The skewness of Bornhuetter-Ferguson

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About the speaker

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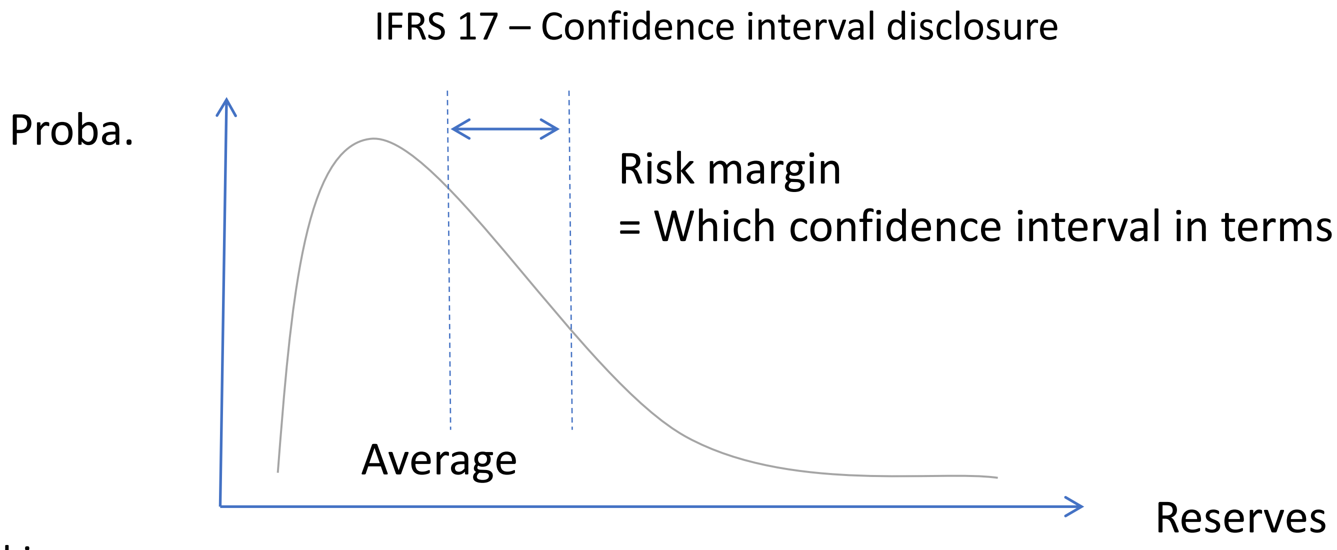
Agenda

- The question
- BF stochastic model
- An extension of the stochastic model
- Skewness per accident year
- Skewness overall
- Numerical examples
- Conclusion

The question



Solution proposed in :
“Probability of sufficiency of Solvency 2 Reserve risk margins: Practical Approximations”
published in ASTIN Bulletin
based on the characteristics of the reserve risk distribution (mean, standard deviation, skewness)

Parameters for Chain-Ladder:
R package Chain-Ladder

What about the parameters for BF ?

=> Mean, standard deviation, skewness

BF stochastic model

From Mack (2008):

UWY	Dvpt					A priori ultimate
	1	2	3	4	5	
1	$C_{1,1}$	$C_{1,2}$	$C_{1,3}$	$C_{1,4}$	$C_{1,5}$	$\hat{U}_1$
2	$C_{2,1}$	$C_{2,2}$	$C_{2,3}$	$C_{2,4}$		$\hat{U}_2$
3	$C_{3,1}$	$C_{3,2}$	$C_{3,3}$			$\hat{U}_3$
4	$C_{4,1}$	$C_{4,2}$				$\hat{U}_4$
5	$C_{5,1}$					$\hat{U}_5$

Reserves

$\hat{R}_i^{BF} = \hat{U}_i(1 - \hat{z}_{n+1-i})$

$\hat{z}_k \in [0;1]$: estimated percentage of the ultimate claims amount which is expected to be known after development year k

Incremental claim

$S_{i,k} = C_{i,k} - C_{i,k-1}$

Mack (2008) assumed the following in relation to the increments $S_{i,k}$:

- BF1: All increments $S_{i,k}$ are independent
- BF2: There are unknown parameters x_i, y_k such that:
  - $E(S_{i,k}) = x_i y_k$
  - $y_1 + \cdots + y_{n+1} = 1$
- BF3: There are unknown proportionality constants s_k^2 with $Var(S_{i,k}) = x_i s_k^2$

An extension of the stochastic model

From Mack (2008):

UWY	Dvpt					A priori ultimate
	1	2	3	4	5	
1	$C_{1,1}$	$C_{1,2}$	$C_{1,3}$	$C_{1,4}$	$C_{1,5}$	$\hat{U}_1$
2	$C_{2,1}$	$C_{2,2}$	$C_{2,3}$	$C_{2,4}$		$\hat{U}_2$
3	$C_{3,1}$	$C_{3,2}$	$C_{3,3}$			$\hat{U}_3$
4	$C_{4,1}$	$C_{4,2}$				$\hat{U}_4$
5	$C_{5,1}$					$\hat{U}_5$

Reserves

$$\hat{R}_i^{BF} = \hat{U}_i(1 - \hat{z}_{n+1-i})$$

Incremental claim

$$S_{i,k} = C_{i,k} - C_{i,k-1}$$

- BF1: All increments $S_{i,k}$ are independent
- BF2: There are unknown parameters x_i, y_k such that:
  - $E(S_{i,k}) = x_i y_k$
  - $y_1 + \cdots + y_{n+1} = 1$
- BF3: There are unknown proportionality constants s_k^2 with $Var(S_{i,k}) = x_i s_k^2$

BF4: There are unknown proportionality constants t_k^3 with $\mathrm{K}(S_{i,k}) = x_i^{3/2} t_k^3$

where $\mathrm{K}(S_{i,k}) = E\left[\left(S_{i,k} - E(S_{i,k})\right)^3\right]$

An extension of the stochastic model

Estimation of the parameters

Following Mack (2008), we have the following with $x_1, \ldots x_n$ known ($E(\hat{U}_i) = x_i$):

$$\hat{y}_k = \sum_{i=1}^{n+1-k} S_{i,k} / \sum_{i=1}^{n+1-k} x_i$$ is a best linear unbiased estimate of y_k

$$\hat{s}_k^2 = \frac{1}{n-k} \sum_{i=1}^{n+1-k} \frac{\left(S_{i,k} - x_i \hat{y}_k\right)^2}{x_i}$$ is an unbiased estimate of s^2_k.

And, following BF4, we get:

$$\hat{t}_k^3 = \frac{1}{n-k} \sum_{i=1}^{n+1-k} \frac{\left(S_{i,k} - x_i \hat{y}_k\right)^3}{x_i^{3/2}}$$ is an unbiased estimate of t^3_k

An extension of the stochastic model

With the parameters $\hat{y}_k, \hat{s}_k^2$ *and* $\hat{t}_k^3$, we can calculate the BF claims reserve by:

$$\hat{R}_i^{BF} = \hat{U}_i(\hat{y}_{n+2-i} + \ldots + \hat{y}_{n+1}) = \hat{U}_i(1 - \hat{z}_{n+1-i}) \text{ with } \hat{z}_k = \hat{y}_1 + \ldots + \hat{y}_k$$

The properties of these estimators are given below:

- $\hat{y}_1, \ldots, \hat{y}_{n+1}$ are pairwise (slightly) negatively correlated as they have to add up to unity.
- $\hat{y}_1, \ldots, \hat{y}_{n+1}$ and $\hat{z}_1, \ldots, \hat{z}_{n+1}$ are independent from $\hat{U}_1, \ldots, \hat{U}_n$
- $E(\hat{U}_i) = E(U_i) = x_i \; for \; 1 \leq i \leq n$

Skewness per accident year

We have:

$$SKEW\left(\hat{R}_i^{BF}\right) = K\left(\hat{R}_i^{BF}\right) - K(R_i)$$

where $\mathrm{K}(\mathrm{R}_i) = E\left[\left(R_i - E(R_i)\right)^3\right]$

Following BF4, we have:

$$K(R_i) = K\left(S_{i,n+2-i}\right) + \ldots + K\left(S_{i,n+1}\right) = x_i^{3/2}\left(t_{n+2-i}^3 + \ldots + t_{n+1}^3\right)$$

which is estimated by:

$$\widehat{K}(R_i) = \widehat{U}_i^{3/2}\left(\hat{t}_{n+2-i}^3 + \ldots + \hat{t}_{n+1}^3\right)$$

Skewness per accident year

As for $K\left(\hat{R}_i^{BF}\right) = K\left[\hat{U}_i(1 - \hat{z}_{n+1-i})\right]$, we use the formula below which holds when X and Y are independent:

$$K(XY) = K(X)K(Y) + K(X)E(Y)[3Var(Y) + E(Y)^2] + K(Y)E(X)[3Var(X) + E(X)^2] + 6E(X)E(Y)Var(Y)Var(X)$$

Using the above formula, we have:

$$K\left(\hat{R}_i^{BF}\right) = -K\left[\hat{U}_i\right]K(\hat{z}_{n+1-i}) + K\left[\hat{U}_i\right]E(1 - \hat{z}_{n+1-i})[3Var(\hat{z}_{n+1-i}) + E(1 - \hat{z}_{n+1-i})^2] - K[\hat{z}_{n+1-i}]E\left(\hat{U}_i\right)\left[3Var\left(\hat{U}_i\right) + E\left(\hat{U}_i\right)^2\right] + 6\,E\left(\hat{U}_i\right)E(1 - \hat{z}_{n+1-i})Var\left(\hat{U}_i\right)Var(\hat{z}_{n+1-i})$$

Estimators of $E\left(\hat{U}_i\right), Var\left(\hat{U}_i\right), E(1 - \hat{z}_{n+1-i})$ *and* $Var(\hat{z}_{n+1-i})$ are provided in the article of Mack (2008).

There remains to estimate $K\left[\hat{U}_i\right]$ *and* $K(\hat{z}_{n+1-i})$

Skewness per accident year

As for $K\left[\widehat{U}_i\right]$, we assume a lognormal distribution with parameters:

$$\sigma = \sqrt{ln\left(1 + \frac{Var(\widehat{U}_i)}{E(\widehat{U}_i)^2}\right)}$$

$$K\left[\widehat{U}_i\right] = \left(2 + exp(\sigma^2)\right)\frac{Var(\widehat{U}_i)^2}{E(\widehat{U}_i)}$$

Note: Can use also pricing distribution

As for $K(\hat{z}_{n+1-i})$, due to the slightly negative correlations between $\hat{y}_1, \ldots, \hat{y}_{n+1}$, we have :

$$K(\hat{z}_k) \approx min[K(\hat{y}_1)+\ldots+K(\hat{y}_k); K(\hat{y}_{k+1})+\ldots+K(\hat{y}_{n+1})]$$

with:

$$K(\hat{y}_k) = \hat{t}_k^3 \sum_{i=1}^{n+1-k} \widehat{U}_i^{3/2} \Bigg/ \left(\sum_{i=1}^{n+1-k} \widehat{U}_i\right)^3$$

Skewness overall

Fleishman polynomials

In order to do the aggregation, we assume that the centralized and normalized copy of the risk value X_i of the i-th class,

$$\widehat{X_i} = \frac{X_i - E(X_i)}{E(X_i)\ CoV_{X_i}}$$

(where CoV denotes the coefficient of variation), is estimated by the Fleishman polynomial structure of a standard normal random variable.

In particular, we consider the following case:

$$\widehat{X_i} = P_2(Z_i) = a_i Z_i + b_i\left(Z_i^2 - 1\right)$$

where Z_i denotes the standard normal distribution – Such a case is suitable for estimating the Skewness of a risk portfolio profile when the confidence level is approximated using Skewness only.

$$\begin{cases} a_i = \sqrt{1 - 2b_i^2} \\ \frac{SKEW(R_i^{BF})}{VAR(R_i^{BF})^{1.5}} = 6b_i - 4b_i^3 \end{cases}$$

Skewness overall

Fleishman polynomials

we define the total reserve value across the portfolio of m risks as:

- $X_\Sigma = \sum_{i=1}^{m} X_i$

where each i-th risk value is approximated by Fleishman polynomial of a standard normal random variable

We compute

$$Var(X_\Sigma) = E\left[\left(\sum_{i=1}^{m} \sigma_i P_2(Z_i)\right)^2\right] = \sum_{i=1}^{m} \sigma_i^2 + 2\sum_{ij} \sigma_i\, \sigma_j E\left[P_2(Z_i)P_2(Z_j)\right]$$

where $E\left[P_2(Z_i)P_2(Z_j)\right] = \rho_{ij}\left(a_i a_j + 2b_i b_j \rho_{ij}\right)$ and $\rho_{ij} = \frac{\hat{z}_{n+1-j}(1-\hat{z}_{n+1-i})}{\hat{z}_{n+1-i}\left(1-\hat{z}_{n+1-j}\right)}$ (see Mack 2008)

The same calculation can be done for the Skewness.

Numerical example

A lot of math … but does it work ?

Casualty NP – Incremental triangle

Triangle 1				Development Year												
Accident Year	Premium	Initial LR	Ui	1	2	3	4	5	6	7	8	9	10	11	12	13
2005	110'940'316	69.6%	77'176'365	4'626'711	7'833'956	9'237'575	4'075'137	10'828'853	2'000'000	1'916'455	1'000'000	2'136'161	873'469	-696'077	1'970'249	2'060'435
2006	136'881'755	54.1%	74'081'078	2'497'127	11'277'229	7'424'316	5'612'201	3'410'710	930'785	952'588	3'311'966	2'527'888	2'177'772	655'961	2'129'708	
2007	148'066'614	74.1%	109'785'557	6'733'949	16'001'488	8'581'545	16'094'884	4'740'896	2'946'401	5'320'553	8'191'596	858'217	1'921'349	2'674'149		
2008	142'419'083	64.8%	92'216'629	5'495'872	10'634'243	9'357'874	3'687'937	5'459'783	7'244'685	3'514'985	1'278'885	1'130'235	2'201'251			
2009	141'000'201	85.5%	120'589'443	5'513'563	11'892'818	13'879'064	5'892'620	6'906'455	19'110'413	4'127'163	4'465'916	4'691'799				
2010	148'748'619	79.4%	118'144'218	4'642'178	9'611'735	16'674'760	7'170'953	13'301'042	7'576'809	1'561'553	5'247'638					
2011	181'023'013	76.1%	137'843'628	5'088'492	12'712'715	12'492'986	16'125'708	17'424'813	9'530'326	6'511'744						
2012	195'545'471	95.0%	185'764'666	5'911'892	14'485'648	15'199'655	20'501'488	15'352'506	10'000'000							
2013	212'536'401	73.2%	155'613'255	7'261'343	13'285'662	16'307'615	12'557'638	11'727'223								
2014	210'377'519	83.5%	175'714'686	5'210'360	23'258'993	10'144'201	23'137'200									
2015	219'950'578	79.3%	174'402'671	6'709'466	14'819'711	20'470'286										
2016	230'357'704	87.3%	201'091'730	13'556'673	13'519'627											
2017	238'706'732	84.9%	202'706'418	8'255'514												

- Fairly volatile triangle
- Extreme Initial LR (2006: 54%, 2012: 95%)

=> Should have a high positive skewness

Numerical example

Triangle 1				Fleishmann polyn.								
i	Ri BF	msep(Ri)	Skewness (Ri)	**bi**	**ai**	k	t^3_k	s^2_k	y_k	$se(y_k)$	z_k	$se(z_k)$
2017	193'654'347	12.0%	0.172	0.0288	0.9992	1	2'386'009	22'883	4.47%	0.35%	4.47%	0.35%
2016	172'363'095	12.2%	0.141	0.0235	0.9994	2	18'165'418	96'444	9.82%	0.77%	14.29%	0.85%
2015	132'336'647	12.9%	0.168	0.0280	0.9992	3	-4'756'396	74'961	9.83%	0.73%	24.12%	1.12%
2014	117'146'970	12.9%	0.151	0.0251	0.9994	4	5'727'410	166'684	9.21%	1.16%	33.33%	1.61%
2013	90'794'602	13.3%	0.073	0.0121	0.9999	5	18'245'877	129'123	8.32%	1.10%	41.65%	1.95%
2012	96'347'577	12.0%	-0.355	-0.0593	0.9965	6	127'640'479	225'383	6.48%	1.57%	48.13%	2.23%
2011	66'978'218	12.3%	0.271	0.0452	0.9980	7	-902'325	23'032	3.28%	0.56%	51.41%	2.16%
2010	52'717'174	11.8%	0.208	0.0346	0.9988	8	4'199'317	51'181	3.97%	0.93%	55.38%	1.95%
2009	50'921'257	11.5%	0.316	0.0528	0.9972	9	-271'596	19'225	2.39%	0.64%	57.77%	1.84%
2008	37'067'585	11.7%	0.303	0.0507	0.9974	10	298	4'798	2.03%	0.37%	59.80%	1.80%
2007	43'021'846	10.7%	0.369	0.0617	0.9962	11	-695'855	25'322	1.01%	0.98%	60.81%	1.51%
2006	27'022'245	10.8%	0.365	0.0610	0.9963	12	168	392	2.71%	0.16%	63.52%	1.50%
2005	26'090'865	10.9%	-	0.0000	1.0000	13	-	-	2.67%	0.00%	66.19%	1.50%
Total	1'106'462'428	5.53%	**0.835**									

Not very volatile using BF

Skewness is high as expected

Numerical example

Casualty Prop – Incremental triangle

Triangle 3				Development Year												
Accident Year	Premium	Initial LR	Ui	1	2	3	4	5	6	7	8	9	10	11	12	13
2005	306'442	90.0%	275'798	13'487	50'621	30'204	33'570	24'290	12'173	5'826	10'832	21'869	10'000	1'048	4'023	4'801
2006	360'077	90.0%	324'069	20'508	56'047	17'758	33'570	24'290	12'173	5'826	10'832	21'869	10'000	17'621	6'244	
2007	526'754	90.0%	474'078	14'156	50'362	53'811	43'285	28'710	31'867	31'430	18'001	12'657	6'330	5'052		
2008	465'852	90.0%	419'267	30'721	41'275	19'637	30'725	28'823	21'106	4'683	8'735	5'000	2'938			
2009	429'724	90.0%	386'751	7'305	30'942	29'580	25'095	17'541	9'402	8'953	4'637	7'104				
2010	381'545	90.0%	343'390	13'601	40'569	22'200	20'380	21'140	10'123	-572	5'787					
2011	367'539	90.0%	330'785	9'657	38'809	47'837	31'987	22'077	6'375	7'262						
2012	347'770	90.0%	312'993	7'085	41'073	55'745	46'024	17'225	19'820							
2013	333'314	90.0%	299'982	18'237	38'891	51'624	39'890	29'183								
2014	382'678	90.0%	344'410	15'008	41'664	40'047	25'834									
2015	375'905	90.0%	338'314	21'592	40'788	26'595										
2016	359'604	90.0%	323'644	17'061	33'068											
2017	388'474	90.0%	349'627	17'291												

- Fairly stable triangle
- Constant Initial LR
- Stable portfolio

=> Should have a low skewness

Numerical example

Triangle 3				Fleishmann polyn.								
i	Ri BF	msep(Ri)	Skewness (Ri)	**bi**	**ai**	k	t^3_k	s^2_k	y_k	se(y_k)	z_k	se(z_k)
2017	333'726	13.2%	0.134	0.0224	0.9995	1	4	107	4.55%	0.49%	4.55%	0.49%
2016	269'832	13.9%	-0.042	-0.0070	1.0000	2	3'703	282	12.08%	0.82%	16.63%	0.96%
2015	247'348	13.4%	-0.197	-0.0328	0.9989	3	6'023	695	10.26%	1.34%	26.89%	1.65%
2014	219'403	13.6%	-0.007	-0.0012	1.0000	4	2'294	292	9.41%	0.91%	36.30%	1.88%
2013	170'900	14.6%	0.097	0.0162	0.9997	5	345	84	6.73%	0.52%	43.03%	1.95%
2012	164'880	14.7%	0.117	0.0195	0.9996	6	244	118	4.29%	0.64%	47.32%	2.06%
2011	166'041	14.2%	-0.155	-0.0259	0.9993	7	3'129	193	2.48%	0.87%	49.80%	2.23%
2010	163'283	14.3%	0.181	0.0302	0.9991	8	-18	50	2.65%	0.47%	52.45%	2.28%
2009	169'810	13.0%	0.008	0.0013	1.0000	9	2'801	310	3.64%	1.29%	56.09%	2.25%
2008	175'869	12.7%	0.265	0.0442	0.9980	10	104	68	1.96%	0.67%	58.05%	2.14%
2007	188'390	10.7%	0.254	0.0423	0.9982	11	2'418	246	2.21%	1.51%	60.26%	1.52%
2006	123'232	10.8%	0.353	0.0589	0.9965	12	-1	3	1.71%	0.23%	61.97%	1.50%
2005	100'075	10.8%	-	0.0000	1.0000	13	-	-	1.74%	0.00%	63.71%	1.50%
Total	2'492'791	5.82%	**0.27**									

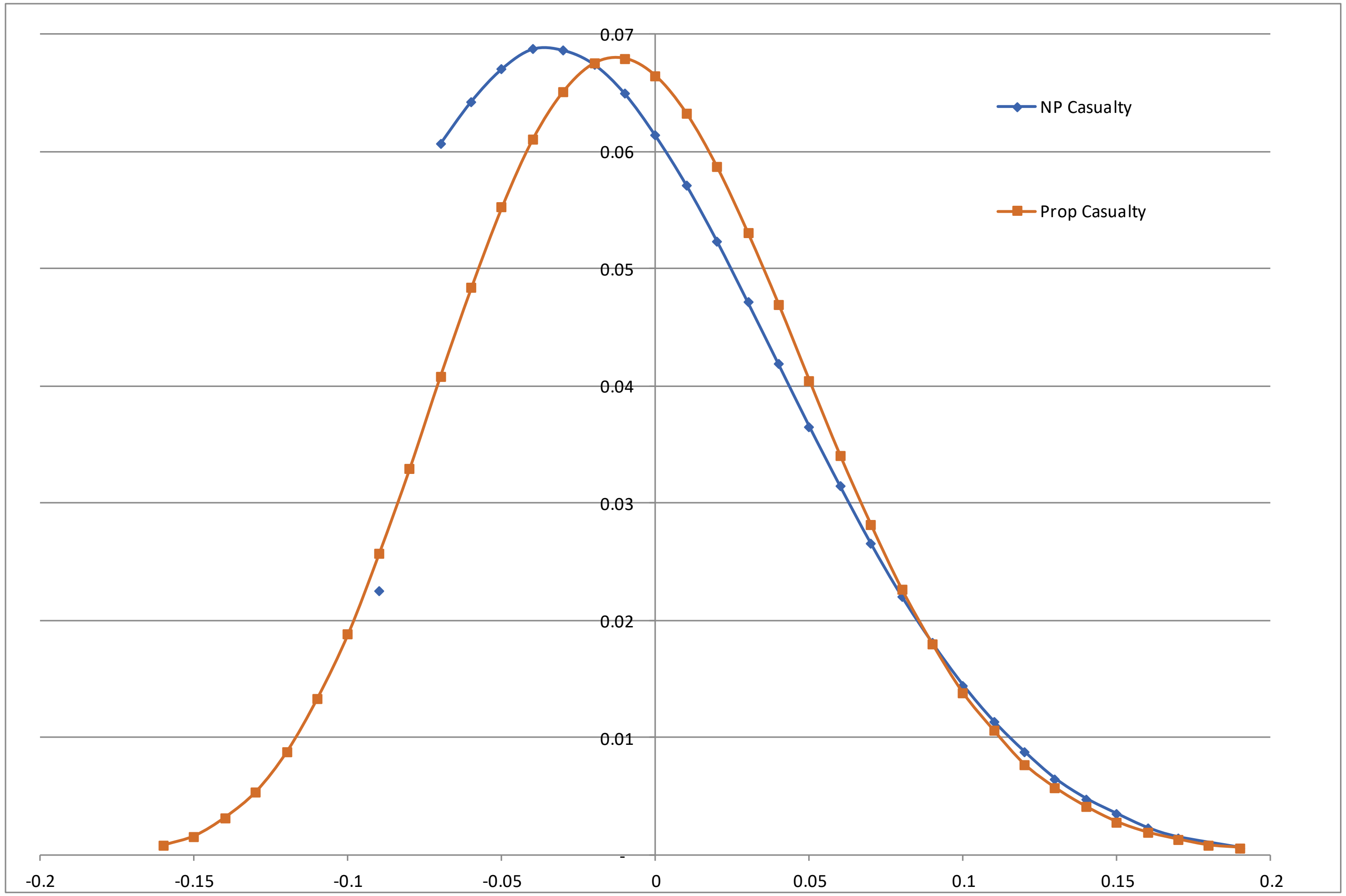
Not very volatile using BF

Skewness is low as expected

Numerical example - Comparison

Skewness comparison is clear

Higher volatility of prop casualty compensates the skewness difference in the tail.



Conclusion

- External knowledge for a priori distribution is crucial
- Simple formulae : Fits with IFRS 17 disclosure requirement
- Next steps: Mix BF and Chain-Ladder skewness parameters

Model available under:
https://drive.google.com/open?id=1iRPEnD8eVOECPyR4oZl_Ezd89agVHYAa

Question ?

Thank you for your attention

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