Multivariate matrix-exponential affine mixtures and their applications in risk theory

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Outline

- Introduction
  - → Matrix-exponential distribution and its properties
- Multivariate matrix-exponential affine mixture
  - → Closure properties
  - → Applications in risk theory
  - → Model calibration

Matrix-exponential distribution

- See e.g. Bladt and Nielsen (2017) for review
- X : Continuous random variable on $[0, \infty)$ with density

$$f(x) = \boldsymbol{\alpha} e^{\boldsymbol{T}x} \boldsymbol{t} \quad \text{for} \quad x \geq 0 \tag{1}$$

  - → $\boldsymbol{\alpha}$: p-dimensional real row vector
  - → $\boldsymbol{T}$: $p \times p$ real square matrix
  - → $\boldsymbol{t}$: p-dimensional real column vector
- Survival function is $\overline{F}(x) = \boldsymbol{\alpha} e^{\boldsymbol{T}x} \boldsymbol{l}$ for $x \geq 0$
  - → $\boldsymbol{l} = (-\boldsymbol{T})^{-1} \boldsymbol{t}$
- The r-th moment is $\mathbb{E}[X^r] = r!\, \boldsymbol{\alpha} (-\boldsymbol{T})^{-(r+1)} \boldsymbol{t}$ for $r \in \mathbb{N}_0$

- Write $X \sim \text{ME}(\boldsymbol{\alpha}, \boldsymbol{T}, \boldsymbol{t})$

- Denote ME as the family of density functions of the form (1)

- Generalization of phase-type distribution

  → But $\boldsymbol{\alpha}$, $\boldsymbol{T}$ and $\boldsymbol{t}$ no longer have probabilistic interpretation

- ME = class of densities with rational Laplace transform

- Let σ_i's be eigenvalues of $\boldsymbol{T}$ and m_i be the multiplicity of σ_i

  → The density of X can be written as a linear combination of $\{x^{k-1}e^{\sigma_i x} : i; 1 \le k \le m_i\}$

Two classes equivalent to ME

- Define the class of *matrix-exponential mixtures* (MEm)

$$\left\{ \sum_{j=1}^{L} c_j f_j : L \in \mathbb{N}_+; f_j \in \text{ME}; c_j \ge 0; \sum_{j=1}^{L} c_j = 1 \right\}$$

- Define the class of *matrix-exponential affine mixtures* (MEam)

$$\left\{ \sum_{j=1}^{L} c_j f_j : L \in \mathbb{N}_+; f_j \in \text{ME}; c_j \in \mathbb{R}; \sum_{j=1}^{L} c_j = 1; \sum_{j=1}^{L} c_j f_j \ge 0 \right\}$$

- **Proposition 1** : ME = MEm = MEam.

- **Proof** :

  → ME ⊆ MEm ⊆ MEam so it remains to prove MEam ⊆ ME

  → Pick an element of MEam where each f_j is $\text{ME}(\boldsymbol{\alpha}_j, \boldsymbol{T}_j, \boldsymbol{t}_j)$

  → Then

$$\sum_{j=1}^{L} c_j f_j(x) = \sum_{j=1}^{L} c_j \boldsymbol{\alpha}_j e^{\boldsymbol{T}_j x} \boldsymbol{t}_j$$
$$= (c_1\boldsymbol{\alpha}_1, \ldots, c_L\boldsymbol{\alpha}_L) \exp\left(\begin{pmatrix} \boldsymbol{T}_1 & & \\ & \ddots & \\ & & \boldsymbol{T}_L \end{pmatrix} x \right) \begin{pmatrix} \boldsymbol{t}_1 \\ \vdots \\ \boldsymbol{t}_L \end{pmatrix}$$

  is an element of ME

Closure of ME under order statistics

- **Proposition 2** (Bladt and Nielsen (2017, Theorem 4.4.14)) :

  If $X_j \sim \text{ME}(\boldsymbol{\alpha}_j, \boldsymbol{T}_j, \boldsymbol{t}_j)$ for $1 \le j \le n$ are independent, then the k-th order statistic $X_{k:n}$ is ME-distributed.

- Easy to see this using Proposition 1

- For example, if $X_1, \ldots, X_n$ follow the same $\text{ME}(\boldsymbol{\alpha}, \boldsymbol{T}, \boldsymbol{t})$ then $X_{k:n}$ has density

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!}\left(1-\overline{F}(x)\right)^{k-1} f(x)\left(\overline{F}(x)\right)^{n-k}$$
$$= \frac{n!}{(k-1)!(n-k)!}\sum_{\ell=0}^{k-1}\binom{k-1}{\ell}(-1)^{\ell} f(x)\left(\overline{F}(x)\right)^{n-\ell-1}$$

→ Here

$$f(x)\left(\overline{F}(x)\right)^{n-\ell-1} = \underbrace{(\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}\otimes\cdots\otimes\boldsymbol{\alpha})}_{n-\ell \text{ blocks}} \exp\left(\underbrace{(\boldsymbol{T}\oplus\boldsymbol{T}\oplus\cdots\oplus\boldsymbol{T})}_{n-\ell \text{ blocks}}\, x\right)\underbrace{(\boldsymbol{t}\otimes\boldsymbol{l}\otimes\cdots\otimes\boldsymbol{l})}_{n-\ell \text{ blocks}}$$

is non-negative and is proportional to an ME density

→ $f_{k:n}$ is MEam

→ $f_{k:n}$ is therefore ME thanks to Proposition 1

Multivariate distributions

- Multivariate distributions are used in modelling multivariate losses
  - → Multivariate risk measures
  - → Premium calculations
  - → Capital allocation
- Desirable properties include e.g.
  - → Ease of solving the above problems
  - → Denseness
  - → Closure properties (e.g. Esscher transform)
  - → Availability of fitting algorithms

- Some multivariate distributions that have enjoyed success
  - → Elliptical : Valdez and Chernih (2003), Furman and Landsman (2006), Landsman, Makov and Shushi (2016, 2018)
  - → Gamma : Furman and Landsman (2005), Furman, Kye and Su (2021)
  - → Phase-type : Cai and Li (2005)
  - → Pareto : Asimit, Furman and Vernic (2010), Sarabia, Gómez-Déniz, Prieto, and Jordá (2016)
  - → Mixed Erlang : Lee and Lin (2012), Willmot and Woo (2015)
- Multivariate ME distribution proposed by Bladt and Nielsen (2010)
  - → Characterized by rational multivariate Laplace transform
  - → Explicit expression for density is unknown

Multivariate ME affine mixture (MMEam)

- Fix $L, M \in \mathbb{N}_+$ and define $\mathscr{S} = \{1, \ldots, L\}^M$
- Let f_j be ME$(\boldsymbol{\alpha}_j, \boldsymbol{T}_j, \boldsymbol{t}_j)$ (with $\boldsymbol{l}_j = (-\boldsymbol{T}_j)^{-1}\boldsymbol{t}_j$) for $j \in \{1, \ldots, L\}$
- **Definition of our proposed** MMEam :

  An MMEam density is a multivariate density function of the form

  $$f(x_1, \ldots, x_M) = \sum_{\boldsymbol{i} \in \mathscr{S}} p_{\boldsymbol{i}} f_{i_1}(x_1) \cdots f_{i_M}(x_M) \quad \text{for} \quad x_1, \ldots, x_M \geq 0.$$

  - → $\boldsymbol{i}$: Abbreviation of $(i_1, \ldots, i_M)$
  - → $\{p_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \mathscr{S}}$: A collection of real numbers such that $\sum_{\boldsymbol{i} \in \mathscr{S}} p_{\boldsymbol{i}} = 1$

- If $(X_1, \ldots, X_M)$ follows an M-variate MMEam density
  - → For $r_1, \ldots, r_M \in \mathbb{N}_0$, the joint moments are

    $$\mathbb{E}\left[\prod_{j=1}^{M} X_j^{r_j}\right] = \sum_{\boldsymbol{i} \in \mathscr{S}} p_{\boldsymbol{i}} \prod_{j=1}^{M} r_j! \, \boldsymbol{\alpha}_{i_j} (-\boldsymbol{T}_{i_j})^{-(r_j+1)} \boldsymbol{t}_{i_j}$$

  - → Any $\widetilde{M}$ sub-components follow an $\widetilde{M}$-variate MMEam density
    - * Each X_j is MEam and hence ME
  - → Density of any $\widetilde{M}$ sub-components conditional on the values of other components is $\widetilde{M}$-variate MMEam
- If $p_{\boldsymbol{i}} \geq 0$ and each f_j is an Erlang(j, λ) density, then MMEam reduces to a multivariate finite Erlang mixture

Closure property 1 : Residual lifetime

- $\boldsymbol{X} = (X_1, \ldots, X_M)$ is MMEam and fix $\boldsymbol{z} = (z_1, \ldots, z_M) \in [0, \infty)^M$
- The density of $\boldsymbol{X} - \boldsymbol{z} | \boldsymbol{X} > \boldsymbol{z}$ is the *residual lifetime* density

$$f_{\boldsymbol{z}}^{\mathsf{RL}}(x_1, \ldots, x_M) = \frac{f(x_1 + z_1, \ldots, x_M + z_M)}{\overline{F}(z_1, \ldots, z_M)}$$

$\rightarrow$ $\overline{F}$ is the joint survival function of $\boldsymbol{X}$

- **Proposition 3** : $f_{\boldsymbol{z}}^{\mathsf{RL}} \in$ MMEam with

$$f_{\boldsymbol{z}}^{\mathsf{RL}}(x_1, \ldots, x_M) = \sum_{\boldsymbol{i} \in \mathscr{S}} p_{\boldsymbol{z}, \boldsymbol{i}}^{\mathsf{RL}} \, f_{z_1, i_1}^{\mathsf{RL}}(x_1) \cdots f_{z_M, i_M}^{\mathsf{RL}}(x_M),$$

where

$$f_{z,j}^{\mathsf{RL}}(x) = \boldsymbol{\alpha}_{z,j}^{\mathsf{RL}} e^{\boldsymbol{T}_j x} \boldsymbol{t}_j, \quad \boldsymbol{\alpha}_{z,j}^{\mathsf{RL}} = \frac{1}{\boldsymbol{\alpha}_j e^{\boldsymbol{T}_j z} \boldsymbol{l}_j} \boldsymbol{\alpha}_j e^{\boldsymbol{T}_j z}.$$

Closure property 2 : Size-biased Esscher transform

- Fix $n_1, \ldots, n_M \in \mathbb{N}_0$ and $\lambda_1, \ldots, \lambda_M \geq 0$
- Let $\boldsymbol{n} = (n_1, \ldots, n_M)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M)$
- The density of a *size-biased Esscher transform* of f is

$$f_{\boldsymbol{n},\boldsymbol{\lambda}}^{\mathsf{ET}}(x_1, \ldots, x_M) = \frac{1}{C_{\boldsymbol{n},\boldsymbol{\lambda}}^{\mathsf{ET}}} x_1^{n_1} \cdots x_M^{n_M} e^{-\lambda_1 x_1 \cdots - \lambda_M x_M} f(x_1, \ldots, x_M)$$

$\rightarrow$ $C_{\boldsymbol{n},\boldsymbol{\lambda}}^{\mathsf{ET}}$ is a normalizing constant

- $\lambda_1 = \cdots = \lambda_M = 0 \Rightarrow$ multivariate size-biased distribution
- $n_1 = \cdots = n_M = 0 \Rightarrow$ multivariate Esscher transform

- **Proposition 4** : $f_{\boldsymbol{n},\boldsymbol{\lambda}}^{\mathsf{ET}} \in$ MMEam with

$$f_{\boldsymbol{n},\boldsymbol{\lambda}}^{\mathsf{ET}}(x_1, \ldots, x_M) = \sum_{\boldsymbol{i} \in \mathscr{S}} p_{\boldsymbol{n},\boldsymbol{\lambda},\boldsymbol{i}}^{\mathsf{ET}} \, f_{n_1,\lambda_1,i_1}^{\mathsf{ET}}(x_1) \cdots f_{n_M,\lambda_M,i_M}^{\mathsf{ET}}(x_M),$$

where

$$f_{n,\lambda,j}^{\mathsf{ET}}(x) = \left(\frac{n!}{C_{n,\lambda,j}^{\mathsf{ET}}} \boldsymbol{\alpha}_j^{[n]} \right) e^{\boldsymbol{T}_j^{[n,\lambda]} x} \boldsymbol{t}_j^{[n]},$$

$$\boldsymbol{\alpha}_j^{[n]} = \underbrace{(\boldsymbol{\alpha}_j, \boldsymbol{0}, \ldots, \boldsymbol{0})}_{n+1 \text{ blocks}}, \quad \boldsymbol{t}_j^{[n]} = \left.\begin{pmatrix} \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \\ \boldsymbol{t}_j \end{pmatrix}\right\} n+1 \text{ blocks},$$

$$\boldsymbol{T}_j^{[n,\lambda]} = \underbrace{\begin{pmatrix} \boldsymbol{T}_j - \lambda \boldsymbol{I} & \boldsymbol{I} & & & \\ & \boldsymbol{T}_j - \lambda \boldsymbol{I} & \boldsymbol{I} & & \\ & & \boldsymbol{T}_j - \lambda \boldsymbol{I} & & \\ & & & \ddots & \\ & & & & \boldsymbol{T}_j - \lambda \boldsymbol{I} \end{pmatrix}}_{n+1 \text{ blocks}}.$$

Property 3 : Aggregation

- **Proposition 5** : The sum $S = \sum_{j=1}^{M} X_j$ has density

$$f_S(y) = \sum_{\boldsymbol{i} \in \mathscr{S}} p_{\boldsymbol{i}} \boldsymbol{\alpha}_{\boldsymbol{i}} e^{\boldsymbol{T}_{\boldsymbol{i}} y} \boldsymbol{t}_{\boldsymbol{i}}$$

which is MEam and hence ME, where

$$\boldsymbol{\alpha}_{\boldsymbol{i}} = (\boldsymbol{\alpha}_{i_1}, \mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}), \quad , \quad \boldsymbol{t}_{\boldsymbol{i}} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \boldsymbol{t}_{i_M} \end{pmatrix},$$

$$\boldsymbol{T}_{\boldsymbol{i}} = \begin{pmatrix} \boldsymbol{T}_{i_1} & \boldsymbol{t}_{i_1}\boldsymbol{\alpha}_{i_2} & & \\ & \boldsymbol{T}_{i_2} & \boldsymbol{t}_{i_2}\boldsymbol{\alpha}_{i_3} & \\ & & \ddots & \\ & & & \boldsymbol{T}_{i_M} \end{pmatrix}.$$

Property 4 : Order statistics

- **Proposition 6** : The j-th order statistic $X_{j:M}^{\mathsf{OS}}$ of $(X_1, \ldots, X_M)$ follows an ME distribution.

- **Sketch of proof** :

$\rightarrow$ $X_{j:M}^{\mathsf{OS}}$ has density

$$f_{j:M}^{\mathsf{OS}}(x) = \sum_{\boldsymbol{i} \in \mathscr{S}} p_{\boldsymbol{i}} f_{\boldsymbol{i},j:M}(x)$$

  * $f_{\boldsymbol{i},j:M}$ is the j-th order statistic of M independent variables with densities $f_{i_1}, f_{i_2}, \ldots, f_{i_M}$

$\rightarrow$ $f_{\boldsymbol{i},j:M} \in$ ME $\Rightarrow$ $f_{j:M}^{\mathsf{OS}} \in$ MEam $\Rightarrow$ $f_{j:M}^{\mathsf{OS}} \in$ ME

Application 1 : Multivariate risk measures

- $\boldsymbol{X} - \boldsymbol{z} | \boldsymbol{X} > \boldsymbol{z}$ can be regarded as *multivariate excess loss*
- **Lemma 1** (Joint moments of excess losses of MMEam) :

$$\mathbb{E}\left[\prod_{j=1}^{M} (X_j - z_j)^{r_j} \,\middle|\, X_1 > z_1, \ldots, X_M > z_M \right]$$
$$= \sum_{\boldsymbol{i} \in \mathscr{S}} p_{\boldsymbol{z},\boldsymbol{i}}^{\mathsf{RL}} \prod_{j=1}^{M} r_j! \, \boldsymbol{\alpha}_{z_j, i_j}^{\mathsf{RL}} (-\boldsymbol{T}_{i_j})^{-(r_j+1)} \boldsymbol{t}_{i_j}.$$

- Recall each X_j is ME

$\Rightarrow$ $\mathsf{V@R}_{\theta_j}(X_j)$ can be easily obtained for $\theta_j \in [0, 1)$

- Define $\mathsf{V@R}_{\boldsymbol{\theta}}(\boldsymbol{X}) = (\mathsf{V@R}_{\theta_1}(X_1), \ldots, \mathsf{V@R}_{\theta_M}(X_M))$

$\rightarrow$ $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_M)$

- Landsman, Makov and Shushi (2016) proposed the *multivariate tail conditional expectation*

$$\text{MTCE}_{\boldsymbol{\theta}}(\boldsymbol{X}) = \mathbb{E}[\boldsymbol{X}|\boldsymbol{X} > \text{V@R}_{\boldsymbol{\theta}}(\boldsymbol{X})]$$

→ M-dimensional vector with j-th component $\mathbb{E}[X_j|\boldsymbol{X} > \text{V@R}_{\boldsymbol{\theta}}(\boldsymbol{X})]$

→ Use Lemma 1 with $\boldsymbol{z} = \text{V@R}_{\boldsymbol{\theta}}(\boldsymbol{X})$

- Landsman, Makov and Shushi (2018) proposed the *multivariate tail covariance*

$$\begin{aligned}&\text{MTCov}_{\boldsymbol{\theta}}(\boldsymbol{X})\\ &= \mathbb{E}[(\boldsymbol{X} - \text{MTCE}_{\boldsymbol{\theta}}(\boldsymbol{X}))^\top(\boldsymbol{X} - \text{MTCE}_{\boldsymbol{\theta}}(\boldsymbol{X}))|\boldsymbol{X} > \text{V@R}_{\boldsymbol{\theta}}(\boldsymbol{X})]\end{aligned}$$

→ M-dimensional square matrix with (j_1, j_2)-th element

$$\begin{aligned}&\mathbb{E}[(X_{j_1} - \mathbb{E}[X_{j_1}|\boldsymbol{X} > \text{V@R}_{\boldsymbol{\theta}}(\boldsymbol{X})])\\ &\quad \times (X_{j_2} - \mathbb{E}[X_{j_2}|\boldsymbol{X} > \text{V@R}_{\boldsymbol{\theta}}(\boldsymbol{X})])|\boldsymbol{X} > \text{V@R}_{\boldsymbol{\theta}}(\boldsymbol{X})]\end{aligned}$$

→ Obtainable via our lemma

Application 2 : Conditional Value-at-Risk

- CoV@R was proposed by Adrian and Brunnermeier (2016)

- CoV@R is the V@R of a loss conditional on a stress scenario

- $\text{CoV@R}^{=}_{\theta_1,\theta_2}(X_2|X_1) = \text{V@R}_{\theta_2}(X_2|X_1 = \text{V@R}_{\theta_1}(X_1))$

  → θ_2-level V@R of X_2 given X_1 is exactly at its θ_1-level V@R

- Mainik and Schaanning (2014) subsequently defined

  $\text{CoV@R}^{>}_{\theta_1,\theta_2}(X_2|X_1) = \text{V@R}_{\theta_2}(X_2|X_1 > \text{V@R}_{\theta_1}(X_1))$

- Stress scenarios can be extended to multiple losses at or exceeding their respective V@R values

- Conditional distributions are ME

  ⇒ These CoV@R's or even CoES can be calculated

- For $S = \sum_{j=1}^{M} X_j$, Di Bernardino, Fernández-Ponce, Palacios-Rodríguez and Rodríguez-Griñolo (2015) defined

  $\text{CoV@R}^{>}_{\theta_1,\theta_S}(S|X_1) = \text{V@R}_{\theta_S}(S|X_1 > \text{V@R}_{\theta_1}(X_1))$

- To determine the distribution $S|X_1 > z_1$ for $z_1 \geq 0$:

  → Write $S = z_1 + (X_1 - z_1) + \sum_{j=2}^{M} X_j$

  → Joint density of $(X_1 - z_1, X_2, \ldots, X_M)$ given $X_1 > z_1$ is MMEam

  → $S - z_1 = (X_1 - z_1) + \sum_{j=2}^{M} X_j$ given $X_1 > z_1$ is ME

Application 3 : Large claims reinsurance

- See e.g. Albrecher, Beirlant and Teugels (2017) for review
- $X_{j:M}^{\mathsf{OS}}$ is the j-th order statistic of the losses $(X_1, \ldots, X_M)$
- Let R be the loss covered by reinsurer
- LCR(k) (k *largest claims reinsurance*) treaty

  → Reinsurer fully covers the largest k losses such that

$$R = \sum_{j=M-k+1}^{M} X_{j:M}^{\mathsf{OS}}$$

- ECOMOR(k) (*Excédent du Coût Moyen Relatif*) treaty

  → Reinsurer covers the losses in excess of the k-th largest loss (where $2 \leq k \leq M$)

  → Like excess-of-loss reinsurance for the $k-1$ largest losses with random deductible being the k-th largest loss so that

$$\begin{aligned} R &= \sum_{j=1}^{M} (X_{j:M}^{\mathsf{OS}} - X_{M-k+1:M}^{\mathsf{OS}})_+ \\ &= \sum_{j=M-k+2}^{M} (X_{j:M}^{\mathsf{OS}} - X_{M-k+1:M}^{\mathsf{OS}}) \\ &= \sum_{j=M-k+2}^{M} X_{j:M}^{\mathsf{OS}} - (k-1) X_{M-k+1:M}^{\mathsf{OS}} \end{aligned}$$

- Each $X_{j:M}^{\mathsf{OS}}$ is ME $\Rightarrow \mathbb{E}[R]$ can be obtained for these contracts

Application 4 : Capital allocation

- See e.g. Denault (2001) for review
- A risk measure $\rho(S)$ of $S = \sum_{j=1}^{M} X_j$ can be regarded as the *risk capital* to be allocated to each risk with an *allocation principle*
- Let K_j be the capital allocated to X_j

  → Preferably satisfies additivity requirement $\rho(S) = \sum_{j=1}^{M} K_j$
- *Covariance-based allocation* (e.g. Cossette, Côté, Marceau and Moutanabbir (2013)) :

$$K_j = \mathbb{E}[X_j] + \frac{\mathsf{Cov}(X_j, S)}{\mathsf{Var}(S)} (\mathsf{TV@R}_\theta(S) - \mathbb{E}[S])$$

  → Additive with $\sum_{j=1}^{M} K_j = \mathsf{TV@R}_\theta(S)$

  → $\mathsf{Cov}(X_j, S) = \mathbb{E}[X_j S] - \mathbb{E}[X_j]\mathbb{E}[S] = \sum_{i=1}^{M} \mathbb{E}[X_j X_i] - \mathbb{E}[X_j]\mathbb{E}[S]$

- TCov *allocation* based on *tail covariance premium* in Furman and Landsman (2006) :

$$K_j = \mathbb{E}[X_j|S > \text{V@R}_\theta(S)] + \beta \text{Cov}(X_j, S|S > \text{V@R}_\theta(S))$$

  → $\sum_{j=1}^M K_j = \mathbb{E}[S|S > \text{V@R}_\theta(S)] + \beta \text{Var}(S|S > \text{V@R}_\theta(S))$ is the *tail variance premium* for the aggregate risk S

- *Tail covariance premium adjusted (*TCPA*) allocation* (Wang (2014)) :

$$K_j = \mathbb{E}[X_j|S > \text{V@R}_\theta(S)] + \beta \frac{\text{Cov}(X_j, S|S > \text{V@R}_\theta(S))}{\sqrt{\text{Var}(S|S > \text{V@R}_\theta(S))}}$$

  → $\sum_{j=1}^M K_j = \mathbb{E}[S|S > \text{V@R}_\theta(S)] + \beta\sqrt{\text{Var}(S|S > \text{V@R}_\theta(S))}$ is the *tail standard deviation premium* discussed in Furman and Landsman (2006)

- Interested in $\mathbb{E}[X_j^k S^h \mathbb{1}\{S > y\}]$ for $k \in \mathbb{N}_+$ and $h \in \mathbb{N}_0$

- **Lemma 2** ($\mathbb{E}[X_j^k S^h \mathbb{1}\{S > y\}]$ for MMEam with $M \geq 2$) :

$$\mathbb{E}[X_j^k S^h \mathbb{1}\{S > y\}] = \sum_{\boldsymbol{i} \in \mathscr{S}} p_{\boldsymbol{i}} k! \left(\boldsymbol{\alpha}_{i_j}^{[k]}, \boldsymbol{0}\right) \left\{ e^{\boldsymbol{A}^{\{k,\boldsymbol{i},j\}} y} \sum_{\ell=0}^{h} \left(-\boldsymbol{A}^{\{k,\boldsymbol{i},j\}}\right)^{-(h-\ell+1)} \frac{h!}{\ell!} y^\ell \right\} \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{t}_{(\boldsymbol{i},j)} \end{pmatrix}.$$

- Extension to multiplicative background risk model (e.g. Franke, Schlesinger and Stapleton (2006)) :

$$(X_1^\dagger, \ldots, X_M^\dagger) = \left(\frac{X_1}{B}, \ldots, \frac{X_M}{B}\right)$$

  → B is assumed independent of $\boldsymbol{X}$

  → Most works in literature assume $X_1, \ldots, X_M$ are mutually independent (e.g. Furman, Kye and Su (2021))

  → We can solve capital allocation when $(X_1, \ldots, X_M)$ is MMEam

Note on fitting of MMEam

- Data : N observations $\mathcal{D} = \{(x_{k,1}, \ldots, x_{k,M})\}_{k=1}^N$ of $(X_1, \ldots, X_M)$

- Fitting can be performed using a two-step procedure

- **1st step** : Estimation of ME marginals

  → For each $j \in \{1, \ldots, M\}$, fit an ME density $\widehat{f}_{X_j}(x) = \widehat{\boldsymbol{\alpha}}_{X_j} e^{\widehat{\boldsymbol{T}}_{X_j} x} \widehat{\boldsymbol{t}}_{X_j}$ (with cdf $\widehat{F}_{X_j}$) to the marginal data $\{x_{k,j}\}_{k=1}^N$

  → May utilize existing algorithms for ME (or its special case of phase-type) fitting, see e.g. Asmussen, Nerman and Olsson (1996), Fackrell (2005), Okamura and Dohi (2016)

- **2nd step** : Construction of dependence using Bernstein copula with empirical mixing weights (Sancetta and Satchell (2004))

$$\widehat{C}(u_1, \ldots, u_M) = \sum_{(h_1,\ldots,h_M)\in\{0,\ldots,A\}^M} \widehat{\zeta}\left(\frac{h_1}{A}, \ldots, \frac{h_M}{A}\right) \prod_{j=1}^{M} \binom{A}{h_j} u_j^{h_j} (1-u_j)^{A-h_j}$$

where

$$\widehat{\zeta}\left(\frac{h_1}{A}, \ldots, \frac{h_M}{A}\right) = \frac{1}{N}\sum_{k=1}^{N} \mathbb{1}\left\{\bigcap_{j=1}^{M}\left\{\widehat{F}_{X_j}(x_{k,j}) \leq \frac{h_j}{A}\right\}\right\}$$

- Fitted joint cumulative distribution function of $(X_1, \ldots, X_M)$ is $\widehat{C}(\widehat{F}_{X_1}(x_1), \ldots, \widehat{F}_{X_M}(x_M))$

- Is the fitted joint density of $(X_1, \ldots, X_M)$ is really MMEam?

- Yes! It can be shown that

$$\widehat{f}(x_1, \ldots, x_m) = \sum_{(h_1,\ldots,h_M)\in\{1,\ldots,A\}^M} \widehat{\phi}_{(h_1,\ldots,h_M)} \prod_{j=1}^{M} \widehat{f}_{X_j,h_j;A}(x_j)$$

→ $$\widehat{\phi}_{(h_1,\ldots,h_M)} = \sum_{(\ell_1,\ldots,\ell_M)\in\{0,1\}^M} (-1)^{M+\ell_1+\ldots+\ell_M} \widehat{\zeta}\left(\frac{h_1-1+\ell_1}{A}, \ldots, \frac{h_M-1+\ell_M}{A}\right)$$

→ $\widehat{f}_{X_j,h_j;A}$ is the density of the h_j-th order statistic of A independent variables with density $\widehat{f}_{X_j}$ and is thus ME

THE END

Thank you !