Optimal reinsurance under terminal value constraints[1]

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[1] Paper available at https://arxiv.org/abs/2203.16108

1. Introduction
   - Motivation
   - Insurance surplus dynamics
   - Objective
   - Connection to mean-variance problem
   - Connection with finance

Motivation

- The choice of reinsurance is important for insurance decisions and risk management in general (e.g. Albrecher, Beirlant, and Teugels 2017)
- This is often a choice made under a number of possible constraints:
  - imposed either externally (regulator) or
  - internally (Board / risk manager)
- "Traditional" actuarial *stability criteria* (Bühlmann (1970)) are used to control *actuarial surplus processes*; these include the probability of ruin (Cramér (1955),Lundberg (1909)), and dividends (Finetti (1957))
- Can optimal portfolio techniques from finance ("à la Merton") be applied here?
  (as, for instance, in Korn (1997), Basak and Shapiro (2001), Korn and Wiese (2008), Gu, Steffensen, and Zheng (2021))
- Furthermore, can an LQ type criterion (mean-variance) be used, *while still staying true to the insurance context*?

1 Introduction

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Insurance surplus dynamics

Surplus without any reinsurance:

$$d\widetilde{X}_t = adt + \sigma dW_t, \quad \widetilde{X}_0 = x < \tilde{k} \tag{1}$$

After reinsurance:

$$d\widetilde{X}_t^\pi = (a - b\pi_t)dt + (1 - \pi_t)\sigma dW_t, \quad \widetilde{X}_0^\pi = x < \tilde{k} \tag{2}$$

Note:

- Reinsurance is non cheap; $b > a$
- If $\pi_t \equiv 1$ then

$$d\widetilde{X}_t^1 = (a - b)dt, \quad \widetilde{X}_0^1 = x < \tilde{k}$$

The Brownian approximation is standard
(see, e.g., Asmussen and Albrecher 2010)

1 Introduction

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Objective

Our objective is (quadratic utility):

$$\max_{\pi \in \Pi} E_{t,x} \left[-\frac{1}{2} \left(\widetilde{k} - \widetilde{X}_T^{\pi} \right)^2 \right], \tag{3}$$

where Π is the set of admissible controls. Note:

- Once $\widetilde{X}_t^{\pi} = \widetilde{k}$, the optimal strategy is to take $\pi_s = 1$ for $s \in [t, T]$
- Hence it makes sense that $x < \tilde{k}$, and that it remains so for $t \geq 0$ so that $X_t^{\pi} \leq \tilde{k}$.
- Hence, there are two important levels:
  - $x = \widetilde{k}$ (an upper bound) and
  - $x = \widetilde{C}$ (a lower solvency threshold which may be strict or not)

There isn't much actuarial literature with such LQ optimisation (an exception is Steffensen 2006 in life)

Furthermore, note that π may be outside the interval $[0, 1]$

- $\pi < 0$: take over more of the risk than originally had
- $\pi > 1$: pass on more risk than he actually had – go short in insurance risk
- $\pi > 1$ is intuitively never optimal (verified by simulations)
- We do not formally consider the additional constraints $0 \leq \pi \leq 1$ ("borrowing and short-selling constraints")

Connection to mean-variance problem

Varying here over $\widetilde{k}$ is equivalent to varying over the mean condition in a mean-variance problem in the form

$$\min_{\pi\in\Pi, E[X_T^\pi]=p} \mathrm{Var}[\widetilde{X}_T^\pi] \text{ for some } p\in\mathbb{R}; \tag{4}$$

a well known objective in the finance literature. However, note:

- the subscript in $E_{t,x}$ denotes a conditional expectation
- in the well-known mean-variance framework, the mean condition is an *unconditional* mean
- so our context can be interpreted as a mean-variance problem from time 0 only, and not at subsequent time points
- in other words, we are in the traditional mean-variance case with pre-commitment (to the mean, around which quadratic distance is measured) and not in the modern version where the variance is actually conditional variance
  (and which requires different optimisation techniques)

1 Introduction

Connection with finance

In finance (think optimal portfolio as in Merton):

- geometric Brownian dynamics for the (market) porfolio value
- usually includes optimal consumption (but not always)
- power utility

What's different here in the insurance context:

- Brownian dynamics for the insurance surplus (additive, can become negative)
- reinsurance is akin to risk investment decision (in symmetry)
- quadratic utility (LQ)

Why not power utility?

- Power utility does not work for processes that can go negative (becomes minus infinity)
- Note we could still use power utility with a strict constraint (as long as $C > 0$); we just did not do that.

Why not exponential utility?

- We could have chosen that, but we were interested in the connection with mean-variance.
- There are also additional interpretations with the quadratic formulation.

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2 Solvency / stability constraints

- Constraints considered in the paper
- In the Merton problem

Constraints considered in the paper

1. No constraint
2. Strict: $X_T^\pi \geq C$
3. Probability (VaR): $\Pr[X_T^\pi \geq C] \geq 1 - \epsilon$
4. Expected shortfall under $\mathbb{P}$: $E[(C - X_T^\pi)_+] \leq \nu$
   - Basel III (and onwards) moved from VaR to ES for solvency assessment
   - ES (vs VaR) has been shown to have desirable properties (e.g., Basak and Shapiro 2001)
5. Expected shortfall under $\mathbb{Q}$; $E[Z_T(C - X_T^\pi)_+] \leq \nu$
   - Does not make sense from a regulatory perspective, but
   - Makes sense from a risk management perspective: puts an upper threshold to the cost of protecting the expected shortfall of $\widetilde{X}_t^\pi$ from $\widetilde{C}$ (up to a risk free discount factor).

Note:

- We retrieve 2. when $\epsilon \to 0$ or when $\nu \to 0$
- Closest "actuarial" literature is Korn and Wiese (2008), but they do not consider 4.-5. and only a limited version of 3. ($C \equiv 0$).

2 Solvency / stability constraints

- Constraints considered in the paper
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In the Merton problem

Why $\mathbb{P}$ vs $\mathbb{Q}$?

- problem easier to communicate under $\mathbb{P}$, but
- solution easier to communicate under $\mathbb{Q}$
- results are model-dependent anyway, so no clear winner
- $\mathbb{Q}$ solution close to $\mathbb{P}$ solution
- power options don't trade anyway

(Gu, Steffensen, and Zheng 2021)

Illustration of final payoff under $\mathbb{P}$

(from Gu, Steffensen, and Zheng 2021)

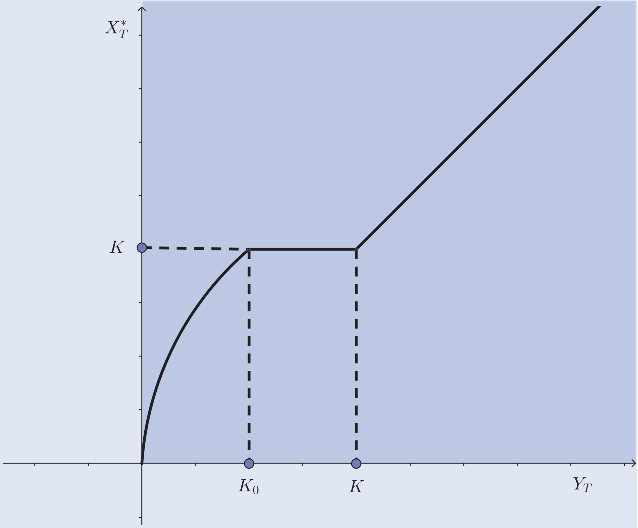
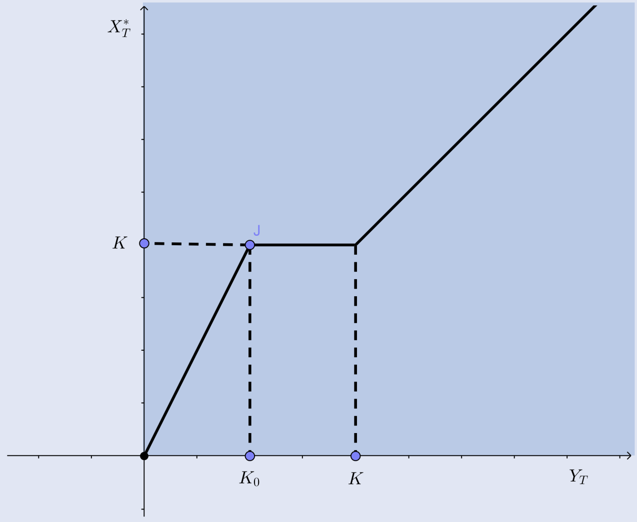


Illustration of final payoff under $\mathbb{Q}$

(from Gu, Steffensen, and Zheng 2021)



3 Optimal strategies

- Auxiliary processes
- Methodology
- Unconstrained problem
- Strict constraint: $X_T^\pi \geq C$
- Probability (VaR) constraint: $\Pr[X_T^\pi \geq C] \geq 1 - \epsilon$
- Expected shortfall constraint under $\mathbb{P}$: $E[(C - X_T^\pi)_+] \leq \nu$
- Expected shortfall constraint under $\mathbb{Q}$: $E[Z_T(C - X_T^\pi)_+] \leq \nu$

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Surplus process

First, for convenience we remove the drift corresponding to a fully reinsured process from our base process,

$$X_t^\pi := \widetilde{X}_t^\pi - (a-b)t,$$

and define associated constants

$$k := \widetilde{k} - (a-b)T,$$
$$C := \widetilde{C} - (a-b)T.$$

We consider the (equivalent) optimiwation problem,

$$\max_{\pi\in\Pi} \mathbb{E}_{t,x}\left[-\frac{1}{2}\left(k - X_T^{\pi}\right)^2\right] \tag{5}$$

with

$$dX_t^{\pi} = (1-\pi_t)(bdt + \sigma dW_t), \quad X(0) = x < \widetilde{k}.$$

In other words, we now focus purely on the residual risk dynamics left after reinsurance:

- dX_t^{π} describes the profit process generated by retaining business $(1-\pi_t)$ at time t, as compared to *full* reinsurance (which is loss making since $b > a$).

The move from X_t^{π} to $\widetilde{X}_t^{\pi}$ is straightforward.

Auxiliary process

Let $Z = \{Z_t, t \in [0, T]\}$ have dynamics

$$dZ_t = \beta Z_t dW_t, \quad Z_0 = 1, \tag{6}$$

with $\beta = -b/\sigma$ and solution,

$$Z_t = e^{-\frac{1}{2}\beta^2 t + \beta W_t}. \quad (\text{Note: } Z_t \to 0 \text{ as } W_t \to \infty). \tag{7}$$

We have that XZ forms a martingale such that, for a given terminal surplus, the path X_t^π is given by

$$X_t^\pi = E\left[\frac{Z_T}{Z_t} X_T^\pi \middle| \mathscr{F}_t\right]. \tag{8}$$

Thus, once we know X_T^π, we can evaluate this expression and then apply Itō in order to deduce the optimal proportion π_t.

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Methodology

There are typically two approaches to optimisation in this context (see, e.g., Korn 1997):

- The martingale method (e.g., Gu, Steffensen, and Zheng 2021)
- The dynamic programming approach (e.g., Kraft and Steffensen 2013)

We use the martingale method, which involves the following steps:

1. Postulate an optimal terminal wealth solution form;
2. Verify the existence of optimal parameters for that form;
3. Verify that the candidate strategy is indeed optimal (here, via path-wise optimisation)

In what follows we mainly present the (verified) terminal wealth. Proofs are available in the paper.

3 Optimal strategies

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- **Unconstrained problem**
- Strict constraint: $X_T^\pi \geq C$
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Unconstrained problem

Optimal terminal wealth is

$$X_T^{\pi_{\lambda_U^*}} := k - \lambda_U^* Z_T, \tag{9}$$

This is the simplest case. Here, the optimal value of λ, λ_U^*, is obtained noting that

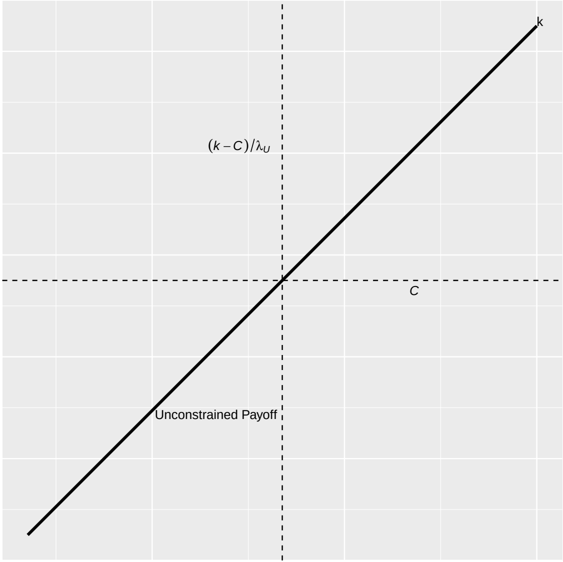
$$V_U(\lambda) := \mathbb{E}[Z_T X_T^{\pi_\lambda}].$$

should be equal to x (budget constraint), and this exists because

$$\frac{\partial}{\partial \lambda} V_U < 0, \quad \lim_{\lambda \to \infty} V_U(\lambda) = -\infty, \quad V_U(0) = k.$$

Since $-\infty < x \leq k$, we conclude that there exists a $\lambda_U^* > 0$ such that $V_U(\lambda_U^*) = x$.

Illustration of final payoff



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Strict constraint: $X_T^{\pi} \geq C$

The optimal terminal wealth is

$$X_T^{\pi_{\lambda_C^*}} := \max(k - \lambda_C^* Z_T, C), \tag{10}$$

characterised by the parameter $\lambda_C^* > 0$.

Here,

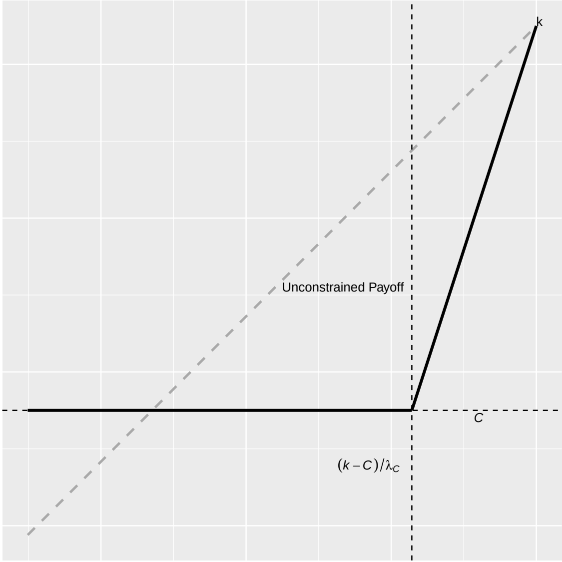
$$\frac{\partial}{\partial \lambda} V_C(\lambda) < 0, \quad \lim_{\lambda \to \infty} V_C(\lambda) = C, \quad V_C(0) = k.$$

Since $C \leq x \leq k$, we conclude that there exists a λ_C^* such that $V_U(\lambda_C^*) = x$. Furthermore, we have that

$$\lambda_C^* > \lambda_U^*, \text{ and that } X_T^{\pi_U} > X_T^{\pi_C} \text{ for } X_T^{\pi_C} > C.$$

Given the “kink” at level C, λ_C^* is adjusted upwards (steeper) to satisfy the budget constraint.

Illustration of final payoff



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Probability (VaR) constraint: $\Pr[X_T^\pi \geq C] \geq 1-\epsilon$

Optimal terminal wealth is of the form

$$X_T^{\pi_{\lambda_P^*, c_P^*}} = \begin{cases} k - \lambda_P^* Z_T, & k - \lambda_P^* Z_T \notin [c_P^*, C], \\ C, & k - \lambda_P^* Z_T \in [c_P^*, C], \end{cases} \tag{11}$$

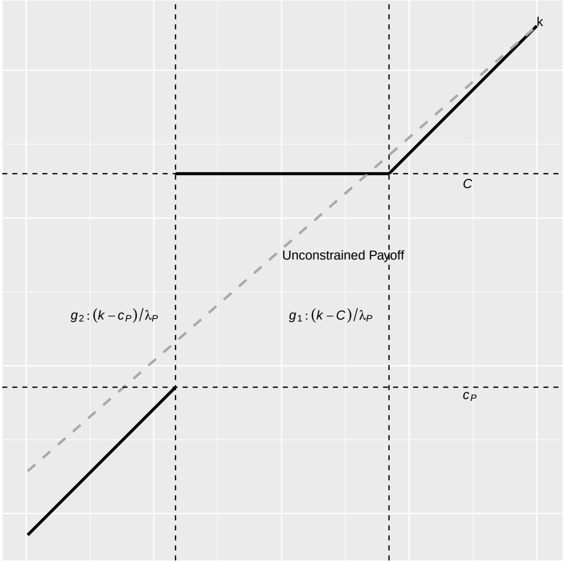
characterised by the constants $\lambda_P^*, c_P^* > 0$ with $c_P^* \leq C \leq k$.

The slope λ_P^* and location of the discontinuity (stemming from c_P^*) are obtained using conditions

$$\begin{cases} \mathbb{P}\left[X_T^{\pi_{\lambda_P^*, c_P^*}} \geq C\right] = 1 - \varepsilon, \\ \mathbb{E}\left[Z_T X_T^{\pi_{\lambda_P^*, c_P^*}}\right] = x, \end{cases}$$

Note that if $\mathbb{P}[X_T^{\pi_U^*} \geq C] \geq 1 - \varepsilon$ then the optimal strategy is the unconstrained one.

Illustration of final payoff



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Expected shortfall constraint under $\mathbb{P}$: $E[(C - X_T^{\pi})_+] \leq \nu$

The optimal terminal wealth becomes

$$X_T^{\pi_S^*} = \begin{cases} k - \lambda_S^* Z_T, & k - \lambda_S^* Z_T \geq C, \\ C, & k - \lambda_S^* Z_T \in [C - \gamma_S^*, C], \\ k - \lambda_S^* Z_T + \gamma_S^*, & k - \lambda_S^* Z_T < C - \gamma_S^*, \end{cases} \tag{12}$$

where the slope and length of the "plateau" stem from the optimal pair

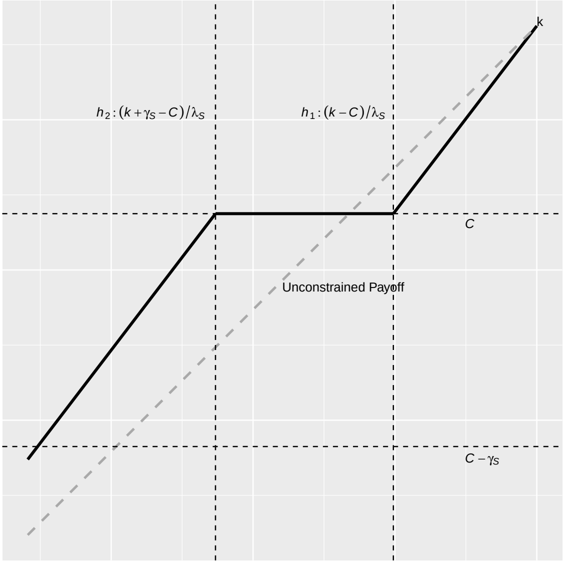
$$(\lambda_S^*, \gamma_S^*),$$

itself determined from conditions

$$\begin{cases} \mathbb{E}\left[\left(C - X_T^{\pi_S^*}\right)_+\right] = \nu, \\ \mathbb{E}\left[Z_T X_T^{\pi_S^*}\right] = x. \end{cases}$$

Again, it could be that π_U^* beats π_S^*.

Illustration of final payoff



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Expected shortfall constraint under $\mathbb{Q}$: $E[Z_T(C - X_T^{\pi})_+] \leq \nu$

The optimal terminal wealth becomes

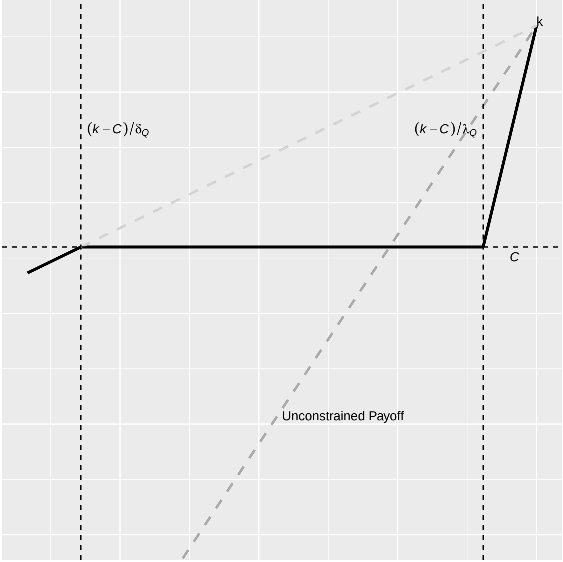
$$X_T^{\pi^*_{\mathbb{Q}}} = \begin{cases} k - \lambda^*_{\mathbb{Q}} Z_T(\omega), & \text{if } Z_T(\omega) \leq \frac{k-C}{\lambda^*_{\mathbb{Q}}}, \\ C, & \text{if } Z_T(\omega) \in \left[\frac{k-C}{\lambda^*_{\mathbb{Q}}}, \frac{k-C}{\delta^*_{\mathbb{Q}}}\right], \\ k - \delta^*_{\mathbb{Q}} Z_T(\omega), & \text{if } Z_T(\omega) > \frac{k-C}{\delta^*_{\mathbb{Q}}}, \end{cases} \tag{13}$$

where the pair $(\lambda^*_{\mathbb{Q}}, \delta^*_{\mathbb{Q}})$ stems from conditions

$$\begin{cases} \mathbb{E}\left[Z_T\left(C - X_T^{\pi^*_{\mathbb{Q}}}\right)_+\right] = \nu, \\ \mathbb{E}\left[Z_T X_T^{\pi^*_{\mathbb{Q}}}\right] = x. \end{cases}$$

Here, the "plateau" is determined by the differential in slopes.

Illustration of final payoff



Note:

- as $\delta \to \lambda$, we retrieve the unconstrained case.
- as $\delta \to 0$, we retrieve the strict constraint case.

Again, it could be that π_U^* beats $\pi_{\mathbb{Q}}^*$.

4 Numerical illustrations

- Parameters
- Comparison of the optimal terminal payoffs
- Traces of the controlled $X_t^{\pi^*}$ and associated π_t^*

Initial parameters

The initial hyperparameters are

$$a = 0.2, \quad b = 0.5, \quad \sigma = 1.2, \text{ and } x = 2$$

for the surplus dynamics, and

$$\tilde{k} = 5, \quad \tilde{C} = 0, \quad \epsilon = 0.01, \text{ and } \nu = 0.1$$

for the solvency constraints, with timeframe $T = 5$.

Optimal parameters

This leads to the following optimal parameters

$$\lambda_U^* = 1.888951 \text{ (unconstrained)}$$

$$\lambda_C^* = 5.828629 \text{ (strict constraint)}$$

$$\begin{cases} \lambda_P^* = 2.159931 \\ c_P^* = -5.725147 \end{cases} \quad \text{(probability (VaR) constraint)}$$

$$\begin{cases} \lambda_S^* = 2.472898 \\ \gamma_S^* = 6.201261 \end{cases} \quad \text{(expected shortfall under } \mathbb{P}\text{)}$$

$$\begin{cases} \lambda_{\mathbb{Q}}^* = 5.199066 \\ \delta_{\mathbb{Q}}^* = 0.6094314 \end{cases} \quad \text{(expected shortfall under } \mathbb{Q}\text{)}$$

4 Numerical illustrations

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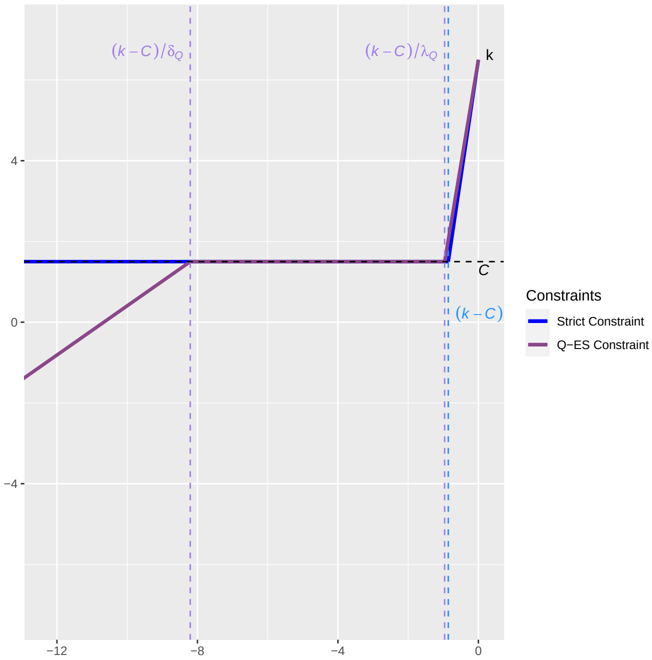
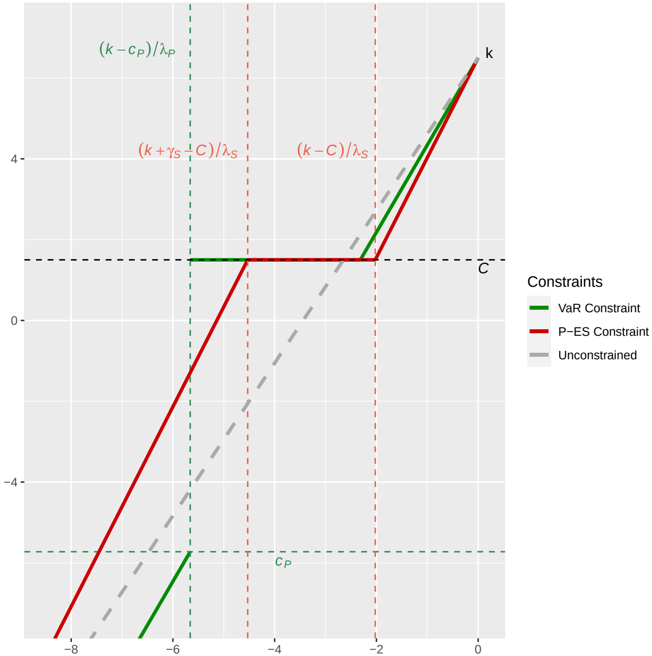
Comparison of the optimal terminal payoffs

- Comparison of Strict constraint payoff and $\mathbb{Q}$-ES constraint payoff with their respective optimal parameters also shows a similar shape, especially for lower values of Z_T.
- Analogously, comparison of the Unconstrained payoff, VaR payoff, and $\mathbb{P}$-ES constraint payoff with their optimal parameters suggests that those are very similar.

This is illustrated in the following figures.


$(k-C)/\delta_Q$
$(k-C)/\lambda_Q$
k
C
$(k-C)$
4
0
−4
−12
−8
−4
0
Constraints
Strict Constraint
Q−ES Constraint


$(k-c_P)/\lambda_P$
$(k+\gamma_S-C)/\lambda_S$
$(k-C)/\lambda_S$
k
C
c_P
4
0
−4
−8
−6
−4
−2
0
Constraints
VaR Constraint
P−ES Constraint
Unconstrained

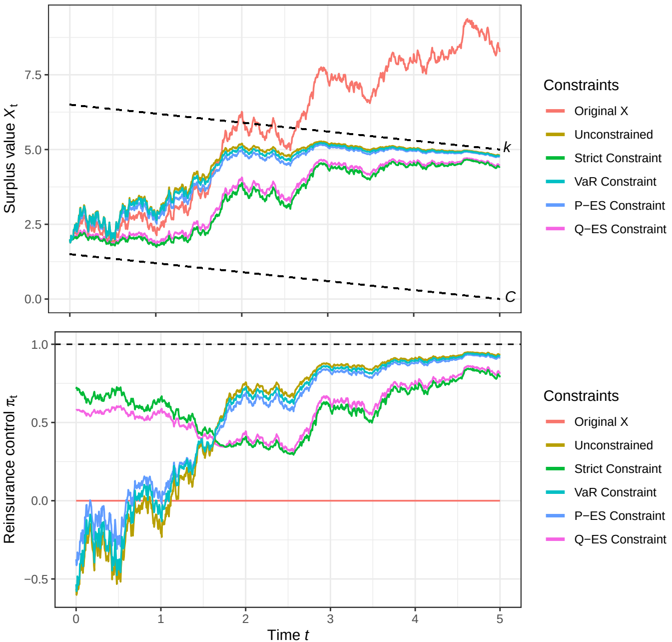
4 Numerical illustrations

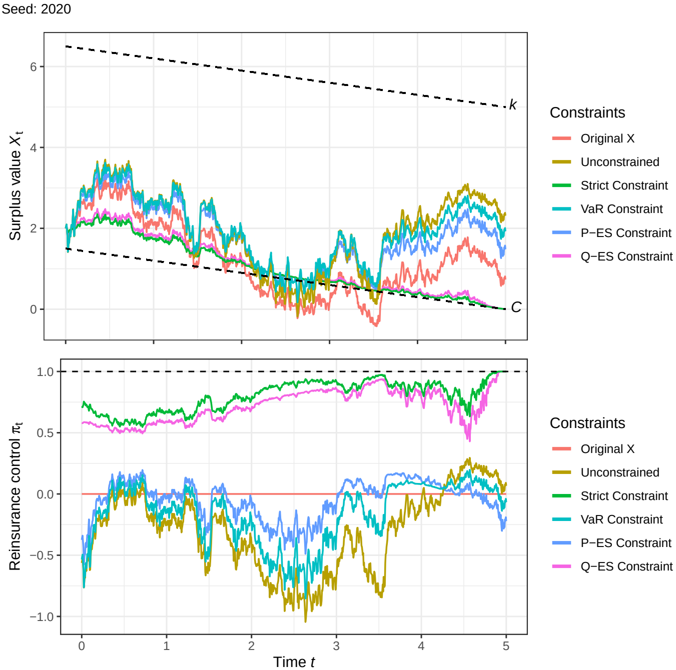
- Parameters
- Comparison of the optimal terminal payoffs
- Traces of the controlled $X_t^{\pi^*}$ and associated π_t^*

Traces of the controlled $X_t^{\pi^*}$ and associated π_t^*

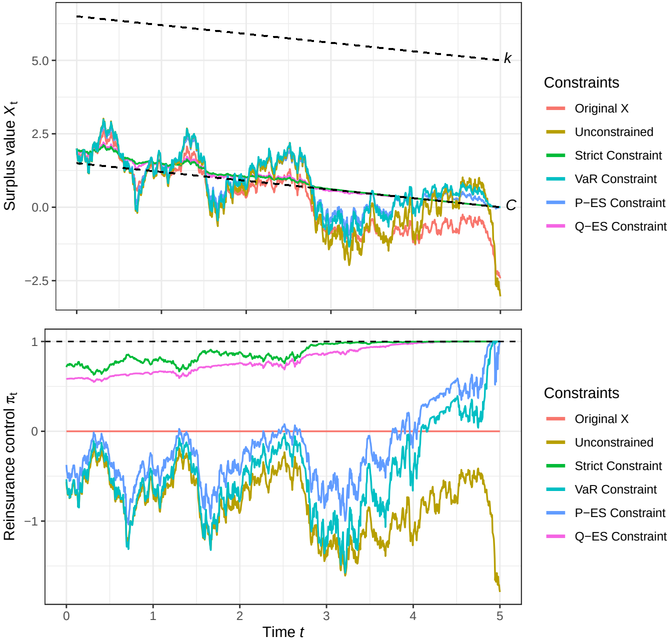
- Here, the controlled path $X_t^{\pi^*}$ and associated reinsurance control π_t^* are simulated by discretising $[0, T]$ over 1000 intervals, using different seeds.
- We compare the uncontrolled surplus X_t (salmon line), with its controlled version under all 5 constraints considered in this paper.
- The paths after application of the optimal control are calculated according to Appendices available in the paper.

Seed: 2

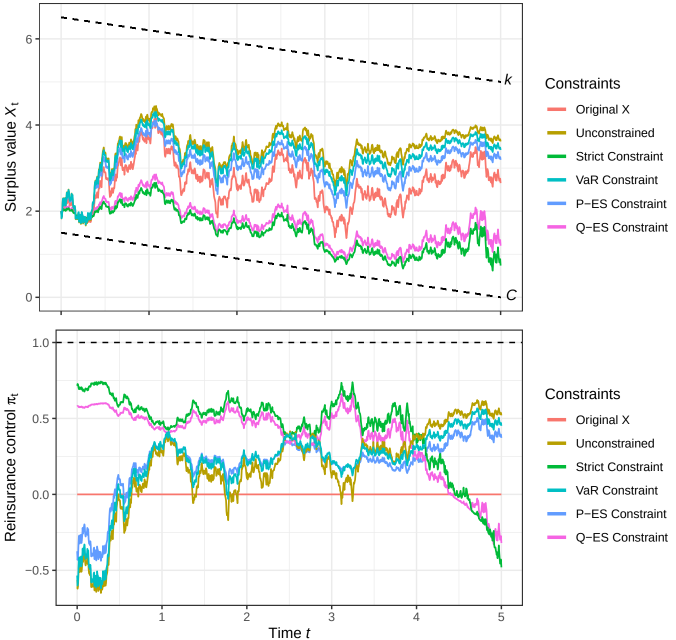

Surplus value X_t
0.0
2.5
5.0
7.5
k
C
Reinsurance control π_t
−0.5
0.0
0.5
1.0
0
1
2
3
4
5
Time t
Constraints
Original X
Unconstrained
Strict Constraint
VaR Constraint
P−ES Constraint
Q−ES Constraint
Constraints
Original X
Unconstrained
Strict Constraint
VaR Constraint
P−ES Constraint
Q−ES Constraint


Seed: 2020
Surplus value X_t
k
C
Constraints
Original X
Unconstrained
Strict Constraint
VaR Constraint
P−ES Constraint
Q−ES Constraint
Reinsurance control π_t
Time t

Seed: 2015


Surplus value X_t
5.0
2.5
0.0
−2.5
k
C
Constraints
Original X
Unconstrained
Strict Constraint
VaR Constraint
P−ES Constraint
Q−ES Constraint
Reinsurance control π_t
1
0
−1
0
1
2
3
4
5
Time t
Constraints
Original X
Unconstrained
Strict Constraint
VaR Constraint
P−ES Constraint
Q−ES Constraint

Seed: 1994


Surplus value X_t
6
4
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0
k
C
Constraints
Original X
Unconstrained
Strict Constraint
VaR Constraint
P−ES Constraint
Q−ES Constraint
Reinsurance control π_t
1.0
0.5
0.0
−0.5
0
1
2
3
4
5
Time t
Constraints
Original X
Unconstrained
Strict Constraint
VaR Constraint
P−ES Constraint
Q−ES Constraint

References I

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