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Three-layer problems and the Generalized Pareto distribution

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About the speaker

- Dr. (rer. nat.) Michael Fackler
- Qualified actuary (DAV), self-employed
- Studied Math at Univ. Munich, Pisa, Oldenburg
- Doctorate in parallel with working: on experience rating, completed 2017
- 10 years with leading reinsurers
- 15+ years as consulting actuary
- Specialized in: non-life reinsurance pricing, dealing with scarce data

Situation

Tail modelling, e.g. for layer pricing, Solvency

- Very scarce loss data
- Helpful information possibly from different sources, e.g. *your portfolio* vs *market benchmark*
- Models not fully specified
- Only easily accessible data bits:

frequencies at thresholds / risk premiums of layers

MTPL example

Task: Pricing of layers from **1** up to **20** (mln USD)

Input:

- A dozen large losses from your portfolio enable you to quote the layer **2 xs 1**, risk premium: **1.04**
- For the whole market someone quoted the layer **5 xs 5**, risk premium: **3**
- Your portfolio supposedly has average exposure, market share is 8%, thus your risk premium for this «market» layer would be: **0.24**

MTPL example

For higher layers you don’t have market quotations or don’t believe them.

Workaround:

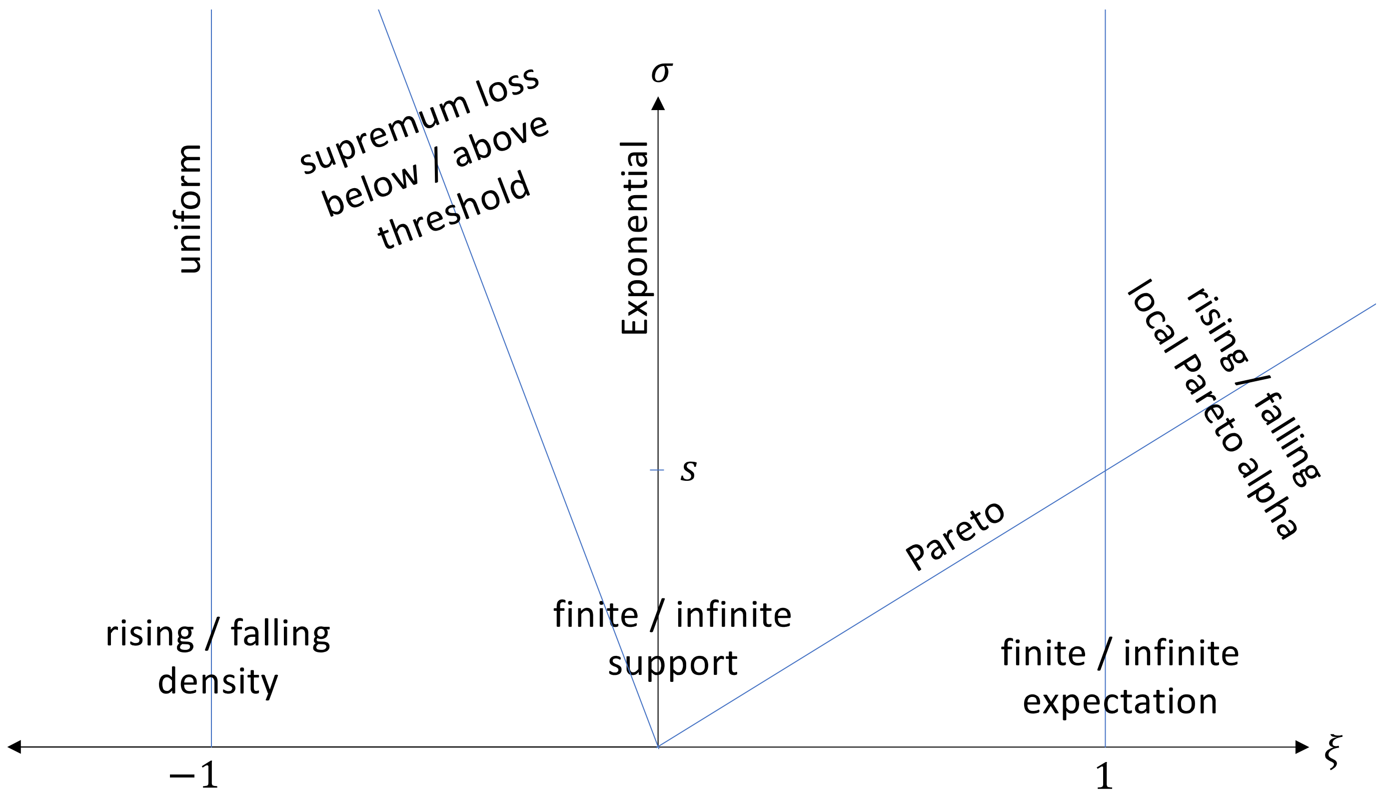
- Maximum desired *payback* period for large events (politically set): **200 years**

General approach

- Be **modest**: no *best*-fit ambitions, a *good-enough* model is fine (*satisficing*)
- Use the Collective Model
- Try to find frequency / severity that reproduce given data bits (moment-matching variant)
- Use the GPD (tail) severity above threshold *s*

$$P(Z > x | Z > s) = \left(\left(1 + \xi \frac{x - s}{\sigma} \right)^{+} \right)^{-\frac{1}{\xi}}$$

GPD map


uniform
supremum loss below / above threshold
σ
Exponential
s
rising / falling local Pareto alpha
Pareto
rising / falling density
finite / infinite support
finite / infinite expectation
−1
1
ξ

Three-layer problems

Given input:

- Risk premiums for 3 layers
- Frequencies for 3 thresholds
- Mixed cases

Heuristics: frequency at threshold = *risk rate on line* of very thin layer

$$RRoL = \frac{risk\ premium}{layer\ limit}$$

MTPL example

Formulate as (mixed) three-layer problem:

- layer **2 xs 1**: *RRoL* = **52%**
- layer **5 xs 5**: *RRoL* = **4.8%**
- threshold **20**: *freq.* = **0.5%**

Theorem

For 3 non-overlapping layers with given RRoL's

$$r_1 > r_2 > r_3 > 0$$

the 3-layer problem can (mostly) be solved:

by a **unique** GPD severity

together with a (unique) frequency f at the attachment point s of the lowest layer

Theorem

- Works also with thresholds or mixed input
- $s = 0$ is possible (ground-up model)
- Top layer may be unlimited ($r_3 = 0$)
- Layers 1 and 2 may overlap (to some extent)
- Uniqueness holds for further situations

Case with no solution:

- Layer 1 is no threshold, Layer 3 is limited
- $r_1 \gg r_2 \approx r_3$

Remarks

- Easy to find numerically
- Special case: one layer with 3 data bits: risk premium, layer entry / exit frequencies
- Single-parameter Pareto solves analogous **2-layer problems**
- GPD solves many real-world **4-layer problems** approximately, piecewise GPD exactly
- Paper gives **model-building recipes** for a variety of scarce-data situations

MTPL example

$s = 1$ (million USD)

- $f = 1.09$
- $\xi = 0.41$ $\quad\quad\quad\quad$ $(\alpha = 2.44)$
- $\sigma = 0.96$

Model risk

… must be high with scarce data, however:

- **Major uncertainty** is expected loss – and possibly the loss count model
- Higher moments of the severity often don’t bear much further uncertainty, in particular for layers in the middle of a program
- GPD is a *choice*, but a good one, both in **practical** and **statistical** sense: other severities are less handy and will often produce very similar output

Parameter-free inequality

Limited layer: limit c, layer severity X,

$$f \geq r \geq g \geq 0$$

with loss frequency f, total loss freq. g, RRoL r:

$$1 - \frac{f-r}{f-g}\frac{r-g}{r} \leq \frac{\mathrm{E}(X^2)}{c\,\mathrm{E}(X)} \leq 1$$

- Interval is narrow for heavy-tailed severity
- Narrower interval for concave cdf
- Analogous bounds for higher moments

Conclusion

The building of models by solving three-layer problems is **powerful** and, in case of very scarce data, an excellent **trade-off** between *statistical ambition* and the *need to get things done*.

Thanks for joining this talk. See the paper for details (colloquium website or SSRN).

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