Continuous partition-of-unity copulas
and their application
to risk management and other fields

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Agenda

1. Introduction & formal framework
2. Construction from given data
3. Case studies
4. Extension to arbitrary dimensions
5. Bibliography / References
1. Introduction & formal framework

Motivation:

- Extension of discrete partition-of-unity copulas to the continuous case
- Construction of new multivariate copulas on the basis of a generalized infinite partition-of-unity approach (extendable to the uncountable infinite case)
- Construction allows for tail-dependence as well as for asymmetry
- Can be easily implemented for risk management purposes
- Particular interest: how to fit such copulas to highly asymmetric data?
1. Introduction & formal framework

Formal framework:

Let $\mathbb{Z}^+ = \{0,1,2,3,\cdots\}$ and suppose that $\{\varphi_i(u)\}_{i \in \mathbb{Z}^+}$ and $\{\psi_j(v)\}_{j \in \mathbb{Z}^+}$ are non-negative maps defined on $(0,1)$ such that:

$$\sum_{i=0}^{\infty} \varphi_i(u) = \sum_{j=0}^{\infty} \psi_j(v) = 1$$ \hspace{1cm} (1)

$$\alpha_i := \int_0^1 \! \varphi_i(u) \, du > 0, \quad \beta_j := \int_0^1 \! \psi_j(v) \, dv > 0, \quad i, j \in \mathbb{Z}^+. \hspace{1cm} (2)$$

- $\{\varphi_i(u)\}_{i \in \mathbb{Z}^+}$ and $\{\psi_j(v)\}_{j \in \mathbb{Z}^+}$ can be thought of representing discrete distributions over $\mathbb{Z}^+$ with parameters $u$ and $v$, resp.
- The sequences $\{\alpha_i\}_{i \in \mathbb{Z}^+}$ and $\{\beta_j\}_{j \in \mathbb{Z}^+}$ represent the probabilities of the corresponding mixed distributions.

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1. Introduction & formal framework

Formal framework:

Let \( \{p_{ij}\}_{i,j \in \mathbb{Z}^+} \) represent the probabilities of an arbitrary discrete bivariate distribution over \( \mathbb{Z}^+ \times \mathbb{Z}^+ \) with marginal distributions given by

\[
p_{i*} = \sum_{j=0}^{\infty} p_{ij} = \alpha_i \text{ and } p_{*j} = \sum_{i=0}^{\infty} p_{ij} = \beta_j \text{ for } i, j \in \mathbb{Z}^+. \tag{3}\]

Then

\[
c(u,v) := \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p_{ij} \frac{\varphi_i(u)}{\alpha_i} \frac{\psi_j(v)}{\beta_j}, \quad u,v \in (0,1) \tag{4}\]

defines the density of a bivariate copula, called (infinite) partition-of-unity copula.
1. Introduction & formal framework

Formal framework:

From a "dual" point of view, we can rewrite (4) as

\[ c(u,v) := \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p_{ij} \frac{\varphi_i(u) \psi_j(v)}{\alpha_i \beta_j} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p_{ij} f_i(u) g_j(v), \ u, v \in (0,1) \]  

(5)

where

\[ f_i(\cdot) = \frac{\varphi_i(\cdot)}{\alpha_i} \] and \[ g_j(\cdot) = \frac{\psi_j(\cdot)}{\beta_j}, \ i, j \in \mathbb{Z}^+ \]  

(6)

denote the densities induced by \( \{\varphi_i(u)\}_{i \in \mathbb{Z}^+} \) and \( \{\psi_j(v)\}_{j \in \mathbb{Z}^+} \). This means that the copula density \( c(u,v) \) can also be seen as a mixture of product densities.
1. Introduction & formal framework

Formal framework:

Example 1 (Binomial distributions – Bernstein copula):

For fixed integers $a, b \geq 2$, consider the family of binomial distributions given by their point masses

$$
\varphi_{a,j}(u) = \begin{cases} 
\binom{a-1}{i} i^i (1-u)^{a-1-i}, & i = 0, \ldots, a-1 \\
0, & i \geq a
\end{cases}
$$

and $\psi_{b,j}(v) = \varphi_{b,j}(v)$ for $i, j \in \mathbb{Z}^+$ and $(u, v) \in (0,1)$.

We have
1. Introduction & formal framework

Formal framework:

Example 1 (Binomial distributions – Bernstein copula):

\[ f_{a,i} = \int_0^1 \varphi_{a,i}(u) du = \frac{1}{a}, \quad \beta_{b,j} = \int_0^1 \psi_{b,j}(v) dv = \frac{1}{b}, \]

where \( f_{a,i} \) and \( g_{b,j} \) are densities of a beta distribution with parameters \( (i, a+1-i) \) and \( (j, b+1-j) \) resp., \( p_{i*} = \frac{1}{a} \) and \( p_{j*} = \frac{1}{b} \), so

\[ c_{a,b}(u,v) = ab \sum_{i=0}^{a} \sum_{j=0}^{b} p_{i,j} \binom{a-1}{i} \binom{b-1}{j} u^{i-1}(1-u)^{a-i} v^{j-1}(1-v)^{b-j}, \quad u, v \in (0,1) \]

which is the density of a bivariate Bernstein copula.
1. Introduction & formal framework

Formal framework:

Example 2 (Negative binomial distributions):

For fixed integers $a, b \geq 2$, consider the family of negative binomial distributions given by their point masses

$$\varphi_{a,i}(u) = \binom{a+i-1}{i} u^i (1-u)^a, \quad (10)$$

and $\psi_{b,j}(v) = \varphi_{b,j}(v)$ for $i, j \in \mathbb{Z}^+$ and $(u,v) \in (0,1)$.

We have
1. Introduction & formal framework

Formal framework:

Example 2 (Negative binomial distributions):

\[
\alpha_{a,j} = \int_0^1 \varphi_{a,j}(u) du = \frac{a}{(a+i)(a+i+1)} \quad \beta_{b,j} = \frac{b}{(b+j)(b+j+1)},
\]

(11)

\(f_{a,i}\) and \(g_{b,j}\) are densities of a beta distribution with parameters \((i+1,a+1)\) and \((j+1,b+1)\), \(p_{i} = \frac{a}{(a+i)(a+1+i)}\), \(p_{j} = \frac{b}{(b+j)(b+1+j)}\), so

\[
c_{a,b}(u,v) = (a+1)(b+1) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p_{ij} \binom{a+i+1}{i} \binom{b+j+1}{j} u^i (1-u)^v v^j (1-v)^b, u, v \in (0,1).
\]

(12)
1. Introduction & formal framework

Formal framework:

Example 2 (Negative binomial distributions):

Negative binomial copulas typically show a tail dependence:

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_\nu(\beta)$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{5}{8}$</td>
<td>$\frac{11}{16}$</td>
<td>$\frac{93}{256}$</td>
<td>$\frac{193}{1024}$</td>
<td>$\frac{793}{2048}$</td>
<td>$\frac{1619}{4096}$</td>
<td>$\frac{26333}{262144}$</td>
<td>$\frac{53381}{5368704}$</td>
<td>$\frac{215955}{268435456}$</td>
</tr>
</tbody>
</table>

with $\lambda_\nu(\beta) = \lim_{\nu \to 1} \frac{1}{1-t} \int_0^1 \int_0^1 c_\nu(u,v) \, du \, dv = \frac{2\Gamma(2,\beta)}{\Gamma^2(\beta)} \int_0^1 \int_0^1 \frac{x^\beta}{(x+y)^{2\beta+1}} \, dx \, dy = 1 - \frac{\left(\frac{2}{\beta}\right)}{4^\beta} - \frac{1}{\sqrt{\pi\beta}}$ 

for large $\beta$. 

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Formal framework:

Example 3 (Poisson distributions):

For fixed $a, b > 0$ consider the family of Poisson distributions given by their point masses

$$\varphi_{a,j}(u) = (1-u)^a \frac{a^j L(u)^j}{j!},$$

(13)

$$L(u) := -\ln(1-u), \quad \psi_{b,j}(v) = \varphi_{b,j}(v), \quad i, j \in \mathbb{Z}^+, \quad (u, v) \in (0,1).$$

We have
1. Introduction & formal framework

Formal framework:

Example 3 (Poisson distributions):

\[
\alpha_{i,j} = \int_0^1 \varphi_{i,j}(u) du = \left( \frac{a}{a+1} \right) \left( 1 - \frac{a}{a+1} \right), \quad \beta_{i,j} = \left( \frac{b}{b+1} \right) \left( 1 - \frac{b}{b+1} \right)
\]  

which correspond to geometric distributions over \( \mathbb{Z}^+ \) with means \( a \) and \( b \),

\[
p_i = \left( \frac{a}{a+1} \right) \left( 1 - \frac{a}{a+1} \right), \quad p_j = \left( \frac{b}{b+1} \right) \left( 1 - \frac{b}{b+1} \right), \quad i, j \in \mathbb{Z}^+
\]  

\[
c_{u,v} = (a+1)(b+1) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p_i \frac{(a+1)^i (b+1)^j}{i! j!} L(u)(1-u)^i L(v)(1-v)^j, \quad u, v \in (0,1).
\]
1. Introduction & formal framework

Formal framework:

- Bernstein copula, $m = 3$; no tail dependence
- Negative binomial copula, $\beta = 3$; $\lambda_0(\beta) = 0.6875$
- Poisson copula, $\gamma = 5$; no tail dependence
1. Introduction & formal framework

Formal framework:

Remark: Sklar’s theorem provides a general method to construct pairs of discrete r.v.’s \((X,Y)\) with joint probabilities \(p_{ij} = P(X = i, Y = j)\) and marginal probabilities \(\{\alpha_i\}_{i \in \mathbb{Z}^+}\) and \(\{\beta_j\}_{j \in \mathbb{Z}^+}\):

Assume quantile functions \(Q_X, Q_Y\) of \(X, Y\) and a pair of r.v’s \((U,V)\) with a given copula \(\tilde{C}\). Then \((X,Y) = (Q_X(U), Q_Y(V))\) has joint probabilities

\[
p_{ij} = P(X = i, Y = j) = P\left(\sum_{k=0}^{i-1} \alpha_k < U \leq \sum_{k=0}^{i-1} \alpha_k, \sum_{k=0}^{j-1} \beta_k < V \leq \sum_{k=0}^{j-1} \beta_k\right)
= \tilde{C}\left(\sum_{k=0}^{i-1} \alpha_k, \sum_{k=0}^{j-1} \beta_k\right) + \tilde{C}\left(\sum_{k=0}^{i-1} \alpha_k, \sum_{k=0}^{j-1} \beta_k\right) - \tilde{C}\left(\sum_{k=0}^{i-1} \alpha_k, \sum_{k=0}^{j-1} \beta_k\right) - \tilde{C}\left(\sum_{k=0}^{i-1} \alpha_k, \sum_{k=0}^{j-1} \beta_k\right).
\] (17)
1. Introduction & formal framework

Formal framework:

Idea: use appropriate continuous extensions $\tilde{C}$ of the empirical copula for modeling the $\{p_{i,j}\}_{i,j \in \mathbb{Z}^+}$ (cf. Bernstein approach).

Lemma 1: Let $(U,V)$ be a pair of rv’s with given copula $\tilde{C}$. Then the $(X,Y)$ with $\{p_{i,j}\}_{i,j \in \mathbb{Z}^+}$ as joint probabilities from Examples 1, 2 and 3 can be constructed as follows (note: $|z| = \min\{x \in \mathbb{R} | x \geq z\}, |z| = \max\{x \in \mathbb{R} | x \leq z\}$):

Example 1: $X = \lfloor aU \rfloor, \ Y = \lfloor bV \rfloor$,

Example 2: $X = \frac{aU}{1-U}, \ Y = \frac{bV}{1-V}$,

Example 3: $X = \frac{-\ln(1-U)}{\ln(a+1)-\ln a}, \ Y = \frac{-\ln(1-V)}{\ln(b+1)-\ln b}$. 

Continuous partition-of-unity copulas and their application to risk management and other fields
2. Construction from given data

Assumptions:

- rv's \((X_i, Y_i), i = 1, \ldots, n\) iid pairs with pairwise copula \(C\)
- continuous marginal distributions (no ties!)
- \(\mathbf{R}_X = (R_{x1}, \ldots, R_{xn})^T\) and \(\mathbf{R}_Y = (R_{y1}, \ldots, R_{yn})^T\) being the ranks of the vectors \(X = (X_1, \ldots, X_n)\) and \(Y = (Y_1, \ldots, Y_n)\), resp.

The empirical copula is usually identified with the point set of relative ranks, i.e. \(\left\{ \frac{r_{11}}{n+1}, \frac{r_{21}}{n+1}, \ldots, \frac{r_{in}}{n+1}, \frac{r_{jn}}{n+1} \right\}\).

For the construction of appropriate \(\{p_{ij}\}_{i,j \in \mathbb{Z}^+}\) we need . . .
2. Construction from given data

Lemma 2: Let \( C_1, \ldots, C_n \) be arbitrary bivariate copulas with densities \( c_1, \ldots, c_n \) and \((U_i, V_i)\) independent random vectors with the copula \( C_i \) for each pair \((U_i, V_i), i = 1, \ldots, n\). Let further \( r_1 = (r_{11}, \ldots, r_{1n})^\top \) and \( r_2 = (r_{21}, \ldots, r_{2n})^\top \) be arbitrary permutations of \((1,2,\ldots,n)^\top\) and the random variable \( I \) follow a discrete uniform distribution over the set \( \{1,2,\ldots,n\} \), independent of the \((U_i, V_i)\) for \( i = 1, \ldots, n \). Then the random vector \((U, V)\) defined by

\[
U := \frac{r_1 - 1}{n} + \frac{U_i}{n}, \quad V := \frac{r_2 - 1}{n} + \frac{V_i}{n}
\]  

(18)

has continuous marginal uniform distributions over \((0,1)\) and density

\[
c(u,v) = n \sum_{k=1}^{n} \left[ \left| \frac{r_{1k} - 1}{n} \right| \right] \left[ \left| \frac{r_{2k} - 1}{n} \right| \right] (u) \cdot 1_{\left[ \frac{r_{1k} - 1}{n}, \frac{r_{1k}}{n} \right]} (u) \cdot 1_{\left[ \frac{r_{2k} - 1}{n}, \frac{r_{2k}}{n} \right]} (v) \cdot c_k (nu - r_{1k} + 1, nv - r_{2k} + 1), \quad u, v \in (0,1).
\]  

(19)
2. Construction from given data

To obtain a realization of $(U,V)$ first select a pair $(r_{yi}, r_{zi})$ from the set of all permutation pairs by a discrete uniform distribution over \{1,2,...,n\} and then draw a sample from $C_i$ rescaled to the interval $\left[ \frac{r_{yi} - 1}{n}, \frac{r_{yi}}{n} \right] \times \left[ \frac{r_{zi} - 1}{n}, \frac{r_{zi}}{n} \right]$. This corresponds to a particular patchwork copula construction, see e.g. Durante et al. (2013).

The following graphs show different realizations of such a construction for $n = 10$ and $r_1 = (3,1,4,2,8,6,5,7,9,10)^T$ and $r_2 = (8,5,7,2,4,6,1,3,9,10)^T$, with local Gaussian copulas for given fixed pairwise correlation $\rho$.
2. Construction from given data

\[ \rho = 0.75 \quad \rho = 0.90 \quad \rho = -0.75 \quad \rho = -0.90 \]
2. Construction from given data

Models of particular interest:

For the rook copula see Cottin and Pfeifer (2014); for the so-called shuffles of M (Fréchet shuffles) see e.g. Nelsen (2007), chapter 3.2.3.

\[ \rho = 0 \quad \text{rook copula} \]

\[ \rho = 1 \quad \text{upper Fréchet shuffle} \]

\[ \rho = -1 \quad \text{lower Fréchet shuffle} \]
3. Case studies

- Data set treated in Cottin and Pfeifer (2014), Example 4.2 and Pfeifer et al. (2019), Section 6.

- Effects of the kind of dependence modeling (w/ or w/o upper tail dependence) on the VaR for the aggregated portfolio with various risk levels; similarly to Maciag et al. (2016)
3. Case studies

scatterplot of original data  
scatterplot of ranks
3. Case studies

5,000 simulated pairs of the data-driven copulas and empirical copula (large points):

- upper Fréchet shuffle
- rook copula
- lower Fréchet shuffle

binomial copula, $a = 22$, $b = 27$
3. Case studies

5,000 simulated pairs of the data-driven copulas and empirical copula (large points):

- upper Fréchet shuffle
- rook copula
- lower Fréchet shuffle
- negative binomial copula, $a = 17$, $b = 22$
3. Case studies

5,000 simulated pairs of the data-driven copulas and empirical copula (large points):

- upper Fréchet shuffle
- rook copula
- lower Fréchet shuffle

Poisson copula, \( a = 17, b = 22 \)
3. Case studies

Q^(u) for aggregated risk based on the largest 100,000 observations from a total of 10^6 simulations, with estimated marginal distributions:
Q^(u) for aggregated risk based on the largest 100,000 observations from a total of 10^6 simulations, with estimated marginal distributions:

empirical quantile functions Q^(u), negative binomial copula
3. Case studies

$Q^\wedge(u)$ for aggregated risk based on the largest 100,000 observations from a total of $10^6$ simulations, with estimated marginal distributions:

![Empirical quantile functions $Q^\wedge(u)$ for Poisson and Fréchet copulas.](image)

Empirical quantile functions $Q^\wedge(u)$, Poisson copula.
4. Extension to arbitrary dimensions

Assumptions:

- \( \{ \varphi_{ki}(u) \}_{i \in \mathbb{Z}^+} \) for \( k = 1, \ldots, d' \) discrete probabilities with
  \[
  \sum_{i=0}^{\infty} \varphi_{ki}(u) = 1 \text{ for } u \in (0, 1) \tag{20}
  \]
  \[
  \int_{0}^{1} \varphi_{ki}(u) du = \alpha_{ki} > 0 \text{ for } i \in \mathbb{Z}^+. \tag{21}
  \]

- \( \{ p_i \}_{i \in \mathbb{Z}^+} \) is a distribution of an arbitrary discrete \( d \)-dimensional random vector \( Z \) over \( \mathbb{Z}^{+d} \) where, with \( i = (i_1, \ldots, i_d) \),
  \[
  P(Z = i) = p_i, \ i \in \mathbb{Z}^{+d}. \tag{22}
  \]

- marginal distributions with
  \[
  P(Z_k = i) = \alpha_{ki}, \ i \in \mathbb{Z}^+, k = 1, \ldots, d. \tag{23}
  \]
4. Extension to arbitrary dimensions

Then

\[ c(u) := \sum_{i \in \mathbb{Z}^+} \rho_i \prod_{k=1}^{d} \varphi_{k, i}(u_k), \quad u = (u_1, \ldots, u_d) \in (0,1)^d \]

(24)

defines the density of a \( d \)-variate copula, which is again called generalized partition-of-unity copula. Alternatively, we can rewrite (24) again as

\[ c(u) = \sum_{i \in \mathbb{Z}^+} \rho_i \prod_{k=1}^{d} f_{k, i}(u_k), \quad u = (u_1, \ldots, u_d) \in (0,1)^d \]

(25)

where the \( f_{k, i}(\cdot) = \frac{\varphi_{k,i}(\cdot)}{\alpha_{ki}} \), \( i \in \mathbb{Z}^+, k = 1, \ldots, d \) denote the Lebesgue densities induced by the \( \{\varphi_{k,i}(u)\}_{i \in \mathbb{Z}^+} \).
5. Bibliography / References


