Optimal reinsurance under terminal value constraints
Benjamin Avanzi, Hayden Lau, Mogens Steffensen

Actuarial Colloquium, Orlando 2022

14 April 2022

1 Introduction

2 Solvency / stability constraints

3 Optimal strategies

4 Numerical illustrations

5 References
1 Introduction

2 Solvency / stability constraints

3 Optimal strategies

4 Numerical illustrations

5 References
Introduction

Motivation

- Insurance surplus dynamics
- Objective
- Connection to mean-variance problem
- Connection with finance
Motivation

- The choice of reinsurance is important for insurance decisions and risk management in general (e.g. Albrecher, Beirlant, and Teugels 2017).
- This is often a choice made under a number of possible constraints:
  - imposed either externally (regulator) or internally (Board / risk manager).
- “Traditional” actuarial stability criteria (Bühlmann (1970)) are used to control actuarial surplus processes; these include the probability of ruin (Cramér (1955), Lundberg (1909)), and dividends (Finetti (1957)).
- Can optimal portfolio techniques from finance (“à la Merton”) be applied here? (as, for instance, in Korn (1997), Basak and Shapiro (2001), Korn and Wiese (2008), Gu, Steffensen, and Zheng (2021)).
- Furthermore, can an LQ type criterion (mean-variance) be used, while still staying true to the insurance context?
Introduction

- Motivation
- Insurance surplus dynamics
- Objective
- Connection to mean-variance problem
- Connection with finance
Insurance surplus dynamics

Surplus without any reinsurance:

\[ d\tilde{X}_t = adt + \sigma dW_t, \quad \tilde{X}_0 = x < \tilde{k} \]  

(1)

After reinsurance:

\[ d\tilde{X}_t^\pi = (a - b\pi_t)dt + (1 - \pi_t)\sigma dW_t, \quad \tilde{X}_0^\pi = x < \tilde{k} \]  

(2)

Note:

- Reinsurance is non cheap; \( b > a \)
- If \( \pi_t \equiv 1 \) then

\[ d\tilde{X}_t^1 = (a - b)dt, \quad \tilde{X}_0^1 = x < \tilde{k} \]

The Brownian approximation is standard (see, e.g., Asmussen and Albrecher 2010)
Introduction

- Motivation
- Insurance surplus dynamics

Objective

- Connection to mean-variance problem
- Connection with finance
Our objective is (quadratic utility):

$$\max_{\pi \in \Pi, x} E_{t,x} \left[ -\frac{1}{2} \left( k - X_{t}^{\pi} \right)^{2} \right],$$  \hspace{1cm} (3)

where $\Pi$ is the set of admissible controls. Note:

- Once $X_{t}^{\pi} = \tilde{k}$, the optimal strategy is to take $\pi_s = 1$ for $s \in [t, T]$
- Hence it makes sense that $x < \tilde{k}$, and that it remains so for $t \geq 0$ so that $X_{t}^{\pi} \leq \tilde{k}$.
- Hence, there are two important levels:
  - $x = \tilde{k}$ (an upper bound) and
  - $x = \tilde{C}$ (a lower solvency threshold which may be strict or not)

There isn’t much actuarial literature with such LQ optimisation (an exception is Steffensen 2006 in life)
Furthermore, note that $\pi$ may be outside the interval $[0, 1]$

- $\pi < 0$: take over more of the risk than originally had
- $\pi > 1$: pass on more risk than he actually had – go short in insurance risk
- $\pi > 1$ is intuitively never optimal (verified by simulations)
- We do not formally consider the additional constraints $0 \leq \pi \leq 1$ ("borrowing and short-selling constraints")
1 Introduction

- Motivation
- Insurance surplus dynamics
- Objective
- Connection to mean-variance problem
- Connection with finance
Connection to mean-variance problem

Varying here over $\tilde{k}$ is equivalent to varying over the mean condition in a mean-variance problem in the form

$$\min_{\pi \in \Pi, E[X^{\pi}_t] = p} \text{Var}[\tilde{X}^{\pi}_T] \text{ for some } p \in \mathbb{R};$$

(4)

a well known objective in the finance literature. However, note:

- the subscript in $E_{t,x}$ denotes a conditional expectation
- in the well-known mean-variance framework, the mean condition is an unconditional mean
- so our context can be interpreted as a mean-variance problem from time 0 only, and not at subsequent time points
- in other words, we are in the traditional mean-variance case with pre-commitment (to the mean, around which quadratic distance is measured) and not in the modern version where the variance is actually conditional variance (and which requires different optimisation techniques)
Introduction

- Motivation
- Insurance surplus dynamics
- Objective
- Connection to mean-variance problem
- Connection with finance
Connection with finance

In finance (think optimal portfolio as in Merton):

- geometric Brownian dynamics for the (market) portfolio value
- usually includes optimal consumption (but not always)
- power utility

What’s different here in the insurance context:

- Brownian dynamics for the insurance surplus (additive, can become negative)
- reinsurance is akin to risk investment decision (in symmetry)
- quadratic utility (LQ)
Why not power utility?

- Power utility does not work for processes that can go negative (becomes minus infinity)
- Note we could still use power utility with a strict constraint (as long as \( C > 0 \)); we just did not do that.

Why not exponential utility?

- We could have chosen that, but we were interested in the connection with mean-variance.
- There are also additional interpretations with the quadratic formulation.
Introduction

Solvency / stability constraints

Optimal strategies

Numerical illustrations

References
2 Solvency / stability constraints

- Constraints considered in the paper
- In the Merton problem


**Constraints considered in the paper**

1. No constraint
2. Strict: $X_T^{\pi} \geq C$
3. Probability (VaR): $\Pr[X_T^{\pi} \geq C] \geq 1 - \epsilon$
4. Expected shortfall under $\mathbb{P}$: $E[(C - X_T^{\pi})_+] \leq \nu$
   - Basel III (and onwards) moved from VaR to ES for solvency assessment
   - ES (vs VaR) has been shown to have desirable properties (e.g., Basak and Shapiro 2001)
5. Expected shortfall under $\mathbb{Q}$: $E[Z_T(C - X_T^{\pi})_+] \leq \nu$
   - Does not make sense from a regulatory perspective, but
   - Makes sense from a risk management perspective: puts an upper threshold to the cost of protecting the expected shortfall of $\tilde{X}_t^{\pi}$ from $\tilde{C}$ (up to a risk free discount factor).

**Note:**

- We retrieve 2. when $\epsilon \to 0$ or when $\nu \to 0$
- Closest “actuarial” literature is Korn and Wiese (2008), but they do not consider 4.-5. and only a limited version of 3. ($C \equiv 0$).
Solvency / stability constraints

- Constraints considered in the paper
- In the Merton problem
In the Merton problem

Why $\mathbb{P}$ vs $\mathbb{Q}$?

- problem easier to communicate under $\mathbb{P}$, but
- solution easier to communicate under $\mathbb{Q}$
- results are model-dependent anyway, so no clear winner
- $\mathbb{Q}$ solution close to $\mathbb{P}$ solution
- power options don’t trade anyway

(Gu, Steffensen, and Zheng 2021)
Illustration of final payoff under $\mathbb{P}$

(from Gu, Steffensen, and Zheng 2021)
Solvency / stability constraints

Illustration of final payoff under $Q$

(from Gu, Steffensen, and Zheng 2021)
1 Introduction

2 Solvency / stability constraints

3 Optimal strategies

4 Numerical illustrations

5 References
3 Optimal strategies

- Auxiliary processes
  - Methodology
  - Unconstrained problem
  - Strict constraint: $X_T^{\pi} \geq C$
  - Probability (VaR) constraint: $\Pr[X_T^{\pi} \geq C] \geq 1 - \epsilon$
  - Expected shortfall constraint under $\mathbb{P}$: $E[(C - X_T^{\pi})_+] \leq \nu$
  - Expected shortfall constraint under $\mathbb{Q}$: $E[Z_T(C - X_T^{\pi})_+] \leq \nu$
Surplus process

First, for convenience we remove the drift corresponding to a fully reinsured process from our base process,

\[ X^\pi_t := \tilde{X}^\pi_t - (a - b)t, \]

and define associated constants

\[ k := \tilde{k} - (a - b)T, \]
\[ C := \tilde{C} - (a - b)T. \]
We consider the (equivalent) optimization problem,

\[
\max_{\pi \in \Pi} \mathbb{E}_{t,x} \left[-\frac{1}{2} (k - X^{\pi}_T)^2\right]
\]  

(5)

with

\[dX^{\pi}_t = (1 - \pi_t)(b dt + \sigma dW_t), \quad X(0) = x < \tilde{k}.
\]

In other words, we now focus purely on the residual risk dynamics left after reinsurance:

- \(dX^{\pi}_t\) describes the profit process generated by retaining business 
  \((1 - \pi_t)\) at time \(t\), as compared to full reinsurance (which is loss making since \(b > a\)).

The move from \(X^{\pi}_t\) to \(\tilde{X}^{\pi}_t\) is straightforward.
Auxiliary process

Let \( Z = \{Z_t, t \in [0, T]\} \) have dynamics

\[
dZ_t = \beta Z_t dW_t, \quad Z_0 = 1, \tag{6}
\]

with \( \beta = -b/\sigma \) and solution,

\[
Z_t = e^{-\frac{1}{2} \beta^2 t + \beta W_t}. \quad \text{(Note: } Z_t \to 0 \text{ as } W_t \to \infty). \tag{7}
\]

We have that \( XZ \) forms a martingale such that, for a given terminal surplus, the path \( X^\pi_t \) is given by

\[
X^\pi_t = E \left[ \frac{Z_T}{Z_t} X^\pi_T \mid \mathcal{F}_t \right]. \tag{8}
\]

Thus, once we know \( X^\pi_T \), we can evaluate this expression and then apply Itô in order to deduce the optimal proportion \( \pi_t \).
Optimal strategies

- Auxiliary processes

Methodology

- Unconstrained problem

- Strict constraint: $X_T^π \geq C$

- Probability (VaR) constraint: $\Pr[X_T^π \geq C] \geq 1 - \epsilon$

- Expected shortfall constraint under $\mathbb{P}$: $E[(C - X_T^π)^+] \leq \nu$

- Expected shortfall constraint under $\mathbb{Q}$: $E[Z_T(C - X_T^π)^+] \leq \nu$
There are typically two approaches to optimisation in this context (see, e.g., Korn 1997):

- The martingale method (e.g., Gu, Steffensen, and Zheng 2021)
- The dynamic programming approach (e.g., Kraft and Steffensen 2013)

We use the martingale method, which involves the following steps:

1. Postulate an optimal terminal wealth solution form;
2. Verify the existence of optimal parameters for that form;
3. Verify that the candidate strategy is indeed optimal (here, via path-wise optimisation)

In what follows we mainly present the (verified) terminal wealth. Proofs are available in the paper.
Optimal strategies

- Auxiliary processes
- Methodology

Unconstrained problem

- Strict constraint: $X_T^π \geq C$
- Probability (VaR) constraint: $\Pr[X_T^π \geq C] \geq 1 - \epsilon$
- Expected shortfall constraint under $\mathbb{P}$: $E[(C - X_T^π)_+] \leq \nu$
- Expected shortfall constraint under $\mathbb{Q}$: $E[Z_T(C - X_T^π)_+] \leq \nu$
Unconstrained problem

Optimal terminal wealth is

\[ X_T^{\pi\lambda_U^*} := k - \lambda_U^* Z_T, \quad (9) \]

This is the simplest case. Here, the optimal value of \( \lambda, \lambda_U^* \), is obtained noting that

\[ V_U(\lambda) := \mathbb{E}[Z_T X_T^{\pi\lambda}]. \]

should be equal to \( x \) (budget constraint), and this exists because

\[ \frac{\partial}{\partial \lambda} V_U < 0, \quad \lim_{\lambda \to \infty} V_U(\lambda) = -\infty, \quad V_U(0) = k. \]

Since \( -\infty < x \leq k \), we conclude that there exists a \( \lambda_U^* > 0 \) such that \( V_U(\lambda_U^*) = x \).
Illustration of final payoff

\[ (k - C)/\lambda_U \]

Unconstrained Payoff

Unconstrained problem
Optimal strategies

- Auxiliary processes
- Methodology
- Unconstrained problem

Strict constraint: $X_T^\pi \geq C$

- Probability (VaR) constraint: $\Pr[X_T^\pi \geq C] \geq 1 - \epsilon$
- Expected shortfall constraint under $\mathbb{P}$: $E[(C - X_T^\pi)_+] \leq \nu$
- Expected shortfall constraint under $\mathbb{Q}$: $E[Z_T(C - X_T^\pi)_+] \leq \nu$
Strict constraint: $X_T^\pi \geq C$

The optimal terminal wealth is

$$X_T^{\pi\lambda^*_C} := \max(k - \lambda^*_C Z_T, C),$$

characterised by the parameter $\lambda^*_C > 0$.

Here,

$$\frac{\partial}{\partial \lambda} V_C(\lambda) < 0, \quad \lim_{\lambda \to \infty} V_C(\lambda) = C, \quad V_C(0) = k.$$ 

Since $C \leq x \leq k$, we conclude that there exists a $\lambda^*_C$ such that $V_U(\lambda^*_C) = x$. Furthermore, we have that

$$\lambda^*_C > \lambda^*_U,$$ 

and that $X_T^{\pi U} > X_T^{\pi C}$ for $X_T^{\pi C} > C$.

Given the “kink” at level $C$, $\lambda^*_C$ is adjusted upwards (steeper) to satisfy the budget constraint.
Illustration of final payoff
### Optimal strategies

- Auxiliary processes
- Methodology
- Unconstrained problem
- Strict constraint: $X^\pi_T \geq C$
- **Probability (VaR) constraint:** $\Pr[X^\pi_T \geq C] \geq 1 - \epsilon$

- Expected shortfall constraint under $\mathbb{P}$: $E[(C - X^\pi_T)_+] \leq \nu$
- Expected shortfall constraint under $\mathbb{Q}$: $E[Z_T(C - X^\pi_T)_+] \leq \nu$
Optimal terminal wealth is of the form

$$X_{T}^{\pi^{*},c^{*}} = \begin{cases} 
  k - \lambda^{*}_{P}Z_{T}, & k - \lambda^{*}_{P}Z_{T} \notin [c^{*}_{P}, C], \\
  C, & k - \lambda^{*}_{P}Z_{T} \in [c^{*}_{P}, C], 
\end{cases} \quad (11)$$

characterised by the constants $\lambda^{*}_{P}, c^{*}_{P} > 0$ with $c^{*}_{P} \leq C \leq k$.

The slope $\lambda^{*}_{P}$ and location of the discontinuity (stemming from $c^{*}_{P}$) are obtained using conditions

$$\begin{aligned}
\mathbb{P} \left[ X_{T}^{\pi^{*},c^{*}} \geq C \right] &= 1 - \epsilon, \\
\mathbb{E} \left[ Z_{T}X_{T}^{\pi^{*},c^{*}} \right] &= x,
\end{aligned}$$

Note that if $\mathbb{P}[X_{T}^{\pi U^{*}} \geq C] \geq 1 - \epsilon$ then the optimal strategy is the unconstrained one.
Optimal strategies

Probability (VaR) constraint: $\Pr[X_{T} \geq C] \geq 1 - \epsilon$

Illustration of final payoff

Unconstrained Payoff

$g_1 : (k - C) / \lambda_P$

$g_2 : (k - c_P) / \lambda_P$

$k$

$c_P$

$C$
Optimal strategies

- Auxiliary processes
- Methodology
- Unconstrained problem
- Strict constraint: $X_T^\pi \geq C$
- Probability (VaR) constraint: $\Pr[X_T^\pi \geq C] \geq 1 - \epsilon$

Expected shortfall constraint under $\mathbb{P}$: $E[(C - X_T^\pi)^+] \leq \nu$

Expected shortfall constraint under $\mathbb{Q}$: $E[Z_T(C - X_T^\pi)^+] \leq \nu$
Expected shortfall constraint under $\mathbb{P}$: $E[(C - X_T^\pi)_+] \leq \nu$

The optimal terminal wealth becomes

$$X_T^{\pi^*} = \begin{cases} 
    k - \lambda_S^* Z_T, & k - \lambda_S^* Z_T \geq C, \\
    C, & k - \lambda_S^* Z_T \in [C - \gamma_S^*, C], \\
    k - \lambda_S^* Z_T + \gamma_S^*, & k - \lambda_S^* Z_T < C - \gamma_S^*, 
\end{cases}$$

(12)

where the slope and length of the “plateau” stem from the optimal pair $(\lambda_S^*, \gamma_S^*)$, itself determined from conditions

$$\begin{cases} 
    E \left[ (C - X_T^{\pi^*})_+ \right] = \nu, \\
    E \left[ Z_T X_T^{\pi^*} \right] = \chi. 
\end{cases}$$

Again, it could be that $\pi_U^*$ beats $\pi_S^*$. 

24/40
Illustration of final payoff

Unconstrained Payoff

\[ h_2 : \frac{(k + \gamma_S - C)}{\lambda_S} \]

\[ h_1 : \frac{(k - C)}{\lambda_S} \]

Optimal strategies

Expected shortfall constraint under \( P \): \( E[(C - X_T^\pi)_+] \leq \nu \)
Optimal strategies

- Auxiliary processes
- Methodology
- Unconstrained problem
- Strict constraint: $X_T^\pi \geq C$
- Probability (VaR) constraint: $\Pr[X_T^\pi \geq C] \geq 1 - \epsilon$
- Expected shortfall constraint under $\mathbb{P}$: $E[(C - X_T^\pi)_+] \leq \nu$
- Expected shortfall constraint under $\mathbb{Q}$: $E[Z_T(C - X_T^\pi)_+] \leq \nu$
Expected shortfall constraint under $Q$:
$$E[Z_T(C - X_T^\pi)_+] \leq \nu$$

The optimal terminal wealth becomes

$$X_T^\pi = \begin{cases} 
  k - \lambda^*_Q Z_T(\omega), & \text{if } Z_T(\omega) \leq \frac{k-C}{\lambda^*_Q}, \\
  C, & \text{if } Z_T(\omega) \in \left[\frac{k-C}{\lambda^*_Q}, \frac{k-C}{\delta^*_Q}\right], \\
  k - \delta^*_Q Z_T(\omega), & \text{if } Z_T(\omega) > \frac{k-C}{\delta^*_Q}, 
\end{cases}$$

(13)

where the pair $(\lambda^*_Q, \delta^*_Q)$ stems from conditions

$$\begin{cases}
  \mathbb{E}\left[Z_T \left(C - X_T^\pi\right)_+\right] = \nu, \\
  \mathbb{E}\left[Z_T X_T^\pi\right] = x.
\end{cases}$$

Here, the “plateau” is determined by the differential in slopes.
Illustration of final payoff
Note:

- as $\delta \to \lambda$, we retrieve the unconstrained case.
- as $\delta \to 0$, we retrieve the strict constraint case.

Again, it could be that $\pi_\mathcal{U}^*$ beats $\pi_\mathcal{Q}^*$. 
4 Numerical illustrations

- Parameters
  - Comparison of the optimal terminal payoffs
  - Traces of the controlled $X_t^{\pi^*}$ and associated $\pi_t^*$
Initial parameters

The initial hyperparameters are

\[ a = 0.2, \quad b = 0.5, \quad \sigma = 1.2, \quad \text{and} \quad x = 2 \]

for the surplus dynamics, and

\[ \tilde{k} = 5, \quad \tilde{C} = 0, \quad \epsilon = 0.01, \quad \text{and} \quad \nu = 0.1 \]

for the solvency constraints, with timeframe \( T = 5 \).
Optimal parameters

This leads to the following optimal parameters

\[
\begin{align*}
\lambda^*_U &= 1.888951 \text{ (unconstrained)} \\
\lambda^*_C &= 5.828629 \text{ (strict constraint)} \\
\begin{cases}
\lambda^*_P = 2.159931 \\
c_P^* = -5.725147
\end{cases} \text{ (probability (VaR) constraint)} \\
\begin{cases}
\lambda^*_S = 2.472898 \\
\gamma^*_S = 6.201261
\end{cases} \text{ (expected shortfall under } \mathbb{P} \text{)} \\
\begin{cases}
\lambda^*_Q = 5.199066 \\
\delta^*_Q = 0.6094314
\end{cases} \text{ (expected shortfall under } \mathbb{Q} \text{)}
\end{align*}
\]
Numerical illustrations

- Parameters
- Comparison of the optimal terminal payoffs
- Traces of the controlled \( X_t^{\pi^*} \) and associated \( \pi_t^* \)
Comparison of the optimal terminal payoffs

- Comparison of Strict constraint payoff and $\mathbb{Q}$-ES constraint payoff with their respective optimal parameters also shows a similar shape, especially for lower values of $Z_T$.
- Analogously, comparison of the Unconstrained payoff, VaR payoff, and $\mathbb{P}$-ES constraint payoff with their optimal parameters suggests that those are very similar.

This is illustrated in the following figures.
Numerical illustrations

Comparison of the optimal terminal payoffs

\[(k - C)/\delta_Q, \quad (k - C)/\lambda_Q\]

Constraints

- Strict Constraint
- Q–ES Constraint
Numerical illustrations

Comparison of the optimal terminal payoffs

\[
\frac{k - c_P}{\lambda_P} \quad \frac{k + \gamma S - C}{\lambda S} \quad \frac{k - C}{\lambda S}
\]

Constraints
- VaR Constraint
- P-ES Constraint
- Unconstrained
4 Numerical illustrations

- Parameters
- Comparison of the optimal terminal payoffs
- Traces of the controlled $X_t^{\pi^*}$ and associated $\pi_t^*$
Traces of the controlled $X_t^{\pi^*}$ and associated $\pi_t^*$

- Here, the controlled path $X_t^{\pi^*}$ and associated reinsurance control $\pi_t^*$ are simulated by discretising $[0, T]$ over 1000 intervals, using different seeds.
- We compare the uncontrolled surplus $X_t$ (salmon line), with its controlled version under all 5 constraints considered in this paper.
- The paths after application of the optimal control are calculated according to Appendices available in the paper.
Numerical illustrations

Traces of the controlled $X_t^{\pi^*}$ and associated $\pi_t^*$

Seed: 2

Constraints
- Original X
- Unconstrained
- Strict Constraint
- VaR Constraint
- P−ES Constraint
- Q−ES Constraint

Constraints
- Original X
- Unconstrained
- Strict Constraint
- VaR Constraint
- P−ES Constraint
- Q−ES Constraint
Numerical illustrations

Traces of the controlled $X_t^{\pi^*}$ and associated $\pi_t^*$

Seed: 2020

Constraints
- Original X
- Unconstrained
- Strict Constraint
- VaR Constraint
- P–ES Constraint
- Q–ES Constraint

Constraints
- Original X
- Unconstrained
- Strict Constraint
- VaR Constraint
- P–ES Constraint
- Q–ES Constraint
Numerical illustrations

Traces of the controlled $X_t^{\pi^*}$ and associated $\pi_t^*$

Seed: 2015

Constraints
- Original X
- Unconstrained
- Strict Constraint
- VaR Constraint
- P–ES Constraint
- Q–ES Constraint

Constraints
- Original X
- Unconstrained
- Strict Constraint
- VaR Constraint
- P–ES Constraint
- Q–ES Constraint
Numerical illustrations

Traces of the controlled $X_t^{\pi^*_t}$ and associated $\pi_t^{*}$

Seed: 1994

Constraints

- Original X
- Unconstrained
- Strict Constraint
- VaR Constraint
- P–ES Constraint
- Q–ES Constraint

Constraints

- Original X
- Unconstrained
- Strict Constraint
- VaR Constraint
- P–ES Constraint
- Q–ES Constraint
1 Introduction

2 Solvency / stability constraints

3 Optimal strategies

4 Numerical illustrations

5 References


References


