Computation of bonus in multi-state life insurance

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With-profit life insurance setup

- Payments written (and guaranteed) on a conservative first order technical basis for financial and insurance risk.

- Policyholder options (surrender and free-policy) satisfy equivalence principles on the first order basis.

- Insurer pools the premium payments from the portfolio and invest on the financial market.

- Realized investment and insurance results leads to systematic surplus in the portfolio.

- Paid back to the individual insured in the form of dividends and ultimately bonus.

- We consider the bonus scheme known as additional benefits.

- The total payments between insurer and insured then consists of:
  
  \[
  \text{Payments written at time 0 + options + bonus}
  \]

  Predetermined guaranteed payments
Liabilities

- Predetermined guaranteed payments: $B^o$
- Unit bonus benefits of some type $B^\dagger$
- Number of additional bonus benefits: $Q$
- Market value of guaranteed payments:
  \[ GB = \mathbb{E} \left[ \int_0^n e^{-\int_0^s r(v) \, dv} \, dB^o(s) \right] \]
- Market value of bonus payments:
  \[ FDB = \mathbb{E} \left[ \int_0^n e^{-\int_0^s r(v) \, dv} Q(s) \, dB^\dagger(s) \right] \]
- Market value of all future payments:
  \[ MV = GB + FDB \]
Calculating GB

- The market value of guaranteed payments:

\[
GB = \int_0^n e^{-\int_0^s f(0, v)dv} a^\circ(0, s) ds
\]  

(1)

- \( f \) is the forward rate curve used to discount future payments
- \( a^\circ \) is the expected cash flow of predetermined guaranteed payments:
  - Calculated as probability-weighted future payments
  - Probability weights \( p_{0j}(0, s) \) obtained by solving Kolmogorov’s forward differential equations.

- Formula (1) applies due to:
  - Simple payment structures
  - An assumption of independence between insurance and financial risk.
Calculating FDB

- Dividends $\delta$ are used to buy $B^\dagger$ on the technical basis.

- Price of one unit: $V^*,\dagger$.

- Number of additional benefits $Q$ develops according to:

$$dQ(t) = \frac{\delta(t)}{V^*,\dagger(t)} \, dt$$

- $\delta(t)$ depends on past financial and insurance paths up to time $t$.

- So we cannot use analytical methods to calculate FDB in general:

$$FDB = \mathbb{E}\left[\int_0^n e^{-\int_0^s r(v) \, dv} Q(s) \, dB^\dagger(s)\right]$$
Simulation method: Scenario-specific bonus cash flows

- We adopt a partly simulation and analytical based approach:
  - Financial scenarios are simulated
  - Analytical method for outstanding insurance risk

- Reformulation of FDB provides the key:

\[
FDB = \mathbb{E} \left[ \int_0^n e^{-\int_0^s r(v) \, dv} a^b(0, s) \, ds \right]
\]

- \( a^b \) is the scenario-specific (expected) bonus cash flow:
  - Calculated as \( Q \)-modified probability-weighted future payments
  - Need method to calculate \( Q \)-modified probabilities \( p_{0j}^Q(0, t) \).
Illustration of simulation method

\[ \text{FDB} \approx \frac{1}{N} \sum_{i=1}^{N} \text{FDB}_i \]
Strategy and class of dividend and investments

- The scenario-specific bonus cash flow depends on the actual specification of the dividend strategy for the whole contract period.

- In practice a **control** variable at the company’s discretion (along with the investment strategy), based on the shape of the insurance business.

- In particular, dividends and investments are based on the balance sheet on a portfolio level at the given time point.

- We call these quantities **shapes**.

- Leads to a class of dividend and investment strategies considered for the remainder.
Assets part of shapes

- Assets is the value process of a portfolio of \( S = (S_0, S_1) \):
  - \( S_0 \) is a bank account with stochastic interest gains \( r \)
  - \( S_1 \) is a risky asset (bonds, stocks, options etc.)

- Self-financed by the premiums (less benefits) paid by the portfolio of insured.

- Assets \( U \) is then defined as

\[
\frac{dU(t)}{dt} = \theta(t) \frac{dS_0(t)}{dt} + \eta(t) \frac{dS_1(t)}{dt} - (a^0(0, t) + a^b(0, t)) dt
\]

where \((\theta, \eta)\) is the investment strategy.

- **Note:** Depends on \( Q \)-modified probabilities \( p_{0j}^Q \) trough \( a^b \).
Liability part of shapes

On individual policy level:

- Expected cash flow of guaranteed payments on the market basis:

  \[ a^g(t, s) = a^\circ(t, s) + Q(t)a^\dagger(t, s). \]

- Technical reserve of guaranteed payments:

  \[ V^*(t) = V^{*, \circ}(t) + Q(t)V^{*, \dagger}(t) \]

Portfolio-wide means:

\[ \bar{a}^g(t, s) = \mathbb{E}[ a^g(t, s) | \mathcal{F}^S(t) ], \]

\[ \bar{V}^*(t) = \mathbb{E}[ V^*(t) | \mathcal{F}^S(t) ]. \]

Based on the law of large numbers w.r.t a collection of independent and comparable insured.
Definition of shapes and controls

The shape of the insurance business $\mathcal{I}$ is given as

$$\mathcal{I}(t) = (U(t), \bar{a}^g(t, \cdot), \tilde{V}^*(t)),$$

while the controls are $(\delta(t), \eta(t))$. Assumptions:

- The investment strategy $\eta$ depends on:
  - The shape of the insurance business $\mathcal{I}$
  - The developments on the financial market $S$.

- The dividend strategy $\delta$ depends on:
  - The shape of the insurance business $\mathcal{I}$
  - The developments on the financial market $S$.
  - The current individual insurance risk (state of the insured and time of free policy conversion, if any).
  - The number of benefits $Q$ currently held, in an affine manner.
Example: Second order interest rates

Idea:

- The technical reserve of guaranteed payments $V^*$ accumulates with the first order interest rate $r^*$
- Dividends may be used to give insured another interest accumulation $r^\delta$ based on the insurance business.
- Mathematically obtained by letting

$$\delta(t) = (r^\delta(t) - r^*(t))V^*(t)$$

- $r^\delta$ may depend on:
  - The shape of the insurance business $I$
  - The financial developments $S$

- Usually $r^\delta \geq r^*$, such that $r^*$ constitutes a minimum interest rate guarantee.
Scenario-based projection model

- We wish to compute FDB within the assumptions on the controls \((\delta, \eta)\) made.

- For each financial scenario, we need
  - The shape of the insurance business \(\mathcal{I}(t)\)
  - The controls \((\delta(t), \eta(t))\)

for all time points \(t \geq 0\).

- In other words, we must **project** the shape of the insurance business and controls into future time points in each financial scenario.

- We call it a scenario-based projection model.
Projecting the shape

The portfolio-wide mean $\bar{a}^g$ of the guaranteed cash flow:

$$\bar{a}^g(t, s) = a^\circ(0, s) + \sum_{j \in \mathcal{J}} p_{0j}^Q(0, t) a_j^{\dagger}(t, s).$$

The portfolio-wide mean technical reserve of guaranteed payments:

$$\bar{V}^*(t) = \bar{V}^{*,\circ}(t) + \sum_{j \in \mathcal{J}} p_{0j}^Q(0, t) V_j^{*,\dagger}(t),$$
The $Q$-modified probabilities $p_{0j}^Q(0, \cdot)$ satisfy the differential equations

$$\frac{d}{dt} p_{0j}^Q(0, t) = \frac{\bar{\delta}_j(t)}{V^{*,\dagger}_j(t)} - p_{0j}^Q(0, t)\mu_j(t) + \sum_{k \in \mathcal{J} \setminus j} p_{0k}^Q(0, t)\mu_{kj}(t),$$

$$p_{0j}^Q(0, 0) = 0.$$

$\bar{\delta}_j$ is a ($Q$-modified) probability-weighted future dividends in the different states.

Corresponds to ordinary probabilities modified with the number of additional benefits $Q$ in the different states.

Ready to formulate numerical procedure to compute FDB.
Numerical procedure - Inputs

Inputs computed independently of the financial scenarios:

(1) The expected cash flow of predetermined guaranteed payments \( a^\circ(0, s) \) for all \( s \geq 0 \).

(2) The portfolio-wide mean technical reserve of predetermined guaranteed payments \( \bar{V}^{*,\circ}(t) \) for all \( t \geq 0 \).

(3) For each \( t \geq 0 \), state-wise expected unit bonus cash flows \( a^{\dagger}_{j}(t, s) \) for all \( s \geq t \) and \( j \in \mathcal{J} \).

(4) State-wise technical unit reserves \( V^{*,\dagger}_{j}(t) \) for all \( t \geq 0 \) and \( j \in \mathcal{J} \).

(5) Transition probabilities \( p_{0j}(0, t) \) for all \( t \geq 0 \) and \( j \in \mathcal{J} \).
Numerical procedure - the heart

1. Simulate $N$ financial scenarios.

2. For each financial scenario $k = 1, \ldots, N$:
   - Initialize with $p_{0j}^{Q,k}(0,0) = 0$ and $U^k(0) = u_0$.
   - Apply a numerical algorithm to solve the coupled stochastic differential equation systems for $p_{0j}^{Q,k}(0, \cdot)$ and $U^k$.
   - Compute the $k$-scenario-specific bonus cash flow $a^{b,k}(0, \cdot)$.

3. Compute the market value of bonus payments FDB via

\[
FDB \approx \frac{1}{N} \sum_{k=1}^{N} \int_0^n \int_0^t e^{-\int_0^t r^k(v) \, dv} a^{b,k}(0, t) \, dt
\]

using an algorithm for numerical integration.
Comments on the numerical procedure

Nice things:

- Efficient to have eliminated the need to simulate insurance risk for every insured.
- The financial scenarios can be applied to all insured.

Not so nice things:

- Require the input $a_j^\dagger(\cdot, \cdot)$ on a two-dimensional time grid.
- We are not able to disentangle future state-specific quantities
- Heavily model dependent in terms of possibilities and flexibility

Solution: The state-independent scenario-based projection model.
State-independent scenario-based projection model

- Goal is to make $Q$ depend only on financial risk, since then

$$p_0^Q(0, t) = Q(t)p_0(0, t),$$

$$a^b(0, t) = Q(t)a^\dagger(0, t).$$

- Obtained by letting $\delta = \tilde{\delta} \times V^{*, \dagger}$, giving:

$$dQ(t) = \tilde{\delta}(t) \, dt$$

where $\tilde{\delta}$ only depends on financial risk ($I$, $S$ and $Q$).

- Projecting the shapes are correspondingly simplified:

$$\tilde{a}^g(t, s) = a^\circ(0, s) + Q(t)a^\dagger(0, s),$$

$$\tilde{V}^*(t) = \tilde{V}^{*, \circ}(t) + Q(t)\tilde{V}^{*, \dagger}(t).$$
Properties of the model

- Everything is now boiled to a (state-independent) calculation of $Q$ and $U$ - significant dimension reduction.

- The state-wise unit bonus cash flows $a_j^\dagger(\cdot, \cdot)$ is now replaced with a single cash flow $a^\dagger(0, \cdot)$

- Do any good (and realistic) examples satisfy the requirements on the restricted class of dividend yields?

- The example of second order interest rates do not fit.

- When examples do not fit, we use the state-independent scenario-based projection model as an approximation.

- We show a numerical study of the error obtained in the second order interest rate example.
Numerical procedure - inputs

Inputs computed independently of the financial scenarios:

1. The expected cash flow of predetermined guaranteed payments $a^\circ(0, s)$.

2. The portfolio-wide mean technical reserve of predetermined guaranteed payments $\bar{V}^*;^\circ(t)$.

3. The expected unit bonus cash flow $a^\dagger(0, s)$.

4. Portfolio-wide mean technical reserve of unit bonus benefits $\bar{V}^*;^\dagger(t)$.

Note:
- All state-dependent quantities in the inputs are gone.
- Cash flows are disentangled from future states - efficiency and flexibility is much higher now.
Numerical procedure - the heart

1. Simulate $N$ financial scenarios.

2. For each financial scenario $k = 1, \ldots, N$:
   - Initialize with $Q^k(0) = 0$ and $U^k(0) = u_0$.
   - Apply a numerical algorithm to solve the coupled stochastic differential equation systems for $Q^k$ and $U^k(\cdot)$.
   - Compute the $k$-scenario-specific bonus cash flow $a^{b,k}(0, t) = Q^k(t) a^\dagger(0, t)$.

3. Calculate the market value of bonus payments FDB via

$$FDB \approx \frac{1}{N} \sum_{k=1}^{N} \int_0^n e^{-\int_0^t r^k(v) \, dv} a^{b,k}(0, t) \, dt$$

using an algorithm for numerical integration.
Numerical Example

Consider 40-year old male with predetermined payments and maximal contract time $n = 70$ (age 110) given by:

- Disability annuity of rate 100,000, until retirement of age 65.
- Life annuity of rate 100,000 while alive, from retirement of age 65.
- Premium payments of rate 46,409.96 while active until retirement of age 65.

![Diagram of the numerical example](image-url)
Financial market, bonus type and controls

- Constant first order interest rate of 1%.
- Stochastic market interest rate following a Vasicek model with mean reversion of 4.3% and volatility 0.015384.
- Risky asset is a zero-coupon bond with expiry at time $n = 70$.
- **Bonus type:** Dividends are used to buy the life annuity only.
- Investment strategy $\eta$ is chosen such that it hedges the interest rate risk on the guaranteed payments on portfolio level.
- Dividends are given as second order interest rates $r^\delta$ with

$$r^\delta(t) = 0.01 + \frac{0.2 \cdot (U(t) - \max\{\tilde{V}^*(t), \tilde{V}^g(t)\})^+}{\tilde{V}^*(t)}.$$

- In the state-independent implementation, we consider

$$\delta(t) = (r^\delta(t) - 0.01) \frac{\tilde{V}^*(t)}{\tilde{V}^*,\dagger(t)} \frac{V^*,\dagger(t)}{\tilde{V}^*,\dagger(t)}.$$
Results

Based on \( N = 10,000 \) financial scenarios and Euler-Maruyama scheme with step length \( h = 1/100 \):

<table>
<thead>
<tr>
<th>GB</th>
<th>FDB</th>
<th>Relative difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>State-dependent</td>
<td>State-independent</td>
<td></td>
</tr>
<tr>
<td>−72,582</td>
<td>72,661</td>
<td>72,663</td>
</tr>
</tbody>
</table>

Let us see what goes on behind the scenes..

Fix a financial scenario:

Short rate scenario

Bond price scenario
Comparison of $Q$-modified probabilities

**Figure:** State-dependent (solid) and state-independent (dashed)
Comparison of bonus cash flow in different states

Figure: State-dependent (solid) and state-independent (dashed)