Multivariate matrix-exponential affine mixtures
and their applications in risk theory

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Outline

• Introduction
  → Matrix-exponential distribution and its properties

• Multivariate matrix-exponential affine mixture
  → Closure properties
  → Applications in risk theory
  → Model calibration

Matrix-exponential distribution

• See e.g. Bladt and Nielsen (2017) for review

• \( X \) : Continuous random variable on \([0, \infty)\) with density
  \[ f(x) = \alpha e^{TxT} \text{ for } x \geq 0 \]  \hspace{1cm} (1)
  → \( \alpha \) : \( p \)-dimensional real row vector
  → \( T \) : \( p \times p \) real square matrix
  → \( t \) : \( p \)-dimensional real column vector

• Survival function is \( F(x) = \alpha e^{TxT} \) for \( x \geq 0 \)
  → \( l = (-T)^{-1} t \)

• The \( r \)-th moment is \( E[X^r] = r! \alpha (-T)^{-(r+1)} t \) for \( r \in \mathbb{N}_0 \)
• Write \( X \sim \text{ME}(\alpha, T, t) \)

• Denote ME as the family of density functions of the form (1)

• Generalization of phase-type distribution

→ But \( \alpha, T \) and \( t \) no longer have probabilistic interpretation

• ME = class of densities with rational Laplace transform

• Let \( \sigma_i \)'s be eigenvalues of \( T \) and \( m_i \) be the multiplicity of \( \sigma_i \)

→ The density of \( X \) can be written as a linear combination of \( \{x^{k-1}e^{\sigma_ix} : i; 1 \leq k \leq m_i\} \)

Two classes equivalent to ME

• Define the class of matrix-exponential mixtures (MEm)

\[
\left\{ \sum_{j=1}^{L} c_j f_j : L \in \mathbb{N}_+; f_j \in \text{ME}; c_j \geq 0; \sum_{j=1}^{L} c_j = 1 \right\}
\]

• Define the class of matrix-exponential affine mixtures (MEam)

\[
\left\{ \sum_{j=1}^{L} c_j f_j : L \in \mathbb{N}_+; f_j \in \text{ME}; c_j \in \mathbb{R}; \sum_{j=1}^{L} c_j = 1; \sum_{j=1}^{L} c_j f_j \geq 0 \right\}
\]

• Proposition 1 : ME = MEm = MEam.

• Proof :

→ ME \( \subseteq \) MEm \( \subseteq \) MEam so it remains to prove MEam \( \subseteq \) ME

→ Pick an element of MEam where each \( f_j \) is ME(\( \alpha_j, T_j, t_j \))

→ Then

\[
\sum_{j=1}^{L} c_j f_j(x) = \sum_{j=1}^{L} c_j \alpha_j e^{T_j x} t_j
\]

\[
= (c_1 \alpha_1, \ldots, c_L \alpha_L) \exp \left( \begin{pmatrix} T_1 & & \\ & \ddots & \\ & & T_L \end{pmatrix} x \right) \begin{pmatrix} t_1 \\ \vdots \\ t_L \end{pmatrix}
\]

is an element of ME
Closure of ME under order statistics

• Proposition 2 (Bladt and Nielsen (2017, Theorem 4.4.14)): If \( X_j \sim \text{ME}(\alpha_j, T_j, t_j) \) for \( 1 \leq j \leq n \) are independent, then the \( k \)-th order statistic \( X_{k:n} \) is ME-distributed.

• Easy to see this using Proposition 1

• For example, if \( X_1, \ldots, X_n \) follow the same ME(\( \alpha, T, t \)) then \( X_{k:n} \) has density

\[
f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} \left( 1 - F(x) \right)^{k-1} f(x) \left( F(x) \right)^{n-k} \]

\[
= \frac{n!}{(k-1)!(n-k)!} \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} (-1)^\ell f(x) \left( F(x) \right)^{n-k-1}
\]

Here

\[
f(x) \left( F(x) \right)^{n-k-1} = (\alpha \otimes \alpha \otimes \cdots \otimes \alpha)^{n-k} \exp \left( (T \oplus T \oplus \cdots \oplus T)x \right) (t \otimes l \otimes \cdots \otimes l)^{n-k} \]

is non-negative and is proportional to an ME density

\( f_{k:n} \) is MEam

\( f_{k:n} \) is therefore ME thanks to Proposition 1

Multivariate distributions

• Multivariate distributions are used in modelling multivariate losses
  → Multivariate risk measures
  → Premium calculations
  → Capital allocation

• Desirable properties include e.g.
  → Ease of solving the above problems
  → Denseness
  → Closure properties (e.g. Esscher transform)
  → Availability of fitting algorithms
• Some multivariate distributions that have enjoyed success
  → Gamma : Furman and Landsman (2005), Furman, Kye and Su (2021)
  → Phase-type : Cai and Li (2005)
  → Pareto : Asimit, Furman and Vernic (2010), Sarabia, Gómez-Déniz, Prieto, and Jordá (2016)

• Multivariate ME distribution proposed by Bladt and Nielsen (2010)
  → Characterized by rational multivariate Laplace transform
  → Explicit expression for density is unknown

**Multivariate ME affine mixture (MMEam)**

• Fix $L, M \in \mathbb{N}_+$ and define $\mathcal{I} = \{1, \ldots, L\}^M$

• Let $f_j$ be $\text{ME}(\alpha_j, T_j, t_j)$ (with $l_j = (-T_j)^{-1}t_j$) for $j \in \{1, \ldots, L\}$

• **Definition of our proposed** MMEam :
  An MMEam density is a multivariate density function of the form
  \[ f(x_1, \ldots, x_M) = \sum_{i \in \mathcal{I}} p_i f_{i_1}(x_1) \cdots f_{i_M}(x_M) \text{ for } x_1, \ldots, x_M \geq 0. \]
  → $i$ : Abbreviation of $(i_1, \ldots, i_M)$
  → $\{p_i\}_{i \in \mathcal{I}}$ : A collection of real numbers such that $\sum_{i \in \mathcal{I}} p_i = 1$

• If $(X_1, \ldots, X_M) \text{ follows an } M\text{-variate MMEam density}$
  → For $r_1, \ldots, r_M \in \mathbb{N}_0$, the joint moments are
    \[ E \left[ \prod_{j=1}^{M} X_j^{r_j} \right] = \sum_{i \in \mathcal{I}} p_i \prod_{j=1}^{M} r_j! \alpha_{i_j}(-T_{i_j})^{-(r_j+1)}t_{i_j} \]
  → Any $\tilde{M}$ sub-components follow an $\tilde{M}\text{-variate MMEam density}$
    * Each $X_j$ is MEam and hence ME
  → Density of any $\tilde{M}$ sub-components conditional on the values of other components is $\tilde{M}\text{-variate MMEam}$

• If $p_i \geq 0$ and each $f_j$ is an $\text{Erlang}(j, \lambda)$ density, then MMEam reduces to a multivariate finite Erlang mixture


Closure property 1 : Residual lifetime

- $X = (X_1, \ldots, X_M)$ is MMEam and fix $z = (z_1, \ldots, z_M) \in [0, \infty)^M$

- The density of $X - z | X > z$ is the residual lifetime density
  $$f^{RL}_z(x_1, \ldots, x_M) = \frac{f(x_1 + z_1, \ldots, x_M + z_M)}{F(z_1, \ldots, z_M)}$$
  $\Rightarrow F$ is the joint survival function of $X$

- Proposition 3 : $f^{RL}_z \in$ MMEam with
  $$f^{RL}_z(x_1, \ldots, x_M) = \sum_{i \in S} p^{RL}_{z, i_1} f^{RL}_{x_1, z_1} \cdots f^{RL}_{x_M, z_M}(x_M),$$
  where
  $$f^{RL}_{z, j}(x) = \alpha^{RL}_{z, j} e^{T_{[z]} x_j}, \quad \alpha^{RL}_{z, j} = \frac{1}{\alpha_j e^{T_{[z]} x_j}}.$$

Closure property 2 : Size-biased Esscher transform

- Fix $n_1, \ldots, n_M \in \mathbb{N}_0$ and $\lambda_1, \ldots, \lambda_M \geq 0$

- Let $\mathbf{n} = (n_1, \ldots, n_M)$ and $\lambda = (\lambda_1, \ldots, \lambda_M)$

- The density of a size-biased Esscher transform of $f$
  $$f^{ET}_{\mathbf{n}, \lambda}(x_1, \ldots, x_M) = \frac{1}{C^{ET}_{\mathbf{n}, \lambda}} \prod_{i=1}^M x_i^{n_i} e^{-\lambda_1 x_1 - \cdots - \lambda_M x_M} f(x_1, \ldots, x_M)$$
  $\Rightarrow C^{ET}_{\mathbf{n}, \lambda}$ is a normalizing constant

- $\lambda_1 = \cdots = \lambda_M = 0 \Rightarrow$ multivariate size-biased distribution

- $n_1 = \cdots = n_M = 0 \Rightarrow$ multivariate Esscher transform

- Proposition 4 : $f^{ET}_{\mathbf{n}, \lambda} \in$ MMEam with
  $$f^{ET}_{\mathbf{n}, \lambda}(x_1, \ldots, x_M) = \sum_{i \in S} p^{ET}_{\mathbf{n}, \lambda, i} f^{ET}_{x_1, z_1, i_1}(x_1) \cdots f^{ET}_{x_M, z_M, i_M}(x_M),$$
  where
  $$f^{ET}_{\mathbf{n}, \lambda, j}(x) = \left(\frac{n!}{C^{ET}_{\mathbf{n}, \lambda, j}} \alpha^{[n]}_{j}\right) \alpha_j^{[n]} e^{T^{[n]} x_j},$$
  $\alpha^{[n]}_{j} = \left(\alpha_j, 0, \ldots, 0\right)$, $T^{[n]} = \begin{pmatrix} 0 \\ t_j^n \\ 0 \\ t_j^n \\ \vdots \\ 0 \\ t_j^n \end{pmatrix}$, $n+1$ blocks,
  $$T^{[n, \lambda]}_j = \begin{pmatrix} T_j - \lambda & I \\ T_j - \lambda & I \\ \vdots \\ T_j - \lambda & I \end{pmatrix},$$
  $n+1$ blocks.
Property 3 : Aggregation

• **Proposition 5**: The sum \( S = \sum_{j=1}^{M} X_j \) has density

\[
f_S(y) = \sum_{i \not\in S} p_i \alpha_i e^{T_i y} t_i
\]

which is MEam and hence ME, where

\[
\alpha_i = (\alpha_{i1}, 0, 0, \ldots, 0), \quad T_i = \begin{pmatrix} T_{i1} & t_{i1} \alpha_{i2} & \cdots & \alpha_{iM} \end{pmatrix},
\]

\[
t_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ t_{iM} \end{pmatrix}.
\]

Property 4 : Order statistics

• **Proposition 6**: The \( j \)-th order statistic \( X_{j:M} \) of \((X_1, \ldots, X_M)\) follows an ME distribution.

• **Sketch of proof**:

\[
\rightarrow X_{j:M} \text{ has density } f_{j:M}(x) = \sum_{i \not\in S} p_i f_{i,j:M}(x)
\]

\* \( f_{i,j:M} \) is the \( j \)-th order statistic of \( M \) independent variables with densities \( f_{i1}, f_{i2}, \ldots, f_{iM} \)

\[
\rightarrow f_{i,j:M} \in ME \Rightarrow f_{j,M}^{OS} \in MEam \Rightarrow f_{j,M}^{OS} \in ME
\]

Application 1 : Multivariate risk measures

• \( X - z \mid X > z \) can be regarded as multivariate excess loss

• **Lemma 1** (Joint moments of excess losses of MMEam):

\[
E \left[ \prod_{j=1}^{M} (X_j - z_j)^{r_j} \mid X_1 > z_1, \ldots, X_M > z_M \right] = \sum_{i \not\in S} p_i^{RL} \prod_{j=1}^{M} r_j \alpha_{j,i}^{RL} (-T_{ij})^{-(r_j + 1)} t_{ij}.
\]

• Recall each \( X_j \) is ME

\[
\Rightarrow V\Omega R_{\theta_j}(X_j) \text{ can be easily obtained for } \theta_j \in [0, 1)
\]

• Define \( V\Omega R_{\theta}(X) = (V\Omega R_{\theta_1}(X_1), \ldots, V\Omega R_{\theta_M}(X_M)) \)

\[
\rightarrow \theta = (\theta_1, \ldots, \theta_M)
\]
Landsman, Makov and Shushi (2016) proposed the **multivariate tail conditional expectation**

\[ \text{MTCE}_\theta(X) = \mathbb{E}[X | X > \text{VaR}_\theta(X)] \]

→ \( M \)-dimensional vector with \( j \)-th component \( \mathbb{E}[X_j | X > \text{VaR}_\theta(X)] \)

→ Use Lemma 1 with \( z = \text{VaR}_\theta(X) \)

Landsman, Makov and Shushi (2018) proposed the **multivariate tail covariance**

\[ \text{MTCov}_\theta(X) = \mathbb{E}[(X - \text{MTCE}_\theta(X))(X - \text{MTCE}_\theta(X)) | X > \text{VaR}_\theta(X)] \]

→ \( M \)-dimensional square matrix with \( (j_1,j_2) \)-th element \( \mathbb{E}[(X_{j_1} - \mathbb{E}[X_{j_1} | X > \text{VaR}_\theta(X)]) \times (X_{j_2} - \mathbb{E}[X_{j_2} | X > \text{VaR}_\theta(X)]) | X > \text{VaR}_\theta(X)] \)

→ Obtainable via our lemma

### Application 2: Conditional Value-at-Risk

- CoV@R was proposed by Adrian and Brunnermeier (2016)
- CoV@R is the VaR of a loss conditional on a stress scenario
- CoV@R\(\theta_1,\theta_2(X_2|X_1) = \text{VaR}_{\theta_2}(X_2|X_1 = \text{VaR}_{\theta_1}(X_1)) \)
  → \( \theta_2 \)-level VaR of \( X_2 \) given \( X_1 \) is exactly at its \( \theta_1 \)-level VaR
- Mainik and Schaanning (2014) subsequently defined
  CoV@R\(\theta_1,\theta_S(X_2|X_1) = \text{VaR}_{\theta_S}(X_2|X_1 > \text{VaR}_{\theta_1}(X_1)) \)
- Stress scenarios can be extended to multiple losses at or exceeding their respective VaR values

- Conditional distributions are ME
  ⇒ These CoV@R’s or even CoES can be calculated

- For \( S = \sum_{j=1}^M X_j \), Di Bernardino, Fernández-Ponce, Palacios-Rodríguez and Rodríguez-Griñolo (2015) defined
  CoV@R\(\theta_1,\theta_S(S|X_1) = \text{VaR}_{\theta_S}(S|X_1 > \text{VaR}_{\theta_1}(X_1)) \)

- To determine the distribution \( S|X_1 > z_1 \) for \( z_1 \geq 0 \):
  → Write \( S = z_1 + (X_1 - z_1) + \sum_{j=2}^M X_j \)
  → Joint density of \( (X_1 - z_1, X_2, \ldots, X_M) \) given \( X_1 > z_1 \) is MMEam
  → \( S - z_1 = (X_1 - z_1) + \sum_{j=2}^M X_j \) given \( X_1 > z_1 \) is ME
Application 3: Large claims reinsurance

- See e.g. Albrecher, Beirlant and Teugels (2017) for review

- $X_{j:M}^\text{OS}$ is the j-th order statistic of the losses $(X_1,\ldots,X_M)$

- Let $R$ be the loss covered by reinsurer

- LCR($k$) (k largest claims reinsurance) treaty
  
  → Reinsurer fully covers the largest $k$ losses such that
  
  $$R = \sum_{j=M-k+1}^{M} X_{j:M}^\text{OS}$$

- ECOMOR($k$) (Excédent du Coût Moyen Relatif) treaty
  
  → Reinsurer covers the losses in excess of the $k$-th largest loss (where $2 \leq k \leq M$)

  → Like excess-of-loss reinsurance for the $k-1$ largest losses with random deductible being the $k$-th largest loss so that

  $$R = \sum_{j=M-k+2}^{M} (X_{j:M}^\text{OS} - X_{M-k+1:M}^\text{OS}) + \sum_{j=M-k+2}^{M} X_{j:M}^\text{OS} - (k-1)X_{M-k+1:M}^\text{OS}$$

- Each $X_{j:M}^\text{OS}$ is ME ⇒ $E[R]$ can be obtained for these contracts

Application 4: Capital allocation

- See e.g. Denault (2001) for review

- A risk measure $\rho(S)$ of $S = \sum_{j=1}^{M} X_j$ can be regarded as the risk capital to be allocated to each risk with an allocation principle

- Let $K_j$ be the capital allocated to $X_j$

  → Preferably satisfies additivity requirement $\rho(S) = \sum_{j=1}^{M} K_j$

  - Covariance-based allocation (e.g. Cossette, Côté, Marceau and Moutanabbir (2013)) :
    
    $$K_j = E[X_j] + \frac{\text{Cov}(X_j,S)}{\text{Var}(S)}(\text{TV@R}_\theta(S) - E[S])$$

  → Additive with $\sum_{j=1}^{M} K_j = \text{TV@R}_\theta(S)$

  → $\text{Cov}(X_j,S) = E[X_jS] - E[X_j]E[S] = \sum_{i=1}^{M} E[X_jX_i] - E[X_j]E[S] = \sum_{i=1}^{M} E[X_i]E[X_j] - E[X_j]E[S]$
TCov allocation based on tail covariance premium in Furman and Landsman (2006):

$$K_j = E[X_j | S > \text{VaR}_\theta(S)] + \beta \text{Cov}(X_j, S | S > \text{VaR}_\theta(S))$$

$$\sum_{j=1}^{M} K_j = E[S | S > \text{VaR}_\theta(S)] + \beta \text{Var}(S | S > \text{VaR}_\theta(S))$$

is the tail variance premium for the aggregate risk $S$.

Tail covariance premium adjusted (TCPA) allocation (Wang (2014)):

$$K_j = E[X_j | S > \text{VaR}_\theta(S)] + \beta \frac{\text{Cov}(X_j, S | S > \text{VaR}_\theta(S))}{\sqrt{\text{Var}(S | S > \text{VaR}_\theta(S))}}$$

$$\sum_{j=1}^{M} K_j = E[S | S > \text{VaR}_\theta(S)] + \beta \frac{\sqrt{\text{Var}(S | S > \text{VaR}_\theta(S))}}{\sqrt{\text{Var}(S | S > \text{VaR}_\theta(S))}}$$

is the tail standard deviation premium discussed in Furman and Landsman (2006).

Interested in $E[X^k_j S^h 1\{S > y\}]$ for $k \in \mathbb{N}_+$ and $h \in \mathbb{N}_0$.

**Lemma 2** ($E[X^k_j S^h 1\{S > y\}]$ for MMEam with $M \geq 2$):

$$E[X^k_j S^h 1\{S > y\}] = \sum_{i \in I} p_{ik} \left( e^{A_{ij}^k y} \sum_{\ell=0}^{h} \binom{h}{\ell} \frac{(-A_{ij}^k)^h}{\ell!} \frac{y^\ell}{\ell!} \right) t(i,j).$$

Extension to multiplicative background risk model (e.g. Franke, Schlesinger and Stapleton (2006)):

$$(X_1^\dagger, \ldots, X_M^\dagger) = \left( \frac{X_1}{B}, \ldots, \frac{X_M}{B} \right)$$

$B$ is assumed independent of $X$.

Most works in literature assume $X_1, \ldots, X_M$ are mutually independent (e.g. Furman, Kye and Su (2021)).

We can solve capital allocation when $(X_1, \ldots, X_M)$ is MMEam.

**Note on fitting of MMEam**

Data: $N$ observations $D = \{(x_{k,1}, \ldots, x_{k,M})\}_{k=1}^{N}$ of $(X_1, \ldots, X_M)$.

Fitting can be performed using a two-step procedure:

1st step: Estimation of ME marginals

- For each $j \in \{1, \ldots, M\}$, fit an ME density $\hat{f}_{X_j}(x) = \alpha_{X_j} e^{\tilde{\alpha}_{X_j} x} \tilde{f}_{X_j}$ (with cdf $\tilde{F}_{X_j}$) to the marginal data $\{(x_{k,j})_{k=1}^{N}\}$.

May utilize existing algorithms for ME (or its special case of phase-type) fitting, see e.g. Asmussen, Nerman and Olsson (1996), Fackrell (2005), Okamura and Dohi (2016).
• 2nd step: Construction of dependence using Bernstein copula with empirical mixing weights (Sancetta and Satchell (2004))

\[ \hat{C}(u_1, \ldots, u_M) = \sum_{(h_1, \ldots, h_M) \in \{0, \ldots, A\}^M} \hat{\zeta} \left( \frac{h_1}{A}, \ldots, \frac{h_M}{A} \right) \prod_{j=1}^{M} \left( \frac{A}{h_j} \right)^{h_j} u_j (1 - u_j)^{A-h_j} \]

where

\[ \hat{\zeta} \left( \frac{h_1}{A}, \ldots, \frac{h_M}{A} \right) = \frac{1}{N} \sum_{k=1}^{N} \mathbb{1} \left\{ \bigcap_{j=1}^{M} \left\{ \hat{F}_{X_j}(x_{k,j}) \leq \frac{h_j}{A} \right\} \right\} \]

• Fitted joint cumulative distribution function of \((X_1, \ldots, X_M)\) is \(\hat{C}(\hat{F}_{X_1}(x_1), \ldots, \hat{F}_{X_M}(x_M))\)

• Is the fitted joint density of \((X_1, \ldots, X_M)\) really MMEam?

YES! It can be shown that

\[ f(x_1, \ldots, x_m) = \sum_{(h_1, \ldots, h_M) \in \{1, \ldots, A\}^M} \hat{\phi}_{(h_1, \ldots, h_M)} \prod_{j=1}^{M} \hat{f}_{X_j,h_j}(x_j) \]

\[ \rightarrow \hat{\phi}_{(h_1, \ldots, h_M)} \]

\[ = \sum_{(\ell_1, \ldots, \ell_M) \in \{0,1\}^M} (-1)^{M+\ell_1+\ldots+\ell_M} \hat{\zeta} \left( \frac{h_1 - 1 + \ell_1}{A}, \ldots, \frac{h_M - 1 + \ell_M}{A} \right) \]

\[ \rightarrow \hat{f}_{X_j,h_j}(x_j) \text{ is the density of the } h_j\text{-th order statistic of } A \text{ independent variables with density } \hat{f}_{X_j} \text{ and is thus ME} \]

THE END

Thank you!