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Continuous partition-of-unity copulas and their application to risk management and other fields

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Agenda

1. Introduction & formal framework
2. Construction from given data
3. Case studies
4. Extension to arbitrary dimensions
5. Bibliography / References

1. Introduction & formal framework

Motivation:

- Extension of discrete partition-of-unity copulas to the continuous case
- Construction of new multivariate copulas on the basis of a generalized infinite partition-of-unity approach (extendable to the uncountable infinite case)
- Construction allows for tail-dependence as well as for asymmetry
- Can be easily implemented for risk management purposes
- Particular interest: how to fit such copulas to highly asymmetric data?

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1. Introduction & formal framework

Formal framework:

Let $\mathbb{Z}^+ = \{0,1,2,3,\cdots\}$ and suppose that $\{\varphi_i(u)\}_{i\in\mathbb{Z}^+}$ and $\{\psi_j(v)\}_{j\in\mathbb{Z}^+}$ are non-negative maps defined on (0,1) such that:

$$\sum_{i=0}^{\infty}\varphi_i(u) = \sum_{j=0}^{\infty}\psi_j(v) = 1 \qquad (1)$$

$$\alpha_i := \int_0^1 \varphi_i(u)\,du > 0,\ \ \beta_j := \int_0^1 \psi_j(v)\,dv > 0,\ \ i,j\in\mathbb{Z}^+. \qquad (2)$$

- $\{\varphi_i(u)\}_{i\in\mathbb{Z}^+}$ and $\{\psi_j(v)\}_{j\in\mathbb{Z}^+}$ can be thought of representing discrete distributions over $\mathbb{Z}^+$ with parameters *u* and *v*, resp.
- The sequences $\{\alpha_i\}_{i\in\mathbb{Z}^+}$ and $\{\beta_j\}_{j\in\mathbb{Z}^+}$ represent the probabilities of the corresponding mixed distributions.

1. Introduction & formal framework

Formal framework:

Let $\left\{p_{ij}\right\}_{i,j\in\mathbb{Z}^+}$ represent the probabilities of an arbitrary discrete bivariate distribution over $\mathbb{Z}^+\times\mathbb{Z}^+$ with marginal distributions given by

$$p_{i\bullet}=\sum_{j=0}^{\infty}p_{ij}=\alpha_i \text{ and } p_{\bullet j}=\sum_{i=0}^{\infty}p_{ij}=\beta_j \text{ for } i,j\in\mathbb{Z}^+. \tag{3}$$

Then

$$c(u,v):=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}p_{ij}\,\frac{\varphi_i(u)}{\alpha_i}\frac{\psi_j(v)}{\beta_j},\; u,v\in(0,1) \tag{4}$$

defines the density of a bivariate copula, called (infinite) partition-of-unity copula.

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1. Introduction & formal framework

Formal framework:

From a "dual" point of view, we can rewrite (4) as

$$c(u,v) := \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p_{ij} \frac{\varphi_i(u)}{\alpha_i}\frac{\psi_j(v)}{\beta_j} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p_{ij} f_i(u) g_j(v),\ u,v \in (0,1) \qquad (5)$$

where

$$f_i(\bullet) = \frac{\varphi_i(\bullet)}{\alpha_i} \quad \text{and} \quad g_j(\bullet) = \frac{\psi_j(\bullet)}{\beta_j},\ i,j \in \mathbb{Z}^+ \qquad (6)$$

denote the densities induced by $\{\varphi_i(u)\}_{i\in\mathbb{Z}^+}$ and $\{\psi_j(v)\}_{j\in\mathbb{Z}^+}$. This means that the copula density $c(u,v)$ can also be seen as a mixture of product densities.

1. Introduction & formal framework

Formal framework:

Example 1 (Binomial distributions – Bernstein copula):

For fixed integers $a,b \geq 2$, consider the family of binomial distributions given by their point masses

$$\varphi_{a,i}(u) = \begin{cases} \binom{a-1}{i} u^i (1-u)^{a-1-i}, & i = 0,\cdots,a-1 \\ 0, & i \geq a \end{cases} \quad (7)$$

and $\psi_{b,j}(v) = \varphi_{b,j}(v)$ for $i,j \in \mathbb{Z}^+$ and $(u,v) \in (0,1)$.

We have

1. Introduction & formal framework

Formal framework:

Example 1 (Binomial distributions – Bernstein copula):

$$\alpha_{a,i} = \int_0^1 \varphi_{a,i}(u)\,du = \frac{1}{a}, \qquad \beta_{b,j} = \int_0^1 \psi_{b,j}(v)\,dv = \frac{1}{b}, \tag{8}$$

$f_{a,i}$ and $g_{b,j}$ are densities of a beta distribution with parameters $(i, a+1-i)$ and $(j, b+1-j)$ resp., $p_{i\bullet} = \frac{1}{a}$ and $p_{\bullet j} = \frac{1}{b}$, so

$$c_{a,b}(u,v) = ab\sum_{i=0}^{a}\sum_{j=0}^{b} p_{ij}\binom{a-1}{i}\binom{b-1}{j}u^{i-1}(1-u)^{a-i}v^{j-1}(1-v)^{b-j},\ u,v \in (0,1) \tag{9}$$

which is the density of a bivariate Bernstein copula.

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1. Introduction & formal framework

Formal framework:

Example 2 (Negative binomial distributions):

For fixed integers $a, b \geq 2$, consider the family of negative binomial distributions given by their point masses

$$\varphi_{a,i}(u) = \binom{a+i-1}{i} u^i (1-u)^a, \tag{10}$$

and $\psi_{b,j}(v) = \varphi_{b,j}(v)$ for $i, j \in \mathbb{Z}^+$ and $(u,v) \in (0,1)$.

We have

1. Introduction & formal framework

Formal framework:

Example 2 (Negative binomial distributions):

$$\alpha_{a,i} = \int_0^1 \varphi_{a,i}(u)\,du = \frac{a}{(a+i)(a+i+1)},\ \beta_{b,j} = \frac{b}{(b+j)(b+j+1)}, \tag{11}$$

$f_{a,i}$ and $g_{b,j}$ are densities of a beta distribution with parameters $(i+1, a+1)$ and $(j+1, b+1)$, $p_{i\bullet} = \dfrac{a}{(a+i)(a+1+i)}$, $p_{\bullet j} = \dfrac{b}{(b+j)(b+1+j)}$, so

$$c_{a,b}(u,v) = (a+1)(b+1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p_{ij}\binom{a+i+1}{i}\binom{b+j+1}{j}u^i(1-u)^a v^j(1-v)^b, u,v \in (0,1). \tag{12}$$

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1. Introduction & formal framework

Formal framework:

Example 2 (Negative binomial distributions):

Negative binomial copulas typically show a tail dependence:

β	1	2	3	4	5	6	7	8	9	10
$\lambda_U(\beta)$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{93}{128}$	$\frac{193}{256}$	$\frac{793}{1024}$	$\frac{1619}{2048}$	$\frac{26333}{32768}$	$\frac{53381}{65536}$	$\frac{215955}{262144}$

with $\lambda_U(\beta) = \lim_{t \uparrow 1} \frac{\int_t^1 \int_t^1 c_\beta(u,v)\,du\,dv}{1-t} = \frac{2\Gamma(2\beta)}{\Gamma^2(\beta)} \cdot \int_0^1 \int_0^1 \frac{x^\beta y^\beta}{(x+y)^{2\beta+1}}\,dx\,dy = 1 - \frac{\binom{2\beta}{\beta}}{4^\beta} \sim 1 - \frac{1}{\sqrt{\pi\beta}}$

for large β.

1. Introduction & formal framework

Formal framework:

Example 3 (Poisson distributions):

For fixed $a, b > 0$ consider the family of Poisson distributions given by their point masses

$$\varphi_{a,i}(u) = (1-u)^a \frac{a^i L(u)^i}{i!}, \tag{13}$$

$L(u) := -\ln(1-u)$, $\psi_{b,j}(v) = \varphi_{b,j}(v)$, $i, j \in \mathbb{Z}^+$, $(u,v) \in (0,1)$.

We have

1. Introduction & formal framework

Formal framework:

Example 3 (Poisson distributions):

$$\alpha_{a,i} = \int_0^1 \varphi_{a,i}(u)\,du = \left(\frac{a}{a+1}\right)^i \left(1-\frac{a}{a+1}\right),\ \beta_{b,j} = \left(\frac{b}{b+1}\right)^j \left(1-\frac{b}{b+1}\right) \tag{14}$$

which correspond to geometric distributions over $\mathbb{Z}^+$ with means a and b,

$$p_{i\bullet} = \left(\frac{a}{a+1}\right)^i \left(1-\frac{a}{a+1}\right) = \frac{a^i}{(a+1)^{i+1}},\ p_{\bullet j} = \frac{b^j}{(b+1)^{j+1}},\ i,j \in \mathbb{Z}^+, \tag{15}$$

$$c_{a,b}(u,v) = (a+1)(b+1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p_{ij}\,\frac{(a+1)^i(b+1)^j}{i!\,j!} L^i(u)(1-u)^a L^j(v)(1-v)^b,\ u,v \in (0,1). \tag{16}$$

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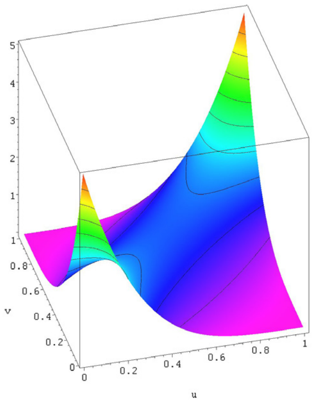
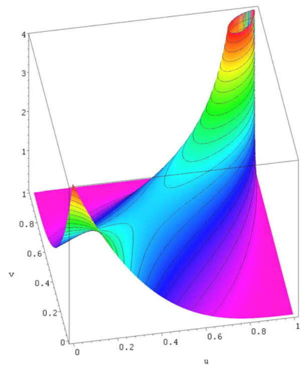
1. Introduction & formal framework

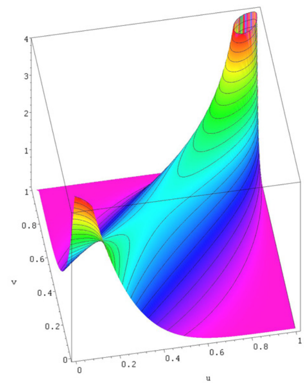
Formal framework:



Bernstein copula, $m = 3$;
no tail dependence



Negative binomial copula, $\beta = 3$;
$\lambda_U(\beta) = 0.6875$



Poisson copula, $\gamma = 5$
no tail dependence

1. Introduction & formal framework

Formal framework:

Remark: Sklar's theorem provides a general method to construct pairs of discrete r.v.'s (X,Y) with joint probabilities $p_{ij} = P(X=i, Y=j)$ and marginal probabilities $\{\alpha_i\}_{i\in\mathbb{Z}^+}$ and $\{\beta_j\}_{j\in\mathbb{Z}^+}$:

Assume quantile functions Q_X, Q_Y of X,Y and a pair of rv's (U,V) with a given copula $\tilde{C}$. Then $(X,Y) = \big(Q_X(U), Q_Y(V)\big)$ has joint probabilities

$$
\begin{aligned}
p_{ij} &= P(X=i, Y=j) = P\left(\sum_{k=0}^{i-1}\alpha_k < U \le \sum_{k=0}^{i}\alpha_k, \sum_{k=0}^{j-1}\beta_k < V \le \sum_{k=0}^{j}\beta_k\right) \\
&= \tilde{C}\left(\sum_{k=0}^{i}\alpha_k, \sum_{k=0}^{j}\beta_k\right) + \tilde{C}\left(\sum_{k=0}^{i-1}\alpha_k, \sum_{k=0}^{j-1}\beta_k\right) - \tilde{C}\left(\sum_{k=0}^{i-1}\alpha_k, \sum_{k=0}^{j}\beta_k\right) - \tilde{C}\left(\sum_{k=0}^{i}\alpha_k, \sum_{k=0}^{j-1}\beta_k\right).
\end{aligned}
\tag{17}
$$

1. Introduction & formal framework

Formal framework:

Idea: use appropriate continuous extensions $\tilde{C}$ of the empirical copula for modeling the $\{p_{ij}\}_{i,j\in\mathbb{Z}^+}$ (cf. Bernstein approach).

Lemma 1: Let (U,V) be a pair of rv's with given copula $\tilde{C}$. Then the (X,Y) with $\{p_{ij}\}_{i,j\in\mathbb{Z}^+}$ as joint probabilities from Examples 1, 2 and 3 can be constructed as follows (note: $\lceil z\rceil=\min\{x\in\mathbb{R}\,|\,x\geq z\},\lfloor z\rfloor=\max\{x\in\mathbb{R}\,|\,x\leq z\}$):

Example 1: $X=\lceil aU\rceil,\ Y=\lceil bV\rceil,$

Example 2: $X=\left\lfloor\dfrac{aU}{1-U}\right\rfloor,\ Y=\left\lfloor\dfrac{bV}{1-V}\right\rfloor,$

Example 3: $X=\left\lfloor\dfrac{-\ln(1-U)}{\ln(a+1)-\ln a}\right\rfloor,\ Y=\left\lfloor\dfrac{-\ln(1-V)}{\ln(b+1)-\ln b}\right\rfloor.$

2. Construction from given data

Assumptions:

- rv's $(X_i, Y_i),\ i = 1,\ldots,n$ iid pairs with pairwise copula C
- continuous marginal distributions (no ties!)
- $\mathbf{R}_\mathrm{X} = (R_{11},\cdots,R_{1n})^T$ and $\mathbf{R}_\mathrm{Y} = (R_{21},\cdots,R_{2n})^T$ being the ranks of the vectors $\mathbf{X} = (X_1,\cdots,X_n)$ and $\mathbf{Y} = (Y_1,\cdots,Y_n)$, resp.

The empirical copula is usually identified with the point set of relative ranks, i.e. $\left\{\left(\frac{r_{11}}{n+1},\frac{r_{21}}{n+1}\right),\cdots,\left(\frac{r_{1n}}{n+1},\frac{r_{2n}}{n+1}\right)\right\}$.

For the construction of appropriate $\left\{p_{ij}\right\}_{i,j\in\mathbb{Z}^+}$ we need . . .

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2. Construction from given data

Lemma 2: Let $C_1,\cdots,C_n$ be arbitrary bivariate copulas with densities $c_1,\cdots,c_n$ and (U_i,V_i) independent random vectors with the copula C_i for each pair (U_i,V_i), $i=1,\cdots,n$. Let further $\mathbf{r}_1=(r_{11},\cdots,r_{1n})^T$ and $\mathbf{r}_2=(r_{21},\cdots,r_{2n})^T$ be arbitrary permutations of $(1,2,\cdots,n)^T$ and the random variable I follow a discrete uniform distribution over the set $\{1,2,\cdots,n\}$, independent of the (U_i,V_i) for $i=1,\cdots,n$. Then the random vector (U,V) defined by

$$U:=\frac{r_{1I}-1+U_I}{n},\ V:=\frac{r_{2I}-1+V_I}{n} \tag{18}$$

has continuous marginal uniform distributions over $(0,1)$ and density

$$c(u,v)=n\sum_{k=1}^{n}\mathbb{1}_{\left[\frac{r_{1k}-1}{n},\frac{r_{1k}}{n}\right]}(u)\cdot\mathbb{1}_{\left[\frac{r_{2k}-1}{n},\frac{r_{2k}}{n}\right]}(v)\cdot c_k\left(nu-r_{1k}+1,nv-r_{2k}+1\right),\ u,v\in(0,1). \tag{19}$$

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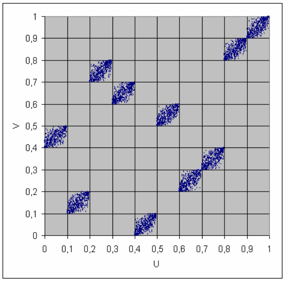
2. Construction from given data

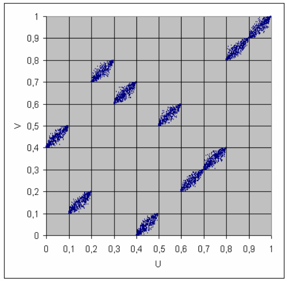
To obtain a realization of (U,V) first select a pair (r_{1i},r_{2i}) from the set of all permutation pairs by a discrete uniform distribution over $\{1,2,\ldots,n\}$ and then draw a sample from C_i rescaled to the interval $\left(\frac{r_{1i}-1}{n},\frac{r_{1i}}{n}\right]\times\left(\frac{r_{2i}-1}{n},\frac{r_{2i}}{n}\right]$.
This corresponds to a particular patchwork copula construction, see e.g. Durante et al. (2013).

The following graphs show different realizations of such a construction for $n=10$ and $\mathbf{r}_1=(3,1,4,2,8,6,5,7,9,10)^T$ and $\mathbf{r}_2=(8,5,7,2,4,6,1,3,9,10)^T$, with local Gaussian copulas for given fixed pairwise correlation ρ:

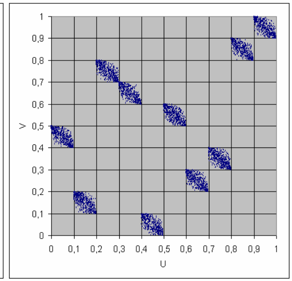
2. Construction from given data



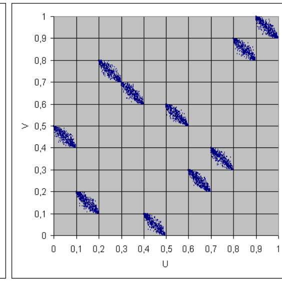
$\rho = 0.75$



$\rho = 0.90$



$\rho = -0.75$



$\rho = -0.90$

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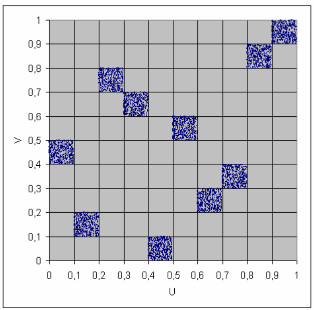
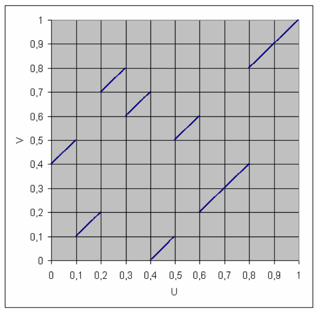
2. Construction from given data

Models of particular interest:

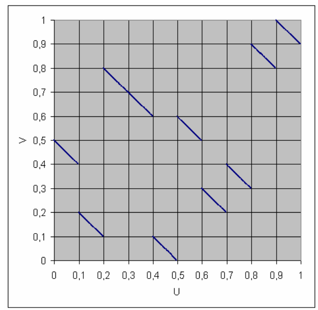
For the rook copula see Cottin and Pfeifer (2014); for the so-called shuffes of M (Fréchet shuffles) see e.g. Nelsen (2007), chapter 3.2.3.



$\rho = 0$
rook copula



$\rho = 1$
upper Fréchet shuffle



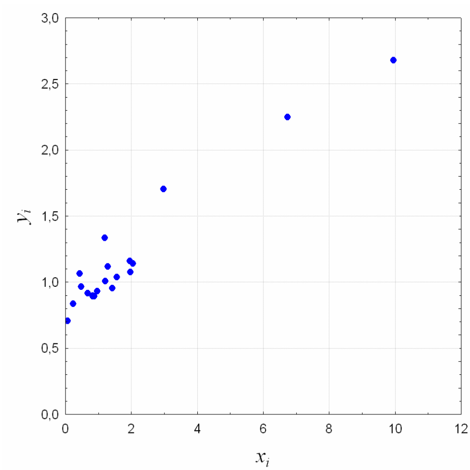
$\rho = -1$
lower Frécht shuffle

3. Case studies

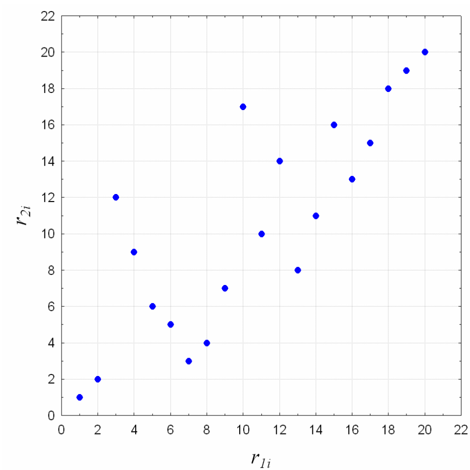
- Data set treated in Cottin and Pfeifer (2014), Example 4.2 and Pfeifer et al. (2019), Section 6.

- Effects of the kind of dependence modeling (w/ or w/o upper tail dependence) on the VaR for the aggregated portfolio with various risk levels; similarly to Maciag et al. (2016)

3. Case studies



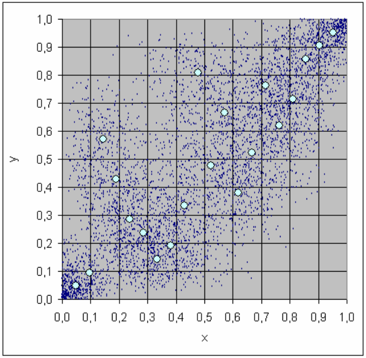
scatterplot of original data

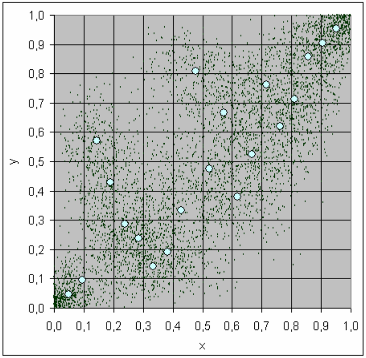


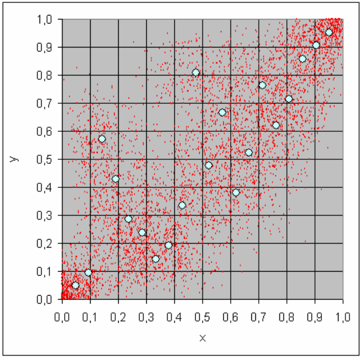
scatterplot of ranks

3. Case studies

5,000 simulated pairs of the data-driven copulas and empirical copula (large points):





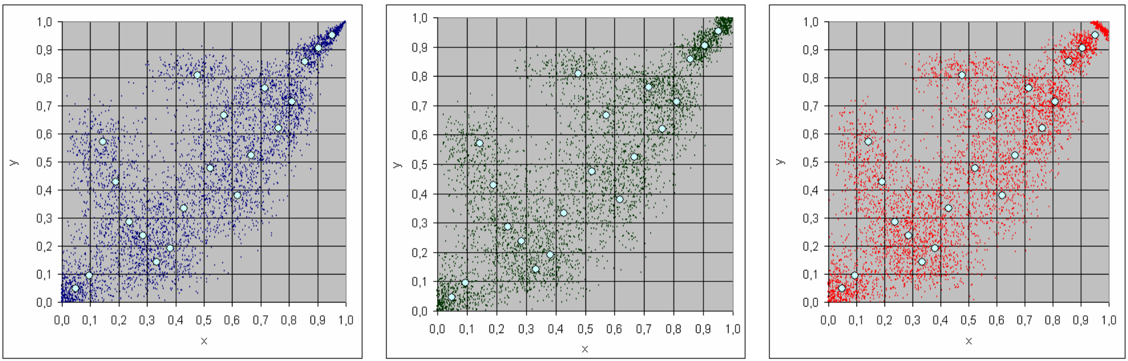


upper Fréchet shuffle | rook copula | lower Fréchet shuffle

binomial copula, $a = 22,\ b = 27$

3. Case studies

5,000 simulated pairs of the data-driven copulas and empirical copula (large points):

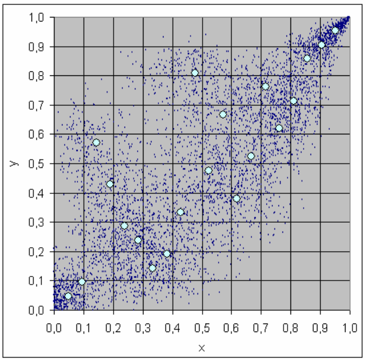


upper Fréchet shuffle rook copula lower Fréchet shuffle

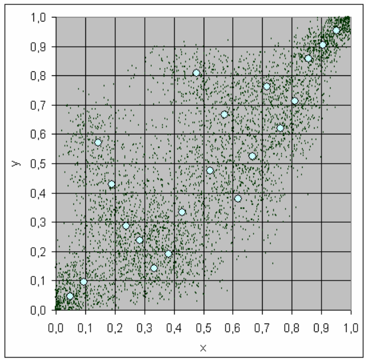
negative binomial copula, $a = 17,\ b = 22$

3. Case studies

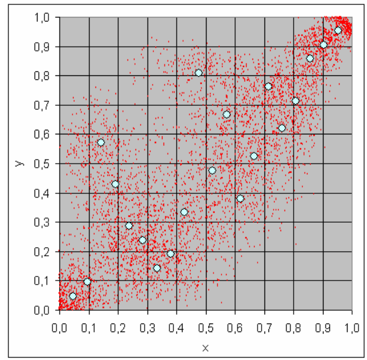
5,000 simulated pairs of the data-driven copulas and empirical copula (large points):



upper Fréchet shuffle



rook copula

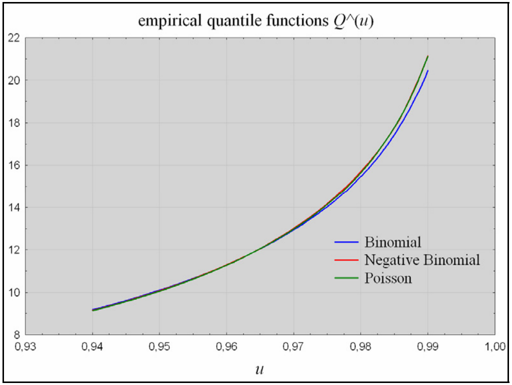


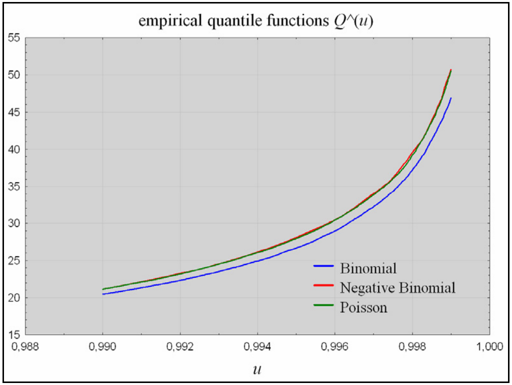
lower Fréchet shuffle

Poisson copula, $a = 17$, $b = 22$

3. Case studies

Q^(u) for aggregated risk based on the largest 100,000 observations from a total of 10^6 simulations, with estimated marginal distributions:





empirical quantile functions $Q\,^\wedge(u)$, upper Fréchet shuffle

3. Case studies

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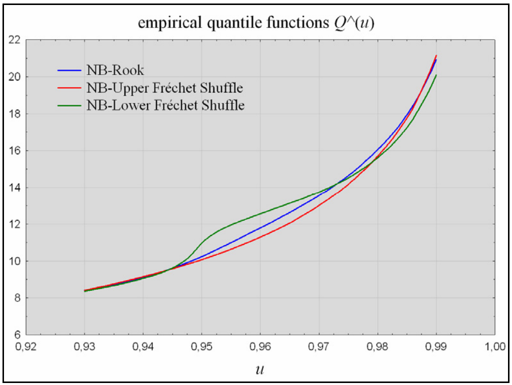
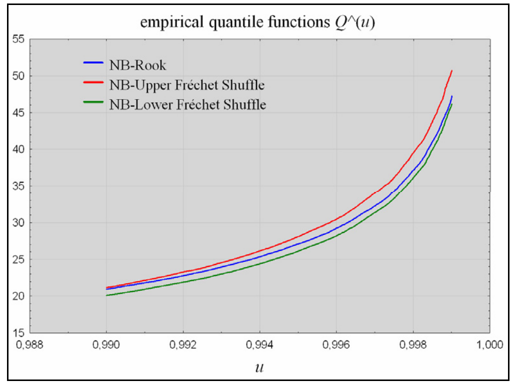
Q^(*u*) for aggregated risk based on the largest 100,000 observations from a total of 10^6 simulations, with estimated marginal distributions:

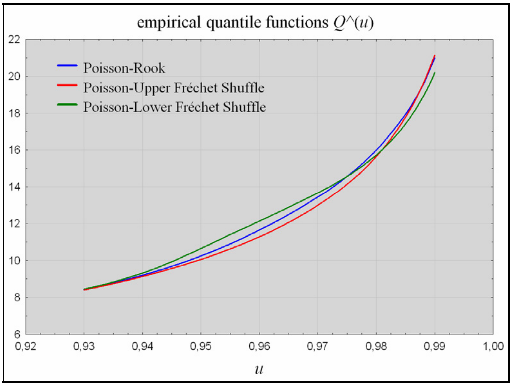


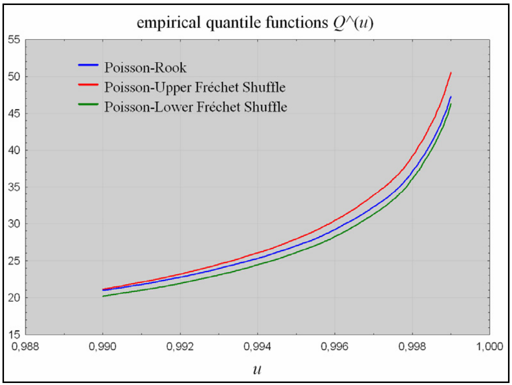


empirical quantile functions $Q^\wedge(u)$, negative binomial copula

3. Case studies

$\hat{Q}(u)$ for aggregated risk based on the largest 100,000 observations from a total of 10^6 simulations, with estimated marginal distributions:





empirical quantile functions $Q^\wedge(u)$, Poisson copula

4. Extension to arbitrary dimensions

Assumptions:

- $\{\varphi_{ki}(u)\}_{i\in\mathbb{Z}^+}$ for $k=1,\ldots,d$ discrete probabilities with

$$\sum_{i=0}^{\infty}\varphi_{ki}(u)=1 \text{ for } u\in(0,1) \tag{20}$$

$$\int_0^1 \varphi_{ki}(u)\,du=\alpha_{ki}>0 \text{ for } i\in\mathbb{Z}^+. \tag{21}$$

- $\{p_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}^{+d}}$ is a distribution of an arbitrary discrete *d*-dimensional random vector **Z** over $\mathbb{Z}^{+d}$ where, with $\mathbf{i}=(i_1,\cdots,i_d)$,

$$P(\mathbf{Z}=\mathbf{i})=p_{\mathbf{i}},\ \mathbf{i}\in\mathbb{Z}^{+d}. \tag{22}$$

- marginal distributions with

$$P(Z_k=i)=\alpha_{ki},\ i\in\mathbb{Z}^+, k=1,\cdots,d. \tag{23}$$

4. Extension to arbitrary dimensions

Then

$$c(\mathbf{u}) := \sum_{\mathbf{i}\in\mathbb{Z}^{+d}} \frac{p_{\mathbf{i}}}{\prod_{k=1}^{d}\alpha_{k,i_k}} \prod_{k=1}^{d} \varphi_{k,i_k}(u_k),\ \mathbf{u} = (u_1,\cdots,u_d) \in (0,1)^d \tag{24}$$

defines the density of a d-variate copula, which is again called *generalized partition-of-unity copula*. Alternatively, we can rewrite (24) again as

$$c(\mathbf{u}) = \sum_{\mathbf{i}\in\mathbb{Z}^{+d}} p_{\mathbf{i}} \prod_{k=1}^{d} f_{k,i_k}(u_k),\ \mathbf{u} = (u_1,\cdots,u_d) \in (0,1)^d \tag{25}$$

where the $f_{ki}(\bullet) = \dfrac{\varphi_{ki}(\bullet)}{\alpha_{ki}},\ i \in \mathbb{Z}^+, k = 1,\cdots,d$ denote the Lebesgue densities induced by the $\{\varphi_{ki}(u)\}_{i\in\mathbb{Z}^+}$.

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