Optimal relativities in a modified Bonus-Malus system with long memory transition rules and frequency-severity dependence

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Design of the automobile insurance

According to e.g. Lemaire (1998), in automobile insurance the insurers tend to utilize

- a **priori** rating factors (e.g. age, sex, marital status, driving experience, car model)
- a **posteriori** or experience rating (e.g. no claim discount)

to

- classify policyholders according to their risks
- adjust the premium charged according to a policyholder's claim history

Bonus-Malus System (BMS)

When adjusting the premium according to claim history :

- Good drivers should have a premium discount (bonus)
- Bad drivers will have an increase in premium (malus)

A BMS can potentially encourage drivers to drive safely

Three major components of a BMS

A BMS consists of 3 major components :

- Bonus-Malus (BM) level : level assigned to a policyholder
- Transition rule : level moves up or down based on claim history
- BM relativity : premium adjustment coefficient in a BM level

Premium charged = base premium x BM relativity

- Base premium depends on a priori (rating factors)
- BM relativity depends on a posteriori (claim history)

Dependence between claim frequency and severity

Independence between frequency and severity is often assumed in credibility and BMS literature

Various statistical models involving dependence have been recently developed, e.g.

- Copula models
  - Czado et al. (2012), Frees et al. (2016)
- Shared or bivariate random effect models
  - Hernández-Bastida et al. (2009), Baumgartner et al. (2015). Cheung et al. (2021), Oh et al. (2020)
- Two-part models
  - Shi et al. (2015), Garrido et al. (2016), Jeong et al. (2017), Park et al. (2018)

Focus of this work

We shall

- Revisit the extension of the BM levels
  - long memory transition rules : consecutive claim free-years
  - e.g. Lemaire (1995), Pitrebois et al. (2003)
- Obtain optimal relativities in two different models
  - frequency-only model
  - dependent collective model e.g. Oh et al. (2020)

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Usual $(-1/+h)$ BMS

The transition rules in a $(-1/+h)$ BMS is such that

- BM level goes down by one for a claim-free year
- BM level goes up by h levels per claim
- Lowest level is 0 ; highest level is z

Note : BM level of a policyholder evolves as a (discrete-time) Markov chain

However, a policyholder who had a claim in the past year (and had his/her premium increased this year) can have his/her premium reduced next year if he/she does not have a claim this coming year

Modified BMS with augmented BM levels

We introduce a "period of the penalty" ("*pen*") and modify the transition rules as

- BM level goes down only when there is no claim for the last consecutive $(1 + pen)$ years
- BM level goes up by h levels per claim (same as the classical BMS)
- Lowest level is 0 ; highest level is z (same as the classical BMS)

$\longrightarrow$ so-called $(-1/+h/pen)$ system

"Augmented" BM levels need to be newly defined so that its transition process has the Markov property

Aggregate claim amount

For the i-th policyholder in the t-th policy year

- N_{it} : the number of claims
- $(Y_{it1}, \ldots, Y_{itN_{it}})$: the vector of associated claim amounts, where Y_{itj} is the j-th claim amount
- $\sum_{j=1}^{N_{it}} Y_{itj}$: aggregate claim amount

Risk characteristics

There are two types of risk characteristics :

- **Observed** risk characteristics (e.g. age, region, model of car)
  - denoted by $\boldsymbol{X}_i$ for the i-th policyholder
  - a priori ratemaking process
  - used to determine "base premium"
  - $\mathcal{K}$ risk classes with characteristics $\boldsymbol{x}_k$ $(k = 1, 2, \ldots, \mathcal{K})$
  - "weight" of the k-th risk class is $w_k := \Pr(\boldsymbol{X} = \boldsymbol{x}_k)$ where $\boldsymbol{X}$ is observed characteristics of a randomly picked person
- **Unobserved** risk characteristics (random component)
  - denoted by Θ_i for the i-th policyholder
  - a posteriori ratemaking process

To model the frequency and the severity of claims, GLM techniques with the assumption of exponential dispersion family for the random components will be applied

Random effect models

For the unobserved risk characteristics, we consider two different models :

- Model 1 : Frequency model with random effect

$$N_{it}|(\Theta_i = \theta_i, \boldsymbol{X}_i = \boldsymbol{x}_i) \overset{\text{i.i.d.}}{\sim} F(\cdot; \lambda_i\theta_i, \psi)$$

- mean parameter is $\lambda_i\theta_i$ with $\lambda_i = \eta^{-1}(\boldsymbol{x}_i\boldsymbol{\beta})$
- Θ_i's are i.i.d. with cdf G and mean $\mathbb{E}[\Theta] = 1$
- λ_i can be regarded as the "base premium"

Random effect models

- Model 2 : Collective risk model with bivariate random effect
  - frequency part is same as Model 1 (but add superscript "[1]" to notations, e.g. Θ_i and $\boldsymbol{X}_i$ are replaced by $\Theta_i^{[1]}$ and $\boldsymbol{X}_i^{[1]}$)
  - severity part follows

$$Y_{itj}|(\Theta_i^{[2]} = \theta_i^{[2]}, \boldsymbol{X}_i^{[2]} = \boldsymbol{x}_i^{[2]}) \overset{\text{i.i.d.}}{\sim} F^{[2]}(\cdot\,;\, \lambda_i^{[2]}\theta_i^{[2]}, \psi^{[2]})$$

  with mean parameter $\lambda_i^{[2]}\theta_i^{[2]}$ where $\lambda_i^{[2]} = \eta_{[2]}^{-1}(\boldsymbol{x}_i^{[2]}\boldsymbol{\beta}^{[2]})$
  - unobserved risk characteristics can be specified by copula

$$(\Theta_i^{[1]}, \Theta_i^{[2]}) \overset{\text{i.i.d.}}{\sim} H = C(G_1, G_2)$$

  - since $\mathbb{E}[\sum_{j=1}^{N_{it}} Y_{itj}|\lambda_i^{[1]}, \lambda_i^{[2]}, \Theta_i^{[1]}, \Theta_i^{[2]}] = \lambda_i^{[1]}\lambda_i^{[2]}\Theta_i^{[1]}\Theta_i^{[2]}$, base premium is $\lambda_i^{[1]}\lambda_i^{[2]}$

Optimal relativities for Model 1 under $(-1/+h)$ BMS

Consider the optimization problem

$$(\tilde{\zeta}(0), \ldots, \tilde{\zeta}(z)) := \underset{(\zeta(0),\ldots,\zeta(z))\in\mathbb{R}^{z+1}}{\operatorname{argmin}} \mathbb{E}[(\Lambda\Theta - \Lambda\zeta(L))^2] \quad (1)$$

- $\mathbb{E}[N_{it}|\lambda_i, \Theta_i] = \lambda_i\Theta_i$ is the "correct" "premium" for the i-th policyholder if we knew Θ_i, where $\lambda_i = \eta^{-1}(\boldsymbol{x}_i\boldsymbol{\beta})$
  $\Rightarrow$ $\Lambda\Theta = \eta^{-1}(\boldsymbol{X}\boldsymbol{\beta})\Theta$ is the "correct" "premium" for a randomly picked policyholder having observed risk characteristics $\boldsymbol{X}$ and unobserved risk characteristics Θ
- L is the BM level for a randomly picked policyholder in a stationary state such that

$$\mathbb{P}(L = \ell) = \sum_{k=1}^{\mathcal{K}} w_k \int \pi_\ell(\lambda_k\theta, \psi) g(\theta) \mathrm{d}\theta$$

where $\pi_\ell(\lambda_k\theta, \psi)$ is the stationary probability that a policyholder with expected frequency $\lambda_k\theta$ is in level ℓ

Optimal relativities for Model 1 under $(-1/+h)$ BMS

- $\zeta(\ell)$ is the relativity associated with the BM level ℓ
- The optimization (1) is about choosing the relativities to minimize the mean squared difference between $\Lambda\Theta$ and the actual "premium" charged $\Lambda\zeta(L)$ when a policyholder is in BM level L

Tan et al. (2015) : The optimal relativities are analytically calculated as

$$\tilde{\zeta}(\ell) := \frac{\mathbb{E}[\Lambda^2\Theta|L=\ell]}{\mathbb{E}[\Lambda^2|L=\ell]} = \frac{\sum_{k=1}^{\mathcal{K}} w_k\lambda_k^2 \int \theta\pi_\ell(\lambda_k\theta,\psi)g(\theta)\mathrm{d}\theta}{\sum_{k=1}^{\mathcal{K}} w_k\lambda_k^2 \int \pi_\ell(\lambda_k\theta,\psi)g(\theta)\mathrm{d}\theta}$$

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Augmented BM levels in a $(-1/+h/pen)$ system

An extended BM level is denoted by $(\ell)_a$:

- ℓ is the BM level occupied
- the subscript a stands for the number of additional claim-free periods (compared to the classical $(-1/+h)$ BMS) required to get rewarded

The state space of the model is

$$\mathcal{A}_{z,pen} := \{(\ell)_0 \big| \ell = 0, \ldots, h-1\} \cup \{(\ell)_0, \ldots, (\ell)_{pen} \big| \ell = h, \ldots, z\}$$

i.e. there are $h + (z - h + 1) \times (1 + pen)$ states

The relativities depend on the BM level ℓ but not the information a which is artificially introduced to make the transitions Markovian, i.e. the relativities are now

$$\zeta^* \left((\ell)_a\right) := \zeta(\ell), \qquad (\ell)_a \in \mathcal{A}_{z,pen}$$

Optimization problem

Define L^* as the extended BM level for a randomly picked policyholder in a stationary state

Under the augmented system, we find optimal relativities as the solution of the optimization problems :

- Under Model 1 (frequency-only),

$$(\tilde{\zeta}(0), \dots, \tilde{\zeta}(z)) := \underset{(\zeta(0),\dots,\zeta(z))\in\mathbb{R}^{z+1}}{\operatorname{argmin}} \mathbb{E}[(\Lambda\Theta - \Lambda\zeta^*(L^*))^2]$$

- Under Model 2 (frequency-severity)

$$(\tilde{\zeta}(0), \dots, \tilde{\zeta}(z)) := \underset{(\zeta(0),\dots,\zeta(z))\in\mathbb{R}^{z+1}}{\operatorname{argmin}} \mathbb{E}[(\Lambda^{[1]}\Lambda^{[2]}\Theta^{[1]}\Theta^{[2]} - \Lambda^{[1]}\Lambda^{[2]}\zeta^*(L^*))^2] \tag{2}$$

Optimal relativities for Model 1 under $(-1/+h/pen)$ BMS

Under Model 1, the optimal relativities are given by

$$\tilde{\zeta}(\ell) := \frac{\mathbb{E}[\Lambda^2\Theta | L^* = (\ell)_0]}{\mathbb{E}[\Lambda^2 | L^* = (\ell)_0]}, \qquad \ell = 0, \ldots, h-1,$$

and

$$\tilde{\zeta}(\ell) := \frac{\sum_{a=0}^{pen} \mathbb{E}[\Lambda^2\Theta | L^* = (\ell)_a]\mathbb{P}(L^* = (\ell)_a)}{\sum_{a=0}^{pen} \mathbb{E}[\Lambda^2 | L^* = (\ell)_a]\mathbb{P}(L^* = (\ell)_a)},$$

for $\ell = h, \ldots, z; a = 0, \ldots, pen$

Comments on Model 2

Recall that $\mathbb{E}[\sum_{j=1}^{N_{it}} Y_{itj}|\lambda_i^{[1]}, \lambda_i^{[2]}, \Theta_i^{[1]}, \Theta_i^{[2]}] = \lambda_i^{[1]}\lambda_i^{[2]}\Theta_i^{[1]}\Theta_i^{[2]}$ is the "correct" premium for the i-th policyholder

$\Rightarrow$ $\Lambda^{[1]}\Lambda^{[2]}\Theta^{[1]}\Theta^{[2]} = \eta_{[1]}^{-1}(\boldsymbol{X}^{[1]}\beta^{[1]})\eta_{[2]}^{-1}(\boldsymbol{X}^{[2]}\beta^{[2]})\Theta^{[1]}\Theta^{[2]}$ is the "correct" premium for a randomly picked policyholder having observed risk characteristics $(\boldsymbol{X}^{[1]}, \boldsymbol{X}^{[2]})$ and unobserved risk characteristics $(\Theta^{[1]}, \Theta^{[2]})$

$\Rightarrow$ The optimization (2) is about choosing the relativities to minimize the mean squared difference between $\Lambda^{[1]}\Lambda^{[2]}\Theta^{[1]}\Theta^{[2]}$ and the actual premium charged $\Lambda^{[1]}\Lambda^{[2]}\zeta^*(L^*)$ when a policyholder is in the extended BM level L^*

Optimal relativities for Model 2 under $(-1/+h/pen)$ BMS

Under Model 2, the optimal relativities are given by

$$\tilde{\zeta}(\ell) := \frac{\mathbb{E}[(\Lambda^{[1]}\Lambda^{[2]})^2\Theta^{[1]}\Theta^{[2]}|L^* = (\ell)_0]}{\mathbb{E}[(\Lambda^{[1]}\Lambda^{[2]})^2|L^* = (\ell)_0]}, \qquad \ell = 0, \dots, h-1,$$

and

$$\tilde{\zeta}(\ell) := \frac{\sum_{a=0}^{pen} \mathbb{E}[(\Lambda^{[1]}\Lambda^{[2]})^2\Theta^{[1]}\Theta^{[2]}|L^* = (\ell)_a]\mathbb{P}(L^* = (\ell)_a)}{\sum_{a=0}^{pen} \mathbb{E}[(\Lambda^{[1]}\Lambda^{[2]})^2|L^* = (\ell)_a]\mathbb{P}(L^* = (\ell)_a)},$$

for $\ell = h, \dots, z; a = 0, \dots, pen$

Note that the optimal relativities are similarly given as in Model 1, but with

- Λ replaced by $\Lambda^{[1]}\Lambda^{[2]}$
- Θ replaced by $\Theta^{[1]}\Theta^{[2]}$

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The effect of the period of penalty

We consider Model 2 and study the effect of "*pen*" on :

- stationary probability

$$\mathbb{P}(L=\ell)=\sum_{\{a|(\ell)_a\in\mathcal{A}_{z,pen}\}}\mathbb{P}(L^*=(\ell)_a)$$

- optimal BM relativity $\tilde{\zeta}(\ell)$
- hypothetical mean square error (HMSE), which is the minimized value of the optimization (2)

Parameters

- Let $z = 9$, i.e. there are 10 BM levels
- Assume one risk class only, i.e. $\mathcal{K} = 1$ and we only have $i = 1$
- $N_{it} | (\Theta_i^{[1]} = \theta_i^{[1]}, \boldsymbol{X}_i^{[1]} = \boldsymbol{x}_i^{[1]}) \sim \mathrm{Poisson}(\lambda_i^{[1]} \theta_i^{[1]})$
- $Y_{itj} | (\Theta_i^{[2]} = \theta_i^{[2]}, \boldsymbol{X}_i^{[2]} = \boldsymbol{x}_i^{[2]}) \sim \mathrm{Gamma}(\lambda_i^{[2]} \theta_i^{[2]}, 1/\psi^{[2]})$ where $\lambda_i^{[2]} \theta_i^{[2]}$ is the mean and $1/\psi^{[2]}$ is the shape parameter
- $\Theta_i^{[k]} \sim \mathrm{Lognormal}(-\sigma_k^2/2, \sigma_k^2)$ for $k = 1, 2$
- Let $\lambda_i^{[1]} = 0.05$, $\lambda_i^{[2]} = e^8$, $1/\psi^{[2]} = 0.67$, $\sigma_1^2 = 0.99$, $\sigma_2^2 = 0.29$
- Assume a Gaussian copula with $\rho = -0.45$ for the bivariate random effect $(\Theta_i^{[1]}, \Theta_i^{[2]})$

$-1/+1/pen$ BMS for $pen = 0, 1, 2, 3$

pen	0		1		2		3	
Level ℓ	$\tilde{\zeta}(\ell)$	$\mathbb{P}(L=\ell)$	$\tilde{\zeta}(\ell)$	$\mathbb{P}(L=\ell)$	$\tilde{\zeta}(\ell)$	$\mathbb{P}(L=\ell)$	$\tilde{\zeta}(\ell)$	$\mathbb{P}(L=\ell)$
9	7.217	0.001	5.749	0.002	4.917	0.004	4.366	0.008
8	6.246	0.000	4.715	0.001	3.889	0.002	3.357	0.002
7	5.573	0.000	4.259	0.001	3.540	0.002	3.073	0.002
6	5.000	0.000	3.834	0.001	3.202	0.002	2.792	0.003
5	4.695	0.001	3.404	0.001	2.854	0.003	2.500	0.004
4	3.805	0.001	2.936	0.002	2.479	0.004	2.184	0.006
3	3.065	0.002	2.407	0.005	2.060	0.008	1.834	0.012
2	2.210	0.007	1.812	0.015	1.591	0.023	1.443	0.030
1	1.369	0.044	1.203	0.073	1.101	0.095	1.026	0.111
0	0.727	0.944	0.692	0.898	0.665	0.858	0.641	0.821
HMSE	14179.63		13189.89		12525.65		12053.38	

Observation

As "*pen*" increases,

- **Stationary probability** : some who occupied BM level 0 move towards higher BM levels
  $\rightarrow$ diversification
- **Optimal BM relativity** $\tilde{\zeta}(\ell)$: decreases (i.e. lower premium per driver for each BM level)
  $\rightarrow$ insurer's premium income is compensated by an increased portion of drivers at higher BM levels
- **HMSE** : decreases
  $\rightarrow$ improvement of prediction power

The end

Thank you !