Efficient computation of expected allocations

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Introduction
Motivation

- Insurance: risk pooling mechanism
- Peer-to-peer insurance
  - Pools risks together
  - Must compute the contribution of each participant to the pool (expected values)
- Insurance companies
  - Compute risk measures at the portfolio level
  - Must compute the contribution of each risk to the portfolio (risk measures)
- Possible difficulties:
  1. Large pools / portfolios
  2. Heterogeneous risks
  3. Dependent risks
Consider a portfolio of \( n \) risks \( X = (X_1, \ldots, X_n) \)

Support for each rv: \( h\mathbb{N}_0 = \{0, h, 2h, \ldots\} \), with \( h > 0 \)

Let \( S \) be the aggregate loss rv, that is, \( S = \sum_{i=1}^{n} X_i \)

**Objective of talk**

1. Provide convenient representations for the values of \( E \left[ X_i \times 1_{\{S=kh\}} \right] \) and \( E \left[ X_i \times 1_{\{S\leq kh\}} \right] \) for \( i \in \{1, \ldots, n\} \) and \( k \in h\mathbb{N}_0 \)

2. Provide efficient computation methods for \( E \left[ X_i \times 1_{\{S=kh\}} \right] \) and \( E \left[ X_i \times 1_{\{S\leq kh\}} \right] \)
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Motivation

Peer-to-peer insurance

- Peer-to-peer insurance pricing schemes: compute the contribution of participant according to risk sharing rule [Denuit, 2020a]
- Conditional mean risk sharing rule [Denuit and Dhaene, 2012]: popular choice
- Satisfies desirable properties [Denuit et al., 2021]
- Price for the $i$th participant is expected contribution of risk $X_i$ given that the actual loss $S$ was $k > 0$

\[
E[X_i|S = k] = \frac{E[X_i \times 1\{S=k\}]}{\Pr(S = k)}, \quad i \in \{1, \ldots, n\}
\]
Consider a portfolio of risks with overall risk measure $TVaR_\kappa(S)$, $\kappa \in (0, 1)$, where $E[S] < \infty$

Problem: determine allocation (risk contribution) of risk $X_i$, $i \in \{1, \ldots, n\}$, to overall risk measure $TVaR_\kappa(S)$

$TVaR_\kappa$-based capital allocation with Euler rule:

$$TVaR_\kappa(X_i; S) = \frac{E[X_i \times 1_{\{S > k_0h\}}] + E[X_i \times 1_{\{S = k_0h\}}] \beta}{1 - \kappa},$$

where $k_0h = VaR_\kappa(S)$,

$$\beta = \begin{cases} \frac{\Pr(S \leq k_0h) - \kappa}{\Pr(S = k_0h)}, & \text{if } \Pr(S = k_0h) > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$E[X_i \times 1_{\{S > k_0h\}}] = E[X_i] - E[X_i \times 1_{\{S \leq k_0h\}}]$$
Common approaches

- For the remainder of this talk, set $h = 1$
- Direct computation. Let $S_{-i} = \sum_{j=1, j \neq i}^{n}$, then
  - Discrete case:
    \[
    E \left[ X_i \times 1_{\{S=k\}} \right] = \sum_{j=0}^{k} j f_{X_i,S_{-i}}(x, s - x)
    \]
    \[
    E \left[ X_i \times 1_{\{S \leq k\}} \right] = \sum_{j=0}^{k} E \left[ X_i \times 1_{\{S=j\}} \right]
    \]
  - Continuous case
    \[
    E \left[ X_i \times 1_{\{S=x\}} \right] = \int_{0}^{x} z f_{X_i,S_{-i}}(z, x - z) \, dz
    \]
    \[
    E \left[ X_i \times 1_{\{S \leq x\}} \right] = \int_{0}^{x} E \left[ X_i \times 1_{\{S=y\}} \right] \, dy
    \]
- Used in, for instance, in [Bargès et al., 2009]
Common approaches

- Size-biased transform: expected allocation is the expected value of a size-bias transformed rv
- Size-biased transform in actuarial science: [Denuit, 2019], [Denuit, 2020b]
- Expected allocations and cumulative expected allocations: [Landsman and Valdez, 2003], [Furman and Landsman, 2005], [Furman and Landsman, 2008], [Denuit and Robert, 2020], [Denuit and Robert, 2021c], [Denuit and Robert, 2021b], [Denuit and Robert, 2021a]
Ordinary generating functions
Method relies on generating functions

Definition (Ordinary generating function)

For a sequence \( \{a_k\}_{k \geq 0} \), the function

\[
A(z) = \sum_{k=0}^{\infty} a_k z^k
\]

is its ordinary generating function (OGF). We use the notation \([z^k] A(z)\) to refer to the coefficient \(a_k, k \in \mathbb{N}_0\).

Generating functions are a "bag" which hold the values of a sequence in a single formula [Sedgewick and Flajolet, 2013]
Important generating function for actuaries: probability generating function

Probability generating functions are OGFs for the sequence of probability masses, that is,

$$P_X(t) = E \left[ t^X \right] = \sum_{k=0}^{\infty} t^k \Pr(X = k)$$

One can recover the values of $\Pr(X = k)$ by differentiating

$$[t^k]P_X(t) = \Pr(X = k) = \frac{1}{k!} \frac{d^k}{dt^k} P_X(t) \bigg|_{t=0}, \quad k \in \mathbb{N}_0$$

Alternatively, use fast Fourier transform (FFT) algorithm
• Multivariate probability generating function

\[ P_{X_1,\ldots,X_n}(t_1,\ldots,t_n) = E \left[ t_1^{X_1} \times \cdots \times t_n^{X_n} \right] \]

• When risks are independent,

\[ P_{X_1,\ldots,X_n}(t_1,\ldots,t_n) = \prod_{j=1}^{n} P_{X_j}(t_j) \]

• For aggregate risk

\[ P_S(t) = E \left[ t^S \right] = E \left[ t^{X_1+\cdots+X_n} \right] = P_{X_1,\ldots,X_n}(t,\ldots,t) \]
Main results
Main result 1

- We present a generating function for the allocations of risk $X_1$

**Theorem**

We have

$$\mathcal{P}_S^{[1]}(t) := \left[ t_1 \times \frac{\partial}{\partial t_1} P_X(t_1, t_2, \ldots, t_n) \right]_{t_1=\cdots=t_n=t} = \sum_{k=0}^{\infty} t^k E \left[ X_1 \times 1_{\{S=k\}} \right].$$

Then, $\mathcal{P}_S^{[1]}(t)$ is the OGF for the sequence of expected allocations $\left\{ E \left[ X_1 \times 1_{\{S=k\}} \right] \right\}_{k \in \mathbb{N}_0}$.

- One can recover the values of $E \left[ X_1 \times 1_{\{S=k\}} \right], k \in \mathbb{N}_0$ by differentiating

$$[t^k] \mathcal{P}_S^{[1]}(t) = E \left[ X_1 \times 1_{\{S=k\}} \right] = \frac{1}{k!} \frac{d^k}{dt^k} \mathcal{P}_S^{[1]}(t) \bigg|_{t=0}, \quad k \in \mathbb{N}_0$$

- Alternatively, use FFT
Main result 2

- We present a generating function for the cumulative allocations of risk $X_1$

### Theorem

Let

$$\frac{P_S^{[1]}(t)}{1-t} = \sum_{k=0}^{\infty} t^k E \left[ X_1 \times 1_{\{S \leq k\}} \right]$$

Then, $P_S^{[1]}(t) / (1 - t)$ is the OGF for the sequence of cumulative expected allocations

$$\{E \left[ X_i \times 1_{\{S \leq kh\}} \right]\}_{k \in \mathbb{N}_0}.$$

- One can recover the values of $E \left[ X_i \times 1_{\{S \leq kh\}} \right], k \in \mathbb{N}_0$ by differentiating

$$[t^k] \left\{ \frac{P_S^{[1]}(t)}{1-t} \right\} = E \left[ X_i \times 1_{\{S \leq kh\}} \right] = \frac{1}{k!} \frac{d^k}{dt^k} \left. \frac{P_S^{[1]}(t)}{1-t} \right|_{t=0}, \quad k \in \mathbb{N}_0$$

- Alternatively, use FFT
Examples – closed form solutions
Independent compound \((a, b, 0)\) distributions

- Let \(M\) be a frequency rv and \(B_1 \sim B_2 \sim \cdots \sim B\) be a sequence of severity rvs (with support \(\mathbb{N}_0\) in this talk)
- Let \(X_1\) be a compound rv,

\[
X_1 = \begin{cases} 
0, & M = 0 \\
\sum_{j=1}^{M} B_j, & M > 0
\end{cases}
\]

- We consider the family of \((a, b, 0)\) distributions where

\[
f_M(k) = (a + b/k)f_M(k-1), \quad k \in \mathbb{N}_1
\]

- Members:
  - Poisson \((a = 0, b = \lambda)\)
  - Binomial \((a = -q/(1-q), b = (n+1)q/(1-q))\)
  - Negative binomial \((a = 1-q, b = (r-1)(1-q))\)
Proposition

Let $X_1$ be a compound rv with frequency rv $M$ with $f_M$ in the $(a, b, 0)$ family of distributions with $|a| < 1$ and discrete severity rv $B$, with $X_1$ independent of $(X_2, \ldots, X_n)$. The OGF for expected allocations is

$$
\mathcal{P}^{[1]}_S (t) = t \mathcal{P}'_{B_1}(t) \mathcal{P}'_{M_1}(\mathcal{P}_{B_1}(t)) \mathcal{P}_{X_2, \ldots, X_n}(t).
$$

Further, if $|a \mathcal{P}_{B_1}(t)| < 1$ for all $|t| < 1$, then

$$
\mathcal{P}^{[1]}_S (t) = t \mathcal{P}'_{B_1}(t) \frac{a + b}{1 - a \mathcal{P}_{B_1}(t)} \mathcal{P}_S(t)
$$
Independent compound \((a, b, 0)\) distributions

- Let \(X_1\) be rv in the class of compound Poisson distributions.
- Severity distribution is discrete with support \(\mathbb{N}_0\).
- We have

\[
\mathcal{P}_S^{[1]}(t) = \lambda_1 t \mathcal{P}'_{B_1}(t) \mathcal{P}_M(\mathcal{P}_{B_1}(t)) \mathcal{P}_{X_2,\ldots,X_n}(t) = \lambda_1 t \mathcal{P}'_{B_1}(t) \mathcal{P}_S(t)
\]

\[
[t^k] \mathcal{P}_S^{[1]}(t) = E[X_1 \times 1_{\{S=k\}}] = \lambda \sum_{l=1}^{k} l f_{B_1}(l) f_S(k-l), \quad k \in \mathbb{N}_1
\]

\[
[t^k] \left\{ \frac{\mathcal{P}_S^{[1]}(t)}{1-t} \right\} = E[X_1 \times 1_{\{S\leq k\}}] = \lambda \sum_{l=1}^{k} E[B_1 \times 1_{\{B_1\leq l\}}] f_S(k-l), \quad k \in \mathbb{N}_1
\]

\[
= \lambda \sum_{l=1}^{k} l f_{B_1}(l) F_S(k-l), \quad k \in \mathbb{N}_1
\]

- Discrete version of [Denuit and Robert, 2020]
Examples – dependent rvs
Poisson common shock

- Let $Y_A \sim \text{Pois}(\lambda_A)$
- Let $X_{ijk} = Y_{ijk} + Y_{ij} + Y_i + Y_0$ for $(i, j, k) \in \{1, 2\}^3$
- Tree dependence structure
Poisson common shock

- Define

\[ S = \sum_{(i,j,k) \in \{1,2\}^3} X_{ijk} \]

- Then \( S \) is a compound Poisson rv and

\[
P_S^{[111]}(t) = \left( \lambda_{111} t + \lambda_{11} t^2 + \lambda_1 t^4 + \lambda_0 t^8 \right) P_S(t)
\]

- We find

\[
E \left[ X_{111} \times 1_{\{S=k\}} \right] = \begin{cases} 
0, & k = 0 \\
\lambda_{111} f_S(k-1), & k = 1 \\
\lambda_{111} f_S(k-1) + \lambda_{11} f_S(k-2), & k = 2,3 \\
\lambda_{111} f_S(k-1) + \lambda_{11} f_S(k-2) + \lambda_1 f_S(k-4), & k = 4, \ldots, 7 \\
\lambda_{111} f_S(k-1) + \lambda_{11} f_S(k-2) + \lambda_1 f_S(k-4) + \lambda_0 f_S(k-8), & k = 8,9, \ldots
\end{cases}
\]
Multivariate negative Binomial

- Multivariate mixed Poisson distribution
- Common mixture $\Theta = (\Theta_1, \ldots, \Theta_n)$ with $E[\Theta_i] = 1$ for $i = 1, \ldots, n$
- Vector of conditionally independent rvs $(X_i|\Theta_i = \theta_i) \sim Poisson(\lambda_i\theta_i)$ for $i = 1, \ldots, n$
- The pgf of $X$ is

$$P_X(t_1, \ldots, t_n) = E_\Theta \left[ e^{\Theta_1\lambda_1(t_1-1)} \cdots e^{\Theta_n\lambda_n(t_n-1)} \right] = M_\Theta(\lambda_1(t_1-1), \ldots, \lambda_n(t_n-1))$$

(2)

where $M_\Theta(t_1, \ldots, t_n)$ is the moment generating function of $\Theta$

- OGF of expected allocations:

$$P^{[1]}_S(t) = \lambda_1 t \left[ \frac{\partial}{\partial x} M_\Theta(x, \lambda_2(t-1), \ldots, \lambda_n(t-1)) \right] \bigg|_{x = \lambda_1(t-1)}$$

(3)
Poisson-gamma common mixture

- Mixture distribution from a bivariate gamma common shock model from [Mathai and Moschopoulos, 1991]
- Define three rvs $Y_i, i \in \{0, 1, 2\}$ with
  - $Y_0 \sim \text{Gamma}(\gamma_0, \beta_0)$
  - $Y_i \sim \text{Gamma}(r_i - \gamma_0, r_i), i \in \{1, 2\}$
  - $0 \leq \gamma_0 \leq \min(r_1, r_2)$
- Let $\Theta_i = \frac{\beta_0}{r_i}Y_0 + Y_i, i = 1, 2$
- $(\Theta_1, \Theta_2)$ is a bivariate Gamma random vector with $\Theta_i \sim \text{Ga}(r_i, r_i), i = 1, 2$, and $\gamma_0$ is a dependence parameter
- $(X_1, X_2)$ is a bivariate negative binomial rv
Poisson-gamma common mixture

- Moment generating functions of mixture random vector

\[ M_{\Theta_1, \Theta_2}(x_1, x_2) = \left(1 - \frac{x_1}{r_1}\right)^{-(r_1-\gamma_0)} \left(1 - \frac{x_2}{r_2}\right)^{-(r_2-\gamma_0)} \left(1 - \frac{x_1}{r_1} - \frac{x_2}{r_2}\right)^{-\gamma_0} \]

- PGF of \( S \)

\[ \mathcal{P}_S(t) = (1 - \zeta_1(t - 1))^{-(r_1-\gamma_0)} (1 - \zeta_2(t - 1))^{-(r_2-\gamma_0)} (1 - \zeta_{12}(t - 1))^{-\gamma_0}, \]

where \( \zeta_1 = \lambda_1 / r_1, \zeta_2 = \lambda_2 / r_2 \) and \( \zeta_{12} = \lambda_1 / r_1 + \lambda_2 / r_2 \)

- \( S \) is the sum of three independent negative binomial rvs with parameters

\( (r_1 - \gamma_0, 1/(1 - \zeta_1)), (r_2 - \gamma_0, 1/(1 - \zeta_2)) \) and \( (\gamma_0, 1/(1 - \zeta_{12})) \)
Poisson-gamma common mixture

- The OGF for expected allocations is

\[
P_S^{[1]}(t) = \lambda_1 t \left( \frac{1 - \gamma_0/r_1}{1 - \zeta_1 (t - 1)} + \frac{\gamma_0/r_1}{1 - \zeta_{12} (t - 1)} \right) P_S(t)
\]

- Expected allocations

\[
[t^k] P_S^{[1]}(t) = E \left[ X_1 \times 1_{\{S = k\}} \right] = \lambda_1 \sum_{j=0}^{k-1} \left[ \left( 1 - \frac{\gamma_0}{r_1} \right) \frac{1}{1 + \zeta_1} \left( \frac{\zeta_1}{1 + \zeta_1} \right)^j + \frac{\gamma_0}{r_1} \frac{1}{1 + \zeta_{12}} \left( \frac{\zeta_{12}}{1 + \zeta_{12}} \right)^j \right] f_S(k - 1 - j)
\]
Fast Fourier transform algorithm
- Characteristic function of \( S \) as

\[
\phi_S(t) := E\left[e^{itS}\right] = \mathcal{P}_S\left(e^{it}\right)
\]

- Characteristic version of OGF for expected allocations

\[
\phi_S^{[1]}(t) := \sum_{k=0}^{\infty} e^{itk} E\left[X_1 \times 1_{\{S=k\}}\right] = \mathcal{P}_S^{[1]}\left(e^{it}\right).
\]
Fast Fourier transform algorithm

- Let $\mu_k^{[1]} = E[X_1 \times 1_{\{S=k\}}]$ for $k = 0, \ldots, k_{\text{max}} - 1$
- Let $\mu^{[1]} = (\mu_0^{[1]}, \ldots, \mu_{k_{\text{max}}}^{[1]})$
- Discrete Fourier transform of $\mu^{[1]}$, noted $\hat{\mu}^{[1]} = (\hat{\mu}_0^{[1]}, \ldots, \hat{\mu}_{k_{\text{max}}-1}^{[1]})$, is

$$\hat{\mu}_j^{[1]} = \mathcal{P}^{[1]}_S \left( e^{i2\pi j/k_{\text{max}}} \right), \quad j = 0, \ldots, k_{\text{max}} - 1$$

(4)

- The inverse DFT can recover the sequence of expected allocations

$$\mu_j^{[1]} = \frac{1}{k_{\text{max}}} \sum_{j=0}^{k_{\text{max}}-1} \text{Re} \left( \hat{\mu}_j^{[1]} e^{-i2\pi j k/k_{\text{max}}} \right), \quad k = 0, \ldots, k_{\text{max}} - 1$$

- See [Embrechts et al., 1993, Embrechts and Frei, 2009] for applications of FFT in actuarial science
Compound Poisson distribution

**Input:** Parameters $\lambda_i, f_{B_i}$ for $i = 1, \ldots, n$.

**Output:** Expected allocations $E[X_i|S = k]$ for $k = 0, \ldots, k_{max} - 1$ and $i = 1, \ldots, n$

1. for $i = 1, \ldots, n$ do
2.     Compute $\hat{f}_{X_i} = \mathcal{P}_{X_i}(\hat{e}_1)$;
3. Compute the DFT of $S$ as the element-wise product $\hat{f}_S = \prod_{i=1}^n \hat{f}_{X_i}$;
4. Compute $f_S$ by taking the inverse DFT of $\hat{f}_S$;
5. for $i = 1, \ldots, n$ do
6.     Compute the DFT $\hat{\phi}_{B_i}$ of the vector $\{(k + 1)f_{B_i}(k + 1)\}_{0 \leq k \leq k_{max} - 1}$;
7.     Compute element-wise $\hat{\mu}^{[i]} = \lambda_i \hat{e}_1 \times \hat{\phi}_{B_i} \times \hat{f}_S$;
8.     Compute $\mu^{[i]}$ as the inverse DFT of $\hat{\mu}^{[i]}$;
9.     Compute $\{E[X_i|S = k]\}_{0 \leq k \leq k_{max} - 1}$ by the element-wise division $\mu^{[1]} / f_S$;
10. Return $\{E[X_i|S = k]\}_{0 \leq k \leq k_{max} - 1}$ for $i = 1, \ldots, n$. 

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Efficient computation of expected allocations
Numerical examples
Application 1: large portfolio

- Consider a portfolio of 10000 risks
- Compound Poisson distributions with mean $\lambda_j$
- Severity rv $B_j \sim NBinom(r_j, q_j)$
- Simulate different parameters for each risk

\[
\begin{align*}
\lambda_j & \sim Exp(10) \\
r_j & \sim Unif\{1, 2, 3, 4, 5, 6\} \\
q_j & \sim Unif([0.4, 0.5])
\end{align*}
\]

- On average, $\lambda_j = 0.1$, $r_j = 3.5$ and $q_j = 0.45$
Application 1: large portfolio

Code (setup)

```r
set.seed(10112021)
n_participants <- 10000
kmax <- 2^13
lam <- list(); fc <- list(); mu <- list()

lambdas <- rexp(n_participants, 10)
rs <- sample(1:6, n_participants, replace = TRUE)
qs <- runif(n_participants, 0.4, 0.5)

for (i in 1:n_participants) {
  lam[[i]] <- lambdas[i]
  fci <- dnbinom(0:(kmax-2), rs[i], qs[i])
  fc[[i]] <- c(fci, 1 - sum(fci))
}
```
Application 1: large portfolio

Code (allocations)

```r
# Code

dft_fx <- list(); phic <- list(); cm <- list()

for(i in 1:n_participants) {
  dft_fx[[i]] <- exp(lam[[i]] * (fft(fc[[i]]) - 1))
  phic[[i]] <- fft(c(1:(kmax-1) * fc[[i]][-1], 0))
}

dft_fs <- Reduce("*", dft_fx)
fs <- Re(fft(dft_fs, inverse = TRUE))/kmax
e1 <- exp(-2i*pi*(0:(kmax-1))/kmax)

for(i in 1:n_participants) {
  dft_mu <- e1 * phic[[i]] * lam[[i]] * dft_fs
  mu[[i]] <- Re(fft(dft_mu, inverse = TRUE))/kmax
  cm[[i]] <- mu[[i]]/fs
}
```

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Efficient computation of expected allocations
### Application 1: Large Portfolio

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<td>0.03</td>
<td>0.03</td>
<td>0.24</td>
<td>0.12</td>
<td>0.47</td>
<td>0.15</td>
<td>0.01</td>
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<tr>
<td>$q_j$</td>
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<td>0.42</td>
<td>0.46</td>
<td>0.45</td>
<td>0.49</td>
<td>0.44</td>
<td>0.44</td>
<td>0.48</td>
</tr>
<tr>
<td>$r_j$</td>
<td>2</td>
<td>6</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>5</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>$E[X_j]$</td>
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<td>0.26</td>
<td>0.03</td>
<td>1.16</td>
<td>0.73</td>
<td>2.99</td>
<td>0.56</td>
<td>0.01</td>
</tr>
</tbody>
</table>

**Table:** First 8 set of parameters
Application 1: large portfolio

- Computes $1600 \times 10000$ conditional means at once
- Takes approximately 16 seconds on a personal computer

Figure: Left: total conditional means. Right: density function of $S$. 
Application 2

- Let $X_j, j \in \{1, \ldots, n\}$: arithmetized Pareto distribution with the moment matching method
- Study $n \in \{1, 3, 100, 1000\}$ to observe behavior of conditional means as $n$ increases
- Set first three risks with fixed parameters
  - Set $(\alpha_1, \alpha_2, \alpha_3) = (1.3, 1.6, 1.9)$
  - Set $\lambda_j = 10(\alpha_j - 1)$ such that $E[X_j] \approx 10, j \in \{1, 2, 3\}$
- Simulate remaining parameters
  - Simulate parameters
    \[
    \begin{cases}
    \alpha_j \sim Unif([1.3, 1.9]) \\
    \lambda_j \sim Unif([5, 15])
    \end{cases}
    \]
  - We have $50/9 \leq E[X_j] \leq 50$ for $j \in \{4, \ldots, 1000\}$
library(actuar)

n <- 1000

xmax <- 2^15
kmax <- 2^20
alphas <- c(alphas, runif(n - 100, 1.3, 1.9))
lambdas <- c(lambdas, runif(n - 100, 5, 15))

phis1000 <- rep(1, kmax)

for(i in 1:n) {
  fx <- discretize(ppareto(x, alphas[i], lambdas[i]), 0, xmax - 1, method = "unbiased", lev = levpareto(x, alphas[i], lambdas[i]))
  phix <- fft(c(fx, rep(0, kmax - xmax)))
  phis1000 <- phis1000 * phix
}

fs1000 <- Re(fft(phis1000, inverse = TRUE))/kmax
```r
phi1 <- exp(-2i*pi*(0:(kmax-1))/kmax)
cm1000 <- list()

for (i in 1:n) {
  fx <- discretize(ppareto(x, alphas[i], lambdas[i]), 0, xmax - 1,
                   method = "unbiased", lev = levpareto(x, alphas[i], lambdas[i]))
  phix <- fft(c(fx, rep(0, kmax - xmax)))
  phi_deriv_x1 <- fft(c((1:(xmax - 1)) * fx[-1], rep(0, kmax - xmax + 1)))
  phism1 <- phis1000 / phix
  agf <- phism1 * phi_deriv_x1 * phi1
  cm1000[[i]] <- (Re(fft(agf, inverse = TRUE))/kmax / fs1000)[1:xmax]
}
```
Application 2: heavy tailed portfolio

- Numerical analysis for $n = 1000$
- Computes $E [X_i \times 1_{\{S=kh\}}]$ for $i \in \{1, \ldots, 1000\}$ and $k \in \{0, \ldots, 1048576\}$
- Takes approximately 9 minutes on a personal computer

**Figure:** Left: total conditional means. Right: density function of $S$. 
Application 2: heavy tailed portfolio

Figure: Cumulative distribution function of conditional means for $n = 3, 100, 1000$. 

$X_1, \alpha_1=1.3, \lambda_1=3$

$X_2, \alpha_2=1.6, \lambda_2=6$

$X_3, \alpha_3=1.9, \lambda_3=9$
Propose new method to compute expected allocations

Practical applications for peer-to-peer insurance and capital allocation

Convenient results for independent compound \((a, b, 0)\) distributions

Convenient results for dependent rvs (is the pgf easy to differentiate?)

Efficient algorithm using FFT, even with large heterogeneous portfolios of heavy-tailed risks
Conclusion

Thanks for your attention!

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References


Panjer recursion versus FFT for compound distributions.

Some applications of the fast Fourier transform algorithm in insurance mathematics
This paper is dedicated to Professor W. S. Jewell on the occasion of his 60th birthday.

Risk capital decomposition for a multivariate dependent gamma portfolio.

