sigma_j^4),$$

where $E\left(C_{i,j}^2 \left(F_{i,j} - f_j\right)^4\right) = \kappa_j^4 < \infty$.

Additional Assumption for the limiting distribution of $(R_{I,n} - \hat{R}_{I,n})_2$

Assumption: (support condition and variance parameters)

The individual development factors $F_{i,j}$ are random variables with support (ε, ∞) for some $\varepsilon > 0$ and the sequence of variance parameters $\left(\sigma_j^2\right)_{j \in \mathbb{N}_0}$ with $\sigma_j^2 \to 0$ as j $\to \infty$ such that

$$\sum_{j=0}^{\infty} (j+1)^2 \frac{\sigma_j^2}{\varepsilon^j} < \infty.$$

Additional Assumption for the limiting distribution of $(R_{I,n} - \hat{R}_{I,n})_2 | Q_{I,n}$

Assumption: (backward conditional moments)

For all $\mathrm{K} \in \mathbb{N}_0, k \geq 0$ and $j, j_1, j_2 \in \{0, \dots, K\}, j_1 \leq j_2$, we have

$$|E(C_{i,j}|C_{i,j+k}) - E(C_{i,j}|C_{i,j+k+1})| \leq a_k X_i$$

and

$$|Cov(C_{i,j_1} C_{i,j_2}|C_{i,j_2+k}) - E(C_{i,j_1} C_{i,j_2}|C_{i,j_2+k+1})| \leq b_k Y_i,$$

where $(X_i)_{i \in \mathbb{Z}}$ and $(Y_i)_{i \in \mathbb{Z}}$ are sequences of non-negative i.i.d. random variables with $E(X_i^{2+\delta}) < \infty$ for some $\delta > 0$ and $E(Y_i^2) < \infty$ and $(a_j)_{j \in \mathbb{N}_0}$ and $(b_j)_{j \in \mathbb{N}_0}$ are non-negative real-valued sequences with and $\sum_{j=0}^{\infty}(j+1)^2 a_j < \infty$ and $\sum_{j=0}^{\infty}(j+1)^2 b_j < \infty$.