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Panjer class revisited: one formula for the distributions of the Panjer (a,b,n) class

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About the speaker

- Dr. (rer. nat.) Michael Fackler
- Qualified actuary (DAV), self-employed
- Studied Math at Univ. Munich, Pisa, Oldenburg
- Doctorate in parallel with working: on experience rating, completed 2017
- 10 years with leading reinsurers
- 15+ years as consulting actuary
- Specialized in: non-life reinsurance pricing, dealing with scarce data

General Panjer Class

- Loss count distributions whose probabilities ultimately satisfy *Panjer's recursion*

$$p_k = \left(a + \frac{b}{k}\right) p_{k-1}, \qquad k > n$$

- Leads to recursive algorithm for aggregate loss
- Index n where recursion starts is called the *order* of the distribution, or say: (a, b, n) *class*.

General Panjer Class

Classified in three papers, which all appeared in the ASTIN Bulletin

- $n = 0$: *Poi*, *Bin*, *NegBin* (Sundt & Jewell 1981)
- $n = 1$: *ETNB*, *Log* (Willmot 1988)
- $n > 1$: *ENB*, *ELog* (Hess, Liewald, Schmidt 2002)

6 basic types (ENB extends ETNB), looking quite diverse, plus variants with first probabilities truncated or modified.

Scope of paper

- Unite the general Panjer class in a *common formula* for the *probabilities*, which emerges from the *Binomial Series*
- Identify *practical* and *intuitive* parameters: a with $s = a + b$; a with $\alpha = s/a$
- Overview of parameterisations in use for distributions of the Panjer class

A sequence

For real a, b (or a, s) the sequence

$$r_k = \frac{1}{k!}\prod_{i=1}^{k}(b + ai) = \frac{1}{k!}\prod_{i=0}^{k-1}(s + ai)$$

satisfies Panjer's recursion for all $k > 0$.

For certain a, s the Binomial Series yields

$$(1 - a)^{-\frac{s}{a}} = \sum_{k=0}^{\infty} r_k$$

Probabilities

For many out of those certain a, s, the r_k ultimately, say for $k \geq m_0(a,s)$, have the same sign. Then, for $m \geq m_0(a,s)$,

$$p_0 = \ \dots = p_{m-1} = 0 \quad \text{and}$$

$$p_k = \frac{r_k}{(1-a)^{-\frac{s}{a}} - \sum_{j=0}^{m-1} r_j}, \qquad k \geq m$$

define a loss count distribution, which belongs to the (a,b,m) class: *Binomial* + *(Extended) NegBin*

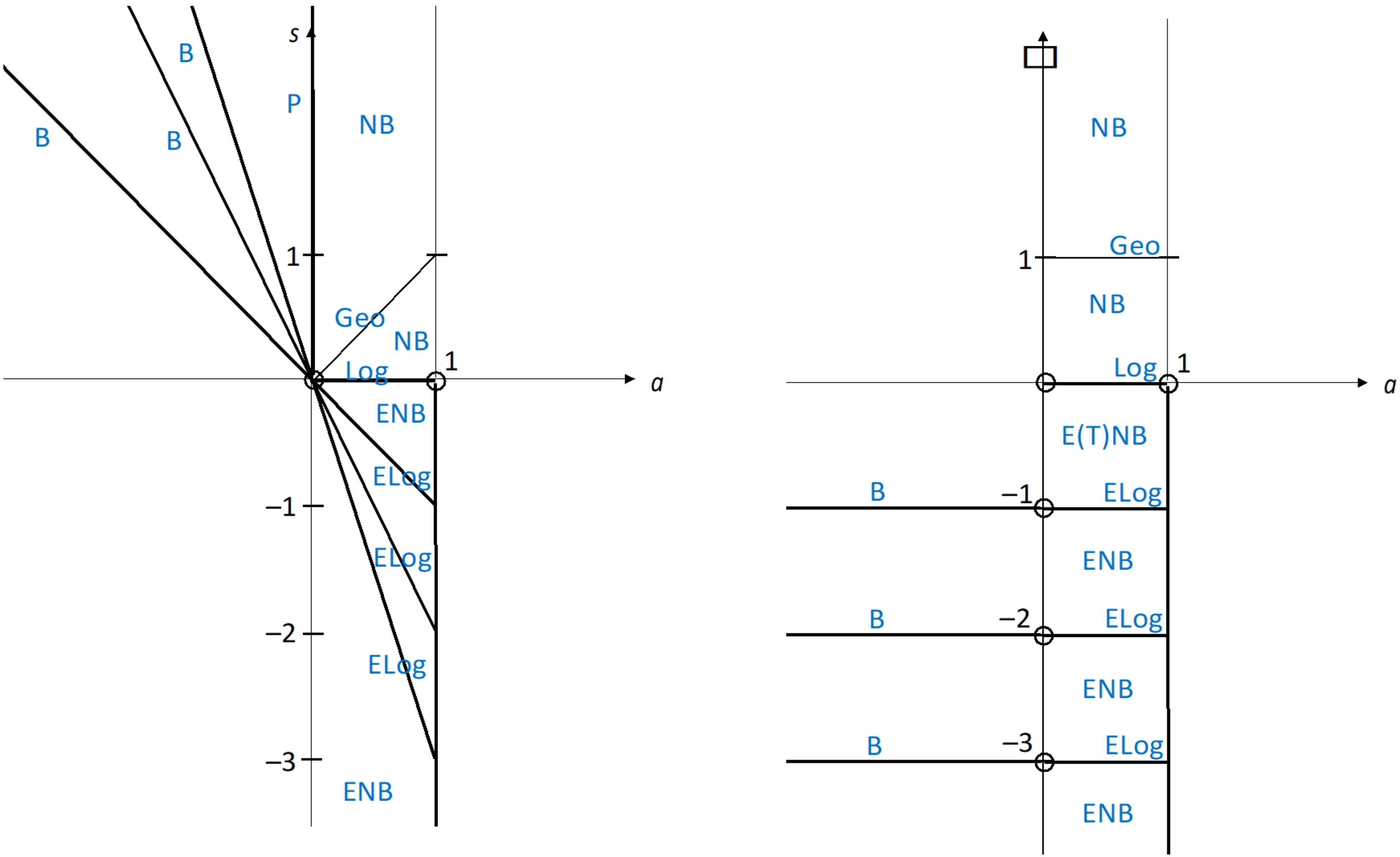
Completing the picture

The probability formula can be *extended* to some a, s beyond the range given by the Binomial Series:

- $a = 0$, via straightforward limit: *Poisson*
- s/a negative integer or $s = 0$, via limit from below: *(Extended) Logarithmic*

For all six distribution types the *pgf* and a compact recursion for the *factorial moments* emerge.

Parameter space



Alternative parameters: overview

Table 1: Panjer $(a,b,0)$ class

	pf	pgf	p_0	$\mathrm{E}(N)$	$\mathrm{Var}(N)$	$\mathrm{CV}^2(N)$	$\mathrm{D}(N)$	$\mathrm{OD}(N)$	$\mathrm{Ct}(N)$	s	a	b	
P	$\frac{\lambda^k}{k!}e^{-\lambda}$	$e^{-\lambda(z-1)}$	$e^{-\lambda}$	λ	λ	$\frac{1}{\lambda}$	1	0	0	λ	0	λ	P
B1	$\binom{n}{k}p^k(1-p)^{n-k}$	$(1-p+pz)^n$	$(1-p)^n$	np	$np(1-p)$	$\frac{1-p}{np}$	$1-p$	$-p$	$-\frac{1}{n}$	$\frac{np}{1-p}$	$\frac{-p}{1-p}$	$\frac{(n+1)p}{1-p}$	B1
B2	$\binom{n}{k}\frac{\lambda^k(n-\lambda)^{n-k}}{n^n}$	$\left(1+\frac{\lambda}{n}(z-1)\right)^n$	$\left(1-\frac{\lambda}{n}\right)^n$	λ	$\lambda\left(1-\frac{\lambda}{n}\right)$	$\frac{1}{\lambda}-\frac{1}{n}$	$1-\frac{\lambda}{n}$	$-\frac{\lambda}{n}$	$-\frac{1}{n}$	$\frac{n\lambda}{n-\lambda}$	$\frac{-\lambda}{n-\lambda}$	$\frac{(n+1)\lambda}{n-\lambda}$	B2
NB1	$\binom{\alpha+k-1}{k}p^\alpha(1-p)^k$	$\left(\frac{1-(1-p)z}{p}\right)^{-\alpha}$	p^α	$\frac{\alpha(1-p)}{p}$	$\frac{\alpha(1-p)}{p^2}$	$\frac{1}{\alpha(1-p)}$	$\frac{1}{p}$	$\frac{1-p}{p}$	$\frac{1}{\alpha}$	$\alpha(1-p)$	$1-p$	$(\alpha-1)(1-p)$	NB1
NB2	$\binom{\alpha+k-1}{k}\left(\frac{\alpha}{\alpha+\lambda}\right)^\alpha\left(\frac{\lambda}{\alpha+\lambda}\right)^k$	$\left(1-\frac{\lambda}{\alpha}(z-1)\right)^{-\alpha}$	$\left(\frac{\alpha}{\alpha+\lambda}\right)^\alpha$	λ	$\lambda\left(1+\frac{\lambda}{\alpha}\right)$	$\frac{1}{\lambda}+\frac{1}{\alpha}$	$1+\frac{\lambda}{\alpha}$	$\frac{\lambda}{\alpha}$	$\frac{1}{\alpha}$	$\frac{\alpha\lambda}{\alpha+\lambda}$	$\frac{\lambda}{\alpha+\lambda}$	$\frac{(\alpha-1)\lambda}{\alpha+\lambda}$	NB2
NB3	$\binom{\alpha+k-1}{k}\frac{\vartheta^\alpha}{(1+\vartheta)^{\alpha+k}}$	$\left(1-\frac{z-1}{\vartheta}\right)^{-\alpha}$	$\left(\frac{\vartheta}{1+\vartheta}\right)^\alpha$	$\frac{\alpha}{\vartheta}$	$\frac{\alpha}{\vartheta}\left(1+\frac{1}{\vartheta}\right)$	$\frac{1+\vartheta}{\alpha}$	$1+\frac{1}{\vartheta}$	$\frac{1}{\vartheta}$	$\frac{1}{\vartheta}$	$\frac{\alpha}{1+\vartheta}$	$\frac{1}{1+\vartheta}$	$\frac{\alpha-1}{1+\vartheta}$	NB3
NB4	$\frac{(1+\beta)^{-k-\lambda/\beta}}{k!}\prod_{i=0}^{k-1}(\lambda+\beta i)$	$(1-\beta(z-1))^{-\frac{\lambda}{\beta}}$	$(1+\beta)^{-\frac{\lambda}{\beta}}$	λ	$\lambda(1+\beta)$	$\frac{1+\beta}{\lambda}$	$1+\beta$	β	$\frac{\beta}{\lambda}$	$\frac{\lambda}{1+\beta}$	$\frac{\beta}{1+\beta}$	$\frac{\lambda-\beta}{1+\beta}$	NB4
BNB1	$\binom{\alpha+k-1}{k}\frac{\beta^k}{(1+\beta)^{\alpha+k}}$	$(1-\beta(z-1))^{-\alpha}$	$(1+\beta)^{-\alpha}$	$\alpha\beta$	$\alpha\beta(1+\beta)$	$\frac{1+\beta}{\alpha\beta}$	$1+\beta$	β	$\frac{1}{\alpha}$	$\frac{\alpha\beta}{1+\beta}$	$\frac{\beta}{1+\beta}$	$\frac{(\alpha-1)\beta}{1+\beta}$	BNB1
BNB2	$(1-a)^\alpha a^k\binom{\alpha+k-1}{k}$	$\left(\frac{1-a}{1-az}\right)^\alpha$	$(1-a)^\alpha$	$\frac{\alpha a}{1-a}$	$\frac{\alpha a}{(1-a)^2}$	$\frac{1}{\alpha a}$	$\frac{1}{1-a}$	$\frac{a}{1-a}$	$\frac{1}{\alpha}$	αa	a	$(\alpha-1)a$	BNB2
Pan1a	$\left(1+\frac{\lambda}{\alpha}\right)^{-\alpha}\frac{\lambda^k}{k!}\prod_{i=0}^{k-1}\frac{\alpha+i}{\alpha+\lambda}$	$\left(1-\frac{\lambda}{\alpha}(z-1)\right)^{-\frac{1}{c}}$	$\left(\frac{\alpha}{\alpha+\lambda}\right)^\alpha$	λ	$\lambda\left(1+\frac{\lambda}{\alpha}\right)$	$\frac{1}{\lambda}+\frac{1}{\alpha}$	$1+\frac{\lambda}{\alpha}$	$\frac{\lambda}{\alpha}$	$\frac{1}{\alpha}$	$\frac{\alpha\lambda}{\alpha+\lambda}$	$\frac{\lambda}{\alpha+\lambda}$	$\frac{(\alpha-1)\lambda}{\alpha+\lambda}$	Pan1a
Pan1b	$(1+c\lambda)^{-\frac{1}{c}}\frac{\lambda^k}{k!}\prod_{i=0}^{k-1}\frac{1+ci}{1+c\lambda}$	$(1-c\lambda(z-1))^{-\frac{1}{c}}$	$(1+c\lambda)^{-\frac{1}{c}}$	λ	$\lambda(1+c\lambda)$	$\frac{1}{\lambda}+c$	$1+c\lambda$	$c\lambda$	c	$\frac{\lambda}{1+c\lambda}$	$\frac{c\lambda}{1+c\lambda}$	$\frac{(1-c)\lambda}{1+c\lambda}$	Pan1b
Pan2a	$(1-a)^{1+\frac{b}{a}}\frac{1}{k!}\prod_{i=1}^{k}(b+ai)$	$\left(\frac{1-a}{1-az}\right)^{1+\frac{b}{a}}$	$(1-a)^{1+\frac{b}{a}}$	$\frac{a+b}{1-a}$	$\frac{a+b}{(1-a)^2}$	$\frac{1}{a+b}$	$\frac{1}{1-a}$	$\frac{a}{1-a}$	$\frac{a}{a+b}$	$a+b$	a	b	Pan2a
Pan2b	$(1-a)^{\frac{s}{a}}\frac{1}{k!}\prod_{i=0}^{k-1}(s+ai)$	$\left(\frac{1-a}{1-az}\right)^{\frac{s}{a}}$	$(1-a)^{\frac{s}{a}}$	$\frac{s}{1-a}$	$\frac{s}{(1-a)^2}$	$\frac{1}{s}$	$\frac{1}{1-a}$	$\frac{a}{1-a}$	$\frac{a}{s}$	s	a	$s-a$	Pan2b
Pan3	$\frac{(1-a)^{\lambda\frac{1-a}{a}}}{k!}\prod_{i=0}^{k-1}(\lambda(1-a)+ai)$	$\left(\frac{1-a}{1-az}\right)^{\lambda\frac{1-a}{a}}$	$(1-a)^{\lambda\frac{1-a}{a}}$	λ	$\frac{\lambda}{1-a}$	$\frac{1}{\lambda(1-a)}$	$\frac{1}{1-a}$	$\frac{a}{1-a}$	$\frac{a}{\lambda(1-a)}$	$\lambda(1-a)$	a	$\lambda(1-a)-a$	Pan3
	pf	pgf	p_0	$\mathrm{E}(N)$	$\mathrm{Var}(N)$	$\mathrm{CV}^2(N)$	$\mathrm{D}(N)$	$\mathrm{OD}(N)$	$\mathrm{Ct}(N)$	s	a	b	

Conclusion

Beyond being *instructive*, the unified view of the general Panjer class can *ease implementation* and use of the models in practice, providing *all-in one formulae* for probabilities and moments.

Thanks for joining this talk. See the paper for details (colloquium website or SSRN).

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