Insurance valuation: A two-step generalised regression approach

IAA Webinar

Karim Barigou, joint work with Valeria Bignozzi and Andreas Tsanakas
May 6, 2022
1. Introduction

2. Fair valuation in a one-period setting

3. Fair valuation in a multi-period setting

4. Numerical example: Equity-linked contracts

5. Conclusion
Introduction
Two types of valuations:

- **Traditional actuarial valuation:**
  \[
  \rho[S] = e^{-r} \mathbb{E}^P [S] + \text{RM}[S]
  \]
  - Based on principle of **diversification** (LLN).
  - Risk margin to cover non-diversified risk.
  - The valuation is performed under the real-world measure $\mathbb{P}$.

- **Financial valuation:**
  \[
  \rho[S] = \mathbb{E}^Q [e^{-r} S]
  \]
  - Based on principle of **no-arbitrage** under a risk-neutral measure $\mathbb{Q}$.
  - Set of feasible $\mathbb{Q}$’s follows from observed market prices.
  - In incomplete markets: **Infinite choice** of measures $\mathbb{Q}$.
**Solvency II**

The assets (liabilities) shall be valued at the amount for which they could be exchanged (transferred or settled), between knowledgeable and willing parties in a transaction under normal market conditions.

- **Liabilities traded** on the financial market:
  Fair value $\rightarrow$ Financial risk-neutral valuation

- **Liabilities independent** of the financial market:
  Fair value $\rightarrow$ Actuarial valuation

- **Liabilities that are partly dependent** on the financial market:
  What is the appropriate fair value?
Insurance liabilities are only partially hedgeable and therefore a two-step hybrid approach is usually considered:

- **Two-step hedge-based approach (Dhaene et al. 2017):**
  \[
  \rho [S] = V^\theta(0) + \pi[S - V^\theta(T)]
  \]
  - First part is typically determined by e.g. quadratic hedging.
  - Second part is priced via a standard actuarial principle.

- **Two-step conditional approach (Pelsser & Stadje 2014):**
  \[
  \rho [S] = \mathbb{E}^Q \left[ \pi [S \mid Y] \right]
  \]
  - Inner step: Actuarial valuation conditional on traded asset prices.
Growing research interest on fair/market-consistent valuation:

- The hedge-based valuation of Dhaene et al. (2017) in a one-period setting was further generalized in a discrete multi-period setting in Barigou et al. (2019) and in continuous-time in Delong et al. (2019).
- Assa & Gospodinov (2018) investigated market-consistent valuation in imperfect markets where the financial valuation is non-linear.
- The two-step conditional valuation was also investigated in Salahnejhad Ghalehjooghi & Pelsser (2021) for participating pension contracts.
- Deelstra et al. (2020) proposed a three-step valuation where the liability is decomposed into hedgeable, diversifiable and residual risk; with a generalization in Linders (2021).
- Fair valuation based on asymmetric hedging is considered in this paper: Barigou et al. (2022) and Chen et al. (2021).
The aim of this paper is twofold:

- We introduce a new valuation framework for insurance liabilities based on a two-step hedging procedure.
  - The resulting hedging strategies produce a residual which has a zero tail risk, as measured by a VaR or Expectile criterion. Hedging cost is then shared between policyholders and shareholders via appropriate cost-of-capital arguments.

- We propose a general backward iterations scheme to determine the valuation and hedging of liabilities in a multi-period framework.
Fair valuation in a one-period setting
• $\mathcal{C} \subseteq L^2(\Omega, \mathcal{F}, \mathbb{P})$ set of contingent claims
• $S \in \mathcal{C}$ insurance liability
• $\mathbf{y} = (1, y_1, \ldots, y_n), \ n \in \mathbb{N}$ value of the financial market assets at time 0
• $\mathbf{Y} = (e^r, Y_1, \ldots, Y_n), \ n \in \mathbb{N}, \ r \geq 0$, value of the financial market assets at time 1
• In this presentation, we assume $r = 0$
• $\mathbf{\beta} = (\beta_0, \ldots, \beta_n) \in \mathbb{R}^{n+1}$ trading strategy

$$\mathbf{\beta} \cdot \mathbf{y} = \sum_{i=0}^{n} \beta_i \cdot y_i \quad \mathbf{\beta} \cdot \mathbf{Y} = \sum_{i=0}^{n} \beta_i \cdot Y_i$$
We can distinguish three types of claims:

- $C^h \subseteq C$ set of perfectly hedgeable claims:

$$S^h = \beta \cdot Y = \sum_{i=0}^{n} \beta_i \cdot Y_i$$

- $C^\perp \subseteq C$ set of claims independent of the financial market:

$$S^\perp \text{ independent of } Y$$

- Hybrid claims (our focus):

$$S \in C \setminus (C^h \cup C^\perp)$$
A valuation is a map $\rho : \mathcal{C} \to \mathbb{R}$, $S \mapsto \rho(S)$, that is

- Normalised: $\rho(0) = 0$
- Translation invariant: $\rho(S + m) = \rho(S) + m$, $\forall S \in \mathcal{C}$, $m \in \mathbb{R}$

\(\rho\) is **market-consistent** if

$$\rho(S + S^h) = \rho(S) + \beta \cdot y,$$

for any $S \in \mathcal{C}$, $S^h = \beta \cdot Y$

\(\rho\) is **actuarial** if

$$\rho(S^\perp) = \mathbb{E}[S^\perp] + \text{RM}(S^\perp)$$

for any claim $S^\perp \in \mathcal{C}^\perp$, where $\text{RM} : \mathcal{C}^\perp \to \mathbb{R}$ is a mapping that does not depend on current asset prices $y$

\(\rho\) is a **fair valuation** (*Dhaene et al. 2017*, *Barigou et al. 2019*) if it is market-consistent and actuarial
Two-step valuation with cost-of-capital approach

Fair valuation of hybrid claims is generally performed in two steps:

- **Quadratic hedging:**
  \[
  \theta = \arg \min_{\beta \in \mathbb{R}^{n+1}} \mathbb{E}[(S - \beta \cdot Y)^2] \Rightarrow \mathbb{E}[S] = \mathbb{E}[\theta \cdot Y]
  \]

- **Risk measure on the residual risk**
  \[
  \text{VaR}_\alpha(S - \theta \cdot Y) \quad \alpha \in (0, 1)
  \]

The fair value then is:

\[
\rho(S) = \theta \cdot y + i \cdot \text{VaR}_\alpha(S - \theta \cdot Y) \quad i \in (0, 1)
\]
Two-step valuation with cost-of-capital approach

Fair valuation of hybrid claims is generally performed in two steps:

- Quadratic hedging:

\[
\theta = \arg \min_{\beta \in \mathbb{R}^{n+1}} \mathbb{E}[(S - \beta \cdot Y)^2] \quad \Rightarrow \quad \mathbb{E}[S] = \mathbb{E}[\theta \cdot Y]
\]

- Risk measure on the residual risk

\[
\text{VaR}_\alpha (S - \theta \cdot Y) \quad \alpha \in (0, 1)
\]

The fair value then is:

\[
\rho(S) = \theta \cdot y + i \cdot \text{VaR}_\alpha (S - \theta \cdot Y) \quad i \in (0, 1)
\]

Cost-of-capital approach

\[
\rho(S) = \mathbb{E}[S] + i \cdot \text{VaR}_\alpha (S - \mathbb{E}[S]) \quad i \in (0, 1)
\]
Two-step valuation with a general loss function

Error function

\[ \ell : \mathbb{R} \rightarrow [0, +\infty) \text{ such that } \ell(x) = 0 \text{ iff } x = 0 \]

\[ \xi = \arg \min_{\beta \in \mathbb{R}^{n+1}} \mathbb{E}[\ell(S - \beta \cdot Y)] \]

- \( \ell_\alpha(x) = \alpha x_+ + (1 - \alpha)x_- \) quantile regression
  \[ \text{VaR}(S - \xi^{(\ell_\alpha)} \cdot Y) = 0 \]

- \( \ell_\tau(x) = \tau(x_+)^2 + (1 - \tau)(x_-)^2 \) expectile regression

We penalize (possibly asymmetrically) distance of the trading portfolio \( \beta \cdot Y \) from the liability \( S \)

- Föllmer & Leukert (2000) \( \rightarrow \) Shortfall risk
- Rockafellar & Uryasev (2013) \( \rightarrow \) Risk quadrangle
**Loss functions**

**Figure 1:** Quantile loss functions (solid lines) and expectile loss functions (dashed curves) for $\alpha \in \{0.4, 0.5, 0.6\}$. 
The main two steps:

- First step (quadratic hedging):

\[
\theta = \arg \min_{\beta \in \mathbb{R}^{n+1}} \mathbb{E}[(S - \beta \cdot Y)^2]
\]

- Second step (hedging residual with loss function \(\ell\)):

\[
\eta = \arg \min_{\beta \in \mathbb{R}^{n+1}} \mathbb{E}[\ell(S - \theta \cdot Y - \beta \cdot Y)]
\]

In total: \(\xi = \arg \min_{\beta \in \mathbb{R}^{n+1}} \mathbb{E}[\ell(S - \beta \cdot Y)] = \theta + \eta\)

Two-step fair valuation

\[
\rho(S) = \theta \cdot y + i \cdot \eta \cdot y, \quad i \in (0, 1)
\]
Quantile hedging is a TVaR deviation minimiser

Assume that we want to hedge \( R(S, \theta) = S - \theta \cdot Y \) and the regulator imposes that \( \text{VaR}_\alpha(R(S, \theta) - \beta \cdot Y) = 0 \) for some trading strategy \( \beta \in \mathcal{B} \).

Two possibilities:

- Invest \( \text{VaR}_\alpha(R(S, \theta)) \) in the risk-free asset. Call this strategy \( \nu \).
- Consider the quantile hedging strategy \( \eta \) such that \( \text{VaR}_\alpha(R(S, \theta) - \eta \cdot Y) = 0 \).

Lemma

The quantile hedging strategy satisfies:

\[
\text{dTVaR}_\alpha(R(S - \theta \cdot Y, \eta)) \leq \text{dTVaR}_\alpha(R(S - \theta \cdot Y, \beta)),
\]

for all hedging strategies \( \beta \) such that \( \text{VaR}_\alpha(R(S, \theta) - \beta \cdot Y) = 0 \). This is in particular the case for \( \beta = \nu \).

N.B: TVaR deviation: \( \text{dTVaR}_\alpha(X) = \text{TVaR}_\alpha(X) - \mathbb{E}(X) \).
Quantile hedging is a TVaR deviation minimiser

Assume that we want to hedge $R(S, \theta) = S - \theta \cdot Y$ and the regulator imposes that $\text{VaR}_\alpha(R(S, \theta) - \beta \cdot Y) = 0$ for some trading strategy $\beta \in \mathcal{B}$. Two possibilities:

- Invest $\text{VaR}_\alpha(R(S, \theta))$ in the risk-free asset. Call this strategy $\nu$.
- Consider the quantile hedging strategy $\eta$ such that $\text{VaR}_\alpha(R(S, \theta) - \eta \cdot Y) = 0$.

Lemma

The quantile hedging strategy satisfies:

$$d\text{TVaR}_\alpha(R(S - \theta \cdot Y, \eta)) \leq d\text{TVaR}_\alpha(R(S - \theta \cdot Y, \beta)),$$

for all hedging strategies $\beta$ such that $\text{VaR}_\alpha(R(S, \theta) - \beta \cdot Y) = 0$. This is in particular the case for $\beta = \nu$.

N.B: TVaR deviation: $d\text{TVaR}_\alpha(X) = \text{TVaR}_\alpha(X) - \mathbb{E}(X)$. 
Equity-linked contracts sold to 1000 policyholders:

- Assets: \( y = (1, 1), \ Y = (1, Y_1), \ Y_1 \sim LN(0.1, 0.2^2) \)
- Mortality: \( N \) is the number of survivors at time 1, \( N \sim Bin(1000; 0.9) \)
- \( K \) is the guarantee level, \( K = 1 \)
- Payoff: \( S = N \times \max (Y_1, K) \)

\( S \) is highly but non-linearly correlated with a tradeable asset \( Y_1 \)
**Figure 2:** Liability $S$ (left) and residual $S - \theta \cdot \mathbf{Y}$ (right) against value of the risky asset $Y_1$. 
Table 1: Investment in risk-free and risky asset, from hedging strategies associated with the two-step valuation of $S$.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>risk-free asset</th>
<th>risky asset</th>
<th>cost of strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>247</td>
<td>709</td>
<td>956</td>
</tr>
<tr>
<td>$\xi$</td>
<td>460</td>
<td>658</td>
<td>1118</td>
</tr>
<tr>
<td>$\xi^{(\ell, \tau)}$</td>
<td>450</td>
<td>663</td>
<td>1113</td>
</tr>
<tr>
<td>$(\text{VaR}_\alpha(R(S, \theta)), 0)$</td>
<td>163</td>
<td>0</td>
<td>163</td>
</tr>
<tr>
<td>$\eta$</td>
<td>213</td>
<td>-52</td>
<td>161</td>
</tr>
<tr>
<td>$\eta^{(\ell, \tau)}$</td>
<td>204</td>
<td>-47</td>
<td>157</td>
</tr>
</tbody>
</table>

Our approach provides

- Higher investment in the risk-free asset (more stringent criterion)
- Higher sensitivity to adverse movement in the risky asset
Figure 3: Densities of residuals, for quadratic hedging of $S$, followed by investing the VaR of the residual in the risk-free asset (black); quantile hedging of $S$ (blue); and expectile hedging of $S$ (red).
The TVaR deviations are given by

\[
d\text{TVaR}_\alpha (R(S, \theta) - \text{VaR}_\alpha (R(S, \theta))) = 194.2
\]
\[
d\text{TVaR}_\alpha (R(R(S, \theta), \eta)) = 182.5
\]
\[
d\text{TVaR}_\alpha (R(R(S, \theta), \eta^{(\ell\tau)})) = 182.6
\]

The fair value of $S$ for a cost-of-capital rate $i = 0.1$:

\[
\phi(S) = \theta \cdot y + i \cdot \text{VaR}_\alpha (R(S, \theta)) = 972.6
\]
\[
\rho(S) = \theta \cdot y + i \cdot \eta \cdot y = 972.4
\]
\[
\rho^{(\ell\tau)}(S) = \theta \cdot y + i \cdot \eta^{(\ell\tau)} \cdot y = 972
\]

- In this example, the impact on the valuation of $S$ is very limited, even though the hedging strategies and, importantly, the statistical behaviour of residuals, are different.
Fair valuation in a multi-period setting
The valuation problem is in general not a one-period static problem but a multi-period dynamic problem.

In this dynamic context, the notion of \textit{time-consistency} is important:

\[ \rho_t[S] = \rho_t[\rho_{t+1}[S]] \]
Objective: Determine a fair dynamic valuation for a claim \( S \) with maturity \( T \).

- The valuation should be time-consistent:

  \[ \rho_0[S] \rightarrow \rho_1[S] \rightarrow \ldots \rightarrow \rho_{T-2}[S] \rightarrow \rho_{T-1}[S] \rightarrow S \]

   Maturity: \( T \)

but also market-consistent and actuarial:

\[ \rho_t[S] = \text{Hedgeable part of } \rho_{t+1}[S] + \text{Risk margin} \]
Fair valuation by iterated two-step valuation

- **Procedure:** Backward iteration scheme:

\[\rho_t(S) := \theta(t + 1) \cdot Y(t) + i \eta(t + 1) \cdot Y(t),\]  

(1)

where

\[\theta(t + 1) = \arg \min_{\beta \in \mathcal{B}(t)} \mathbb{E}_t \left[ (\rho_{t+1}(S) - \beta \cdot Y(t + 1))^2 \right]\]

\[\eta(t + 1) = \arg \min_{\beta \in \mathcal{B}(t)} \mathbb{E}_t \left[ \ell_{\alpha}(\rho_{t+1}(S) - \theta(t + 1) \cdot Y(t + 1) - \beta \cdot Y(t + 1)) \right].\]

- **Notation:** \(Y(t) = (e^{rt}, Y_1(t), \ldots, Y_n(t))\) the vector of asset prices at time \(t \in \{1, 2, \ldots, T\}\), \(\mathcal{B}(t)\) the set of all real-valued \(\mathcal{F}_t\)-measurable trading strategies.
Properties

- The valuation is **market-consistent, actuarial and time-consistent**.
- Yearly solvency constraints are satisfied:

\[
\text{VaR}_{\alpha,t}(\rho_{t+1}(S) - \theta(t+1) \cdot Y(t+1) - \eta(t+1) \cdot Y(t+1)) = 0, \\
\text{for } t \in \{0, 1, \ldots, T - 1\}.
\]
General algorithm

**Algorithm 1** Backward resolution of the dynamic fair valuation problem

1: \( \rho_T \leftarrow S \)
2: **for** \( t = T - 1, T - 2, \ldots, 0 \) **do**
3: \( g_{t+1} = \arg \min_{g \in G} \frac{1}{M} \sum_{i=1}^{M} \left( \rho_{t+1}^{(i)}(S) - g(Z^{(i)}(t)) \cdot Y^{(i)}(t+1) \right)^2 \)
4: \( h_{t+1} = \arg \min_{g \in G} \frac{1}{M} \sum_{i=1}^{M} \ell(\alpha \left( \rho_{t+1}^{(i)}(S) - g(Z^{(i)}(t)) \cdot Y^{(i)}(t+1) \right)) \)
5: 
6: **end for**

Assumptions:

- Risk drivers are markovian
- Hedging strategies are non-linear functions \( g \) of risk drivers at time \( t \).
Numerical example: Equity-linked contracts
Application: Equity-linked life-insurance contracts

- Payoff at maturity \( T \):
  \[
  S = N(T) \times \max \left( Y^{(1)}(T), K \right),
  \]
  with
  - \( N(T) \): Number of survivors at time \( T \).
  - \( Y^{(1)}(T) \): Stock at time \( T \).
  - \( K \): fixed guarantee level.

- Stock and force of mortality dynamics:
  \[
  dY^{(1)}(t) = Y^{(1)}(t) (\mu dt + \sigma dW_1(t))
  \]
  \[
  d\lambda_x(t) = c\lambda_x(t) dt + \xi dW_2(t)
  \]
  with \( W_1(t) \) and \( W_2(t) \) correlated Brownian motions.

- Financial market: Risk-free asset \( Y^{(0)}(t) = e^{rt} \) and risky asset \( Y^{(1)}(t) \).
We have two neural networks:

\[
g_{t+1} : \mathbb{R}^2 \to \mathbb{R}^2, \ (Y_1(t), \ N(t)) \mapsto g_{t+1}(Y_1(t), \ N(t)) = \theta(t + 1),
\]

\[
h_{t+1} : \mathbb{R}^2 \to \mathbb{R}^2, \ (Y_1(t), \ N(t)) \mapsto h_{t+1}(Y_1(t), \ N(t)) = \xi(t + 1),
\]

Specifications:

- \( N = 200.000 \) samples,
- Parameters for the financial market are \( r = 0.01, \mu = 0.02, \sigma = 0.1, K = 1, \delta = -0.5 \) and \( Y^{(1)}(0) = 1 \),
- The mortality parameters follow from Luciano et al. (2017) and correspond to UK male individuals who are aged 55 at time 0.
- \( l_x = 1000 \) initial contracts at time 0 with a maturity of \( T = 10 \) years.
Application: Equity-linked life-insurance contracts

**Figure 4:** Left: Evolution of the fair valuation from time 0 to maturity time $T = 10$. Right: Histogram of the final payoff $S = N(T) \times \max \left( Y^{(1)}(T), K \right)$. Shades in the fan represent prediction intervals at the 50%, 80% and 95% level.
Application: Equity-linked life-insurance contracts

\[ RB(t) = \tilde{\xi}(t+1) \cdot Y(t) - \tilde{\xi}(t) \cdot Y(t), \quad \forall t \in \{1, \ldots, T - 1\} \]

**Figure 5:** Left: Rebalancing cost of the hedging portfolio at any rebalancing times \( t = 1, \ldots, T - 1 \). Right: total rebalancing cost. Shades in the fan represent prediction intervals at the 50%, 80% and 95% level.
Figure 6: Number of asset units bought at time $t = 5$ in the risk-free asset and risky asset under the quantile hedging strategy as function of the asset price $Y^{(1)}(5)$. This strategy corresponds to the expression $h_6(Y_1(5), N(t)) = \xi(6)$, with fixed mortality $N(t) = \mathbb{E}[N(5)]$. 
Conclusion
There is no general agreement on the valuation of the “residual part”:

- Esscher valuation, e.g. Deelstra et al. (2020)
- Standard deviation principle, e.g. Delong et al. (2019), Barigou & Delong (2022)
- Cost-of-capital principle, e.g. Pelsser (2011)

We proposed to quantile-hedge the residual part.

- Leads a VaR-neutral portfolio.
- The hedging portfolio is a TVaR deviation risk minimiser.
- The residual risk can still be hedged by moving from quadratic to quantile objective.

We proposed a simulation-based algorithm that is valid for general loss functions $\ell$ and non-linear optimisers.
Conclusion

- There is no general agreement on the valuation of the “residual part”:
  - Esscher valuation, e.g. Deelstra et al. (2020)
  - Standard deviation principle, e.g. Delong et al. (2019), Barigou & Delong (2022)
  - Cost-of-capital principle, e.g. Pelsser (2011)
- We proposed to quantile-hedge the residual part.
  - Leads a VaR-neutral portfolio.
  - The hedging portfolio is a TVaR deviation risk minimiser.
  - The residual risk can still be hedged by moving from quadratic to quantile objective.
- We proposed a simulation-based algorithm that is valid for general loss functions $\ell$ and non-linear optimisers.
Conclusion

There is no general agreement on the valuation of the “residual part”:

- Esscher valuation, e.g. Deelstra et al. (2020)
- Standard deviation principle, e.g. Delong et al. (2019), Barigou & Delong (2022)
- Cost-of-capital principle, e.g. Pelsser (2011)

We proposed to quantile-hedge the residual part.
- Leads a VaR-neutral portfolio.
- The hedging portfolio is a TVaR deviation risk minimiser.
- The residual risk can still be hedged by moving from quadratic to quantile objective.

We proposed a simulation-based algorithm that is valid for general loss functions $\ell$ and non-linear optimisers.
Thank you for your attention! Any questions?


Contact: karim.barigou@univ-lyon1.fr


