Stable dividends are optimal under linear-quadratic optimization

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Agenda

• Motivation
• LQ optimization
• Classical optimization
• Compare and benchmarks
Motivation

Stability criteria

- Expected present value of dividends until ruin. [de Finetti(1957)]
Motivation

Stability criteria

• Expected present value of dividends until ruin. [de Finetti(1957)]
  
  - Optimal dividend strategies unreasonable in practise.
Motivation

Stability criteria

• Expected present value of dividends. [de Finetti(1957)]
  - Optimal dividend strategies unreasonable in practice.

• Stable dividends. [Avanzi, Tu, Wong (2016)]
Motivation

Stability criteria

• Expected present value of dividends. [de Finetti(1957)]
  - Optimal dividend strategies unreasonable in practise.

• Stable dividends. [Avanzi, Tu, Wong (2016)]
  - Optimal strategies rarely have this property.
Motivation

Stable dividends - Affine dividend strategy.
Motivation

Stable dividends - Affine dividend strategy.

- Not optimal for maximizing expected present value of dividends.

- Optimal in the linear quadratic optimal control problem.

[Cairns(2000)]
Setup

Surplus model

\[ dX(t) = c \, dt + \sigma \, dW(t) - l(t, X(t)) \, dt, \]
Setup

Surplus model

\[ dX(t) = c \, dt + \sigma \, dW(t) - l(t, X(t)) \, dt, \]

- \( c \) deterministic variation.
- \( W(t) \) Brownian motion.
- \( l(t, X(t)) \) continuous dividend process.
Linear-Quadratic objective

Value function

\[ V(t, X(t)) = \min_{l} \mathbb{E}_{t,x} \left[ \frac{1}{2} \int_{t}^{T} e^{-\delta(s-t)} \left( l(s, X(s)) - l_0 - l_1 X(s) \right)^2 ds 
+ \frac{1}{2} \int_{t}^{T} e^{-\delta(s-t)} \gamma \left( X(s) - x_0 \right)^2 ds 
+ \lambda \left( X(T) - k \right)^2 \right] \]
Linear-Quadratic objective

Value function

\[ V(t, X(t)) = \min_l \mathbb{E}_{t,x} \left[ \frac{1}{2} \int_t^T e^{-\delta(s-t)} \left( l(s, X(s)) - l_0 - l_1 X(s) \right)^2 ds \right. \]

\[ \left. + \frac{1}{2} \int_t^T e^{-\delta(s-t)} \gamma (X(s) - x_0)^2 ds \right. \]

\[ \left. + \lambda \left( X(T) - k \right)^2 \right] \]

Satisfy Hamilton-Jacobi-Bellman equation.
Linear-Quadratic objective

Value function

\[ V(t, X(t)) = \min_l \mathbb{E}_{t,x} \left[ \frac{1}{2} \int_t^T e^{-\delta(s-t)} \left( l(s, X(s)) - l_0 - l_1 X(s) \right)^2 ds + \frac{1}{2} \int_t^T e^{-\delta(s-t)} \gamma \left( X(s) - x_0 \right)^2 ds + \lambda \left( X(T) - k \right)^2 \right] \]

Satisfy Hamilton-Jacobi-Bellman equation.

Optimal dividend strategy \( l^*(t, x) = l_0 + l_1 x + V_x(t, x) \).
Linear-Quadratic objective

Guess

\[ V(t, x) = q(t)x^2 + p(t)x + r(t), \]

where the functions \( q, p, r \) satisfy ODEs.
Linear-Quadratic objective

Guess

\[ V(t, x) = q(t)x^2 + p(t)x + r(t), \]

where the functions \( q, p, r \) satisfy ODEs.

Optimal dividend strategy \( l^*(t, x) = l_0 + p(t) + (l_1 + 2q(t))x. \)
Classical objective

Dividend payouts to shareholders

$$E_{t,x} \left[ \int_{t}^{\tau_x} e^{-\tilde{\delta}(s-t)} dD(s) \right],$$

where $D$ is dividends and $\tau_x$ is the time of ruin.
Classical objective

Dividend payouts to shareholders

\[ \mathbb{E}_{t,x} \left[ \int_{t}^{\tau_x} e^{-\tilde{\delta}(s-t)} dD(s) \right], \]

where \( D \) is dividends and \( \tau_x \) is the time of ruin.

Optimal dividend strategy is the barrier strategy.
Comparing objectives

Comparing the classical optimization problem and the LQ problem.

- Understanding different objectives.
- Optimal dividend strategies.
Numerical
Affine dividends

[Avanzi and Wong(2012)] and [Albrecher and Cani(2017)] maximize expected present value of affine dividends until ruin

$$\max_{\tilde{l}_0, \tilde{l}_1} \mathbb{E}_{t,x} \left[ \int_t^{\tau_x} e^{-\tilde{\delta}(s-t)} \left( \tilde{l}_0 + \tilde{l}_1 X(s) \right) ds \right],$$

and $X$ is mean reverting around $\frac{c-\tilde{l}_0}{\tilde{l}_1}$. 
Numerical
Numerical

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<th>$\sigma$</th>
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<td>$0.5b^*$</td>
<td>0</td>
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Numerical
Relative difference in initial surplus

\( \xi \) - Relative difference in initial surplus for present value of future dividends to be equal

\[
V^{LQ}(t, X(t)(1 + \xi)) = V^{b}(t, X(t))
\]
References

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