

On valuation and risk management at the interface of insurance and finance

Thomas Møller

Laboratory of Actuarial Mathematics

University of Copenhagen

Universitetsparken 5

DK-2100 Copenhagen Ø

Denmark

Phone: +45 35 32 07 94

Fax: +45 35 32 07 72

e-mail: tmoller@math.ku.dk

Abstract. This paper reviews methods for hedging and valuation of insurance claims with an inherent financial risk, with special emphasis on quadratic hedging approaches and indifference pricing principles and their applications in insurance. It thus addresses aspects of the interplay between finance and insurance, an area which has gained considerable attention during the past years, in practice as well as in theory. Products combining insurance risk and financial risk have gained considerable market shares. Special attention is paid to unit-linked life insurance contracts, and it is demonstrated how these contracts can be valued and hedged by using traditional methods as well as more recent methods such as risk-minimization, mean-variance hedging and super-replication.

Key words: Actuarial valuation principles, financial risk, financial market, hedging, indifference pricing, unit-linked contracts, financial stop-loss contract.

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1 Introduction

During the past years, new insurance products that combine elements of insurance risk and financial risk have appeared; examples are unit-linked life insurance contracts, catastrophe insurance futures and bonds, and integrated risk-management solutions. This paper describes some of these new products in detail and discusses how they can be valued and hedged. This discussion includes a review of some recent theoretical results from the interface of insurance and finance. Focus will be on specific developments involving methods for hedging and valuation of risk in incomplete financial markets, and the aim is not to give a complete overview of the area. As an example, it is demonstrated how the risk in a portfolio of unit-linked life insurance contracts can be analysed by applying various methods such as risk-minimization, mean-variance hedging and super-replication. The results are compared with an approach proposed by Brennan and Schwartz (1979a, 1979b) that combines traditional law of large number considerations and financial mathematics.

Insurance background

The two fields of insurance and finance started as separate areas. At its very origin, the theory of insurance was mainly concerned with the computation of premiums for life insurance contracts. An overview of the early history of life insurance can be found in Braun (1937), and, according to this exposition, the first known social welfare programs with elements of life insurance are the Roman *Collegia* which date back at least to AD 133. The first primitive mortality tables were published in 1662 by John Graunt (1620–1674), who worked with only 7 different age groups. The first mortality table, where the expected number of survivors from year to year is given, is due to the astronomer Edmund Halley (1656–1742). These tables allowed for more precise predictions about portfolios of independent lives and were essential for computation of premiums for various life insurance contracts. In his book on evaluation of annuities on life from 1725, Abraham de Moivre (1667–1754) suggested methods for evaluation of life insurance contracts, combining interest and mortality under very simple assumptions about the mortality.

In 1738, Daniel Bernoulli (1700–1782) argued that risks¹ should not be measured by their expectations and hence laid the foundation for modern utility theory. Using examples related to gambling, he explained that the preferences of an individual may depend on his economic situation and, more specifically, that in some situations it could be reasonable for a poorer individual to prefer one uncertain future payment to another (more) uncertain payment with a larger expected value, whereas a wealthier person would prefer the payment with the largest expected value. This observation was also of importance for insurance in general, since it explained for example why individuals may accept to buy insurance contracts at a price which exceeds the expected value of the payment from the contract.

Financial background

Bachelier (1900) proposed to describe fluctuations in the price of a stock by a Brownian motion by assuming that the change in the value of the stock in a time interval of length h was normally distributed with mean αh and variance $\sigma^2 h$ and that changes

in non-overlapping intervals were stochastically independent. Samuelson (1965) advocated a framework where the stock price was modeled by a geometric Brownian motion, i.e. the exponential function of a Brownian motion, which had the advantage that it did not generate negative prices. Within this framework, and assuming in addition that money could be deposited on a savings account, Black and Scholes (1973) and Merton (1973) introduced the idea that options on stocks should really be priced such that no sure profits could arise from composing portfolios of long and short positions in the underlying stock and in the option itself. Assuming that the option price was a function of time and the current value of the stock, they obtained the celebrated Black-Scholes formula for European call options. This pricing formula has the at first glance surprising feature that it does not involve the expected return of the underlying stock. Cox, Ross and Rubinstein (1979) investigated a simple discrete time model, where the change in the value of the stock between two trading times can attain two different values only. In that setting, they derived option prices and obtained the pricing formulas of Black, Scholes and Merton as limiting cases by letting the length of the time intervals between trading times tend to 0. Building on concepts and ideas in Harrison and Kreps (1979) for discrete time models, Harrison and Pliska (1981) gave a mathematical theory for pricing of options under continuous trading and clarified the role of martingale theory in the pricing of options and its connection to key concepts such as absence of arbitrage and completeness.

Interplay between insurance and finance

The emergence of products combining financial and insurance risk (e.g. so-called unit-linked insurance contracts, various catastrophe futures and options and financial stop-loss reinsurance contracts) has forced the two fields of insurance and finance to search for combinations and unification of methodologies and basic principles. A survey of aspects of the growing interplay between the two fields is given in Embrechts (1996), who mentions institutional issues such as the increasing collaboration between insurance companies and banks (e.g. the construction of so-called “financial supermarkets”) and the deregulation of insurance markets as two further important aspects.

The present paper is organized as follows. Section 2 gives an overview of valuations techniques in life and non-life insurance, and Section 3 introduces main concepts related to financial valuation principles. In Section 4, some specific examples of interplay between the two fields of finance and insurance are mentioned. Section 5.1 studies applications in insurance of various hedging criteria including risk-minimization, mean-variance hedging and super-replication. Finally, Section 5.2 reviews results on indifference pricing of insurance contracts and presents some new results on actuarial premium calculations principles adapted to financial models.

2 Classical valuation of insurance contracts

Traditionally, actuarial theory is divided into life insurance mathematics and non-life insurance mathematics. In addition to historical aspects, there are fundamental differences between the two areas, for example in respect of time horizon of the

individual contract (for life insurance extending up to 50 years, whereas for non-life insurance typically limited to one year). These are reflected e.g. in the principles that are applied for the calculation of premiums. In this section, we review some notions and key concepts of life and non-life insurance, placing focus on the valuation techniques used there.

2.1 Life insurance

We recall some classical and basic concepts from life insurance; some recent introductory expositions to the area are Gerber (1986) and Norberg (2000).

Consider a portfolio of n lives aged y , say, to be insured at time 0 with i.i.d. remaining life times T_1, \dots, T_n , and assume that there exists a continuous function (called the hazard rate function) μ_{y+t} such that the survival probability is of the form ${}_t p_y = P(T_1 > t) = \exp(-\int_0^t \mu_{y+u} du)$. A *pure endowment* contract with sum insured K and term T stipulates that the amount K (the insurance benefit) is payable at time T contingent on survival of the policy-holder. Assume that the contract is paid by a single premium κ , say, at time 0. Assume furthermore that the seller of the contract (the insurance company) invests the premium κ in some asset which pays a rate of return $r = (r_t)_{0 \leq t \leq T}$ during $[0, T]$. For the i 'th policy-holder, the obligation of the insurance company is now given by the *present value*

$$H_i = 1_{\{T_i > T\}} K e^{-\int_0^T r_t dt}, \quad (2.1)$$

which is obtained by discounting the amount payable at T , $1_{\{T_i > T\}} K$, using the rate of return r . Note that (2.1) is a random variable. The fundamental *principle of equivalence* now states that the premiums should be chosen such that the present values of premiums and benefits balance on average. If we assume in addition that r is stochastically independent of the remaining life times, the principle of equivalence states that

$$\kappa = E[H_i] = {}_T p_y K E[e^{-\int_0^T r_t dt}] \quad (2.2)$$

for the single premium case. Since life insurance portfolios are often very large, this principle can be partly justified by using the law of large numbers. Indeed, as the size n of the portfolio is increased, the relative number of survivors $\frac{1}{n} \sum_{i=1}^n 1_{\{T_i > T\}}$ converges a.s. towards the probability ${}_T p_y$ of survival to T by the strong law of large numbers since the lifetimes T_1, \dots, T_n are stochastically independent. Thus, for n sufficiently large, the actual number of survivors $\sum_{i=1}^n 1_{\{T_i > T\}}$ will be “approximately” equal to the expected number, $n {}_T p_y$. Accumulating the premiums $n\kappa$ with interest now leads to

$$n\kappa e^{\int_0^T r_t dt} = n {}_T p_y K E[e^{-\int_0^T r_t dt}] e^{\int_0^T r_t dt} \approx \sum_{i=1}^n 1_{\{T_i > T\}} K E[e^{-\int_0^T r_t dt}] e^{\int_0^T r_t dt}. \quad (2.3)$$

In particular, when r is non-random, the expression on the right is equal to the amount to be paid to the policy-holders. So in the case of a deterministic rate of return, the principle of equivalence is justified directly by use of the law of large

numbers which essentially guarantees that the actual number of survivors is “close” to the expected number.

The problem becomes much more delicate in the more realistic situation where r is a stochastic process, and it follows immediately from (2.3) that the simple accumulation of the premium κ will not in general generate the amount to be paid, since $e^{-\int_0^T r_t dt}$ may differ considerably from its expected value. One way of dealing with this problem is to replace the “true” rate of return process r in (2.2) with some deterministic rate of return process r' which is such that the single premium $n\kappa$ accumulated by the true rate of return r is larger than K times the expected number of survivors with a large probability. The excess (if any) should then be added to the amount paid to the policy-holder and is known as *bonus*, see e.g. Ramlau-Hansen (1991) and Norberg (1999) and references therein. However, this approach really raises the problem of whether it is reasonable to assume the existence of *any* deterministic and strictly positive r' which over a very long time horizon has the property that it will be larger than the actual return on investments with a very large probability. In particular, this is an extremely relevant discussion when one thinks of the historically low interest rates observed in the late 1990s. An alternative to this approach is therefore to replace r by the so-called short rate of interest and then replace the last term in (2.2) by the price on the financial market of a financial asset which pays one unit at time T , a so-called *zero coupon bond*; see Persson (1998).

2.2 Non-life insurance

In comparison to the valuation principles in life insurance, discounting plays a much less prominent role in the classical non-life insurance premium calculation principles; see e.g. Bühlmann (1970) and Gerber (1979) for standard textbooks on the mathematics of these principles. This difference can be partly explained by the relatively short time horizons of most non-life insurance contracts, which typically change from year to year.

Let H denote some claim payable at a fixed time T , say. A premium calculation principle is a mapping which assigns to each claim a number, called the premium. One class of classical actuarial valuation principles applied in non-life insurance can be directly and somewhat pragmatically motivated from the law of large numbers. These principles prescribe charging a premium $\tilde{u}(H)$ which is equal to the expected value $E[H]$ of the claim augmented by some amount $A(H)$, the so-called *safety-loading*, i.e.

$$\tilde{u}(H) = E[H] + A(H). \quad (2.4)$$

The most important examples of such premium calculation principles are: $A(H) = 0$ (*the net premium principle* or the *principle of equivalence*), $A(H) = aE[H]$ (*the expected value principle*), $A(H) = a(\text{Var}[H])^{1/2}$ (*the standard deviation principle*), $A(H) = a\text{Var}[H]$ (*the variance principle*) and $A(H) = aE[(H - E[H])^+]$ (*the semi-variance principle*). Of these, the standard deviation principle seems to be the most widely used principle in practice. Bühlmann (1970) mentions the fact that it

is linear up to scaling as one possible explanation for its popularity, but judges its theoretical properties to be inferior to those of the variance principle.

Another interesting class of premium calculation principles consists of the so-called *zero increase expected utility principles*, which are derived as follows. Let u be a utility function, i.e. $u'(x) \geq 0$ and $u''(x) \leq 0$ for any $x \in \mathbb{R}$, and let V_0 denote the insurer's initial capital at time 0 (possibly random, e.g. depending on the result of other business). The zero (increase expected) utility premium of H under u and initial capital V_0 is the solution $\tilde{u}(H)$ to the equation

$$\mathbb{E}[u(V_0 + \tilde{u}(H) - H)] = \mathbb{E}[u(V_0)], \quad (2.5)$$

which states that the expected utility of the final wealth $V_0 + \tilde{u}(H) - H$ from selling the claim H at the premium $\tilde{u}(H)$ should equal the expected utility of V_0 ; the latter may be interpreted as the wealth associated with not selling the claim H . The zero utility premium defined by (2.5) is often also called the *fair premium*, since selling the claim leaves the expected utility unaffected, i.e. it leads neither to an increase nor a decrease in expected utility. The most prominent example is probably the so-called *exponential principle* which is obtained for the exponential utility function $u(x) = \frac{1}{a}(1 - e^{-ax})$. In particular, when V_0 is constant P -a.s., the solution to (2.5) does not depend on V_0 and is given by

$$\tilde{u}(H) = \frac{1}{a} \log(\mathbb{E}[e^{aH}]).$$

Another frequently used utility function is the quadratic utility function which is defined by $u(x) = x - \frac{x^2}{2s}$, $x \leq s$, and $u(x) = \frac{s}{2}$ for $x > s$. For a more complete survey of utility functions in insurance (and finance), see e.g. Gerber and Pafumi (1998).

An alternative principle is the so-called *Esscher principle*, which states that

$$\tilde{u}(H) = \frac{\mathbb{E}[He^{aH}]}{\mathbb{E}[e^{aH}]}. \quad (2.6)$$

This principle basically amounts to an exponential scaling of the claim H .

Other premium calculation principles worth mentioning are generalizations of the so-called *maximal loss principle*. For $\varepsilon \in [0, 1]$ and $p \in [0, 1]$, the (generalized) $(1 - \varepsilon)$ -percentile principle states that the premium should be computed as

$$\tilde{u}(H) = p\mathbb{E}[H] + (1 - p)F^{-1}(1 - \varepsilon),$$

where F is the distribution function of H and F^{-1} is its generalized inverse, i.e. $F^{-1}(y) = \inf\{x | F(x) \geq y\}$. Thus the premium is a weighted average of the expected value of H and the $(1 - \varepsilon)$ -percentile of the distribution of H . In particular, the maximal loss principle is obtained for $\varepsilon = 0$ and $p = 0$.

For a detailed investigation of the above mentioned principles and several other premium calculation principles, see e.g. Goovaerts, De Vylder and Haezendonck (1984) and Heilmann (1987).

3 Financial valuation principles

We recall some basic notation and concepts from financial mathematics. Standard textbooks are Duffie (1996) and Lamberton and Lapeyre (1996); see also Hull (1997) for an exposition including some more institutional aspects. Let T denote a fixed finite time horizon and consider a financial market consisting of two traded assets, a stock and a savings account with price processes $S = (S_t)_{0 \leq t \leq T}$ and $B = (B_t)_{0 \leq t \leq T}$, respectively, which are defined on some probability space (Ω, \mathcal{F}, P) , and introduce the discounted price processes $X = S/B$ and $X^0 = B/B \equiv 1$. In this setting, a *trading strategy* (or dynamical portfolio strategy) is a 2-dimensional process $\varphi = (\vartheta_t, \eta_t)_{0 \leq t \leq T}$ satisfying certain integrability conditions (which will be indicated later), and where ϑ is predictable and η is adapted with respect to some filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ which describes the evolution of available information. The pair $\varphi_t = (\vartheta_t, \eta_t)$ is the *portfolio* held at time t , that is, ϑ_t is the number of shares of the stock held at t and η_t is the discounted amount invested in the savings account. Thus, the discounted value at time t of φ_t is given by $V_t(\varphi) = \vartheta_t X_t + \eta_t$. A strategy φ is said to be *self-financing* if

$$V_t(\varphi) = V_0(\varphi) + \int_0^t \vartheta_s dX_s. \quad (3.1)$$

Here, $V_0(\varphi)$ can be interpreted as the amount invested at time 0 and $\int_0^t \vartheta_s dX_s$ as the accumulated trading gains generated by φ up to and including time t . Thus, for a self-financing strategy φ , the current value of the portfolio φ_t at time t is exactly the initially invested amount plus trading gains, so that no in- or outflow of capital has taken place during $(0, t]$. A contract (or *claim*) specifying the discounted (\mathcal{F}_T -measurable) payoff H at time T is said to be *attainable* if there exists a self-financing strategy φ such that $V_T(\varphi) = H$ a.s., that is, if H coincides with the terminal value of a self-financing strategy. Thus a claim is attainable if and only if it can be represented as a constant H_0 plus a stochastic integral with respect to the discounted stock price process

$$H = H_0 + \int_0^T \vartheta_s^H dX_s. \quad (3.2)$$

The initial investment $V_0(\varphi) = H_0$ needed for this perfect replication of H is also called the unique no-arbitrage price of H . To see that any other price will lead to an arbitrage possibility, i.e. to a risk-free gain, suppose that the price of H at time 0 is given by $H_0 + \epsilon$, where $\epsilon > 0$. A risk-free gain of ϵ can now be generated in the following way:

- Sell the claim H at time 0 and receive $H_0 + \epsilon$. Thus, at time T we have to pay H to the buyer of the claim.
- Invest H_0 via the self-financing strategy $\varphi = (\vartheta, \eta)$ defined by taking $\vartheta_t = \vartheta_t^H$ and by choosing η_t such that (3.1) is satisfied, i.e.

$$\eta_t = V_0(\varphi) + \int_0^t \vartheta_s dX_s - \vartheta_t X_t.$$

- The value of the portfolio φ_T at time T is now exactly equal to H , see (3.1) and (3.2), which is to be paid to the buyer of the contract.

The net result of these transactions is the gain ϵ . (If the price of H is $H_0 + \epsilon$, with $\epsilon < 0$, the gain $-\epsilon$ can be obtained by buying H and using the hedging strategy $-\varphi$.) The argument illustrates how the amount H_0 received at time 0 can be transformed into the amount H at time T by following a self-financing strategy, so that H_0 is indeed the only reasonable price.

A financial market is said to be *complete* if all claims are attainable. One example of a complete market with continuous trading is the so-called Black-Scholes model which consists of two assets, a stock whose price process is described by a geometric Brownian motion and a savings account which pays a deterministic and constant rate of return. An example with discrete time trading is the Cox-Ross-Rubinstein model described above, which is also known as the binomial model. One important feature of complete markets admitting no arbitrage possibilities is the existence of a unique risk-neutral measure. A risk-neutral measure is a probability measure Q which is equivalent² to P and which is such that X is a (local) Q -martingale. From the general theory of stochastic calculus it follows that $\int \vartheta^H dX$ is also a local Q -martingale under certain conditions on ϑ^H . Furthermore, if ϑ^H is sufficiently integrable for $\int \vartheta^H dX$ to be a true Q -martingale, then it follows from (3.2) that the no-arbitrage price of H is $H_0 = E_Q[H]$, since in this case $E_Q[\int_0^T \vartheta^H dX] = 0$.

If there exist claims that are not attainable, i.e. claims which do not allow a representation of the form (3.2) and hence cannot be replicated by means of any self-financing trading strategy, then the market is said to be *incomplete*; in this case there are infinitely many risk-neutral measures. The completeness property is often lost when we move on to more general models than the ones described above. In the discrete time case, incompleteness occurs already if we replace the binomial model with a *trinomial* model, i.e. a model where the change in the value of the stock between two trading times can attain three different values. An example of an incomplete model under continuous trading is obtained by adding to the geometric Brownian motion a Poisson-driven jump component, say. Another class of examples of incomplete markets consists of models where claims are allowed to depend on more uncertainty than the one generated by the financial market. Pricing of non-attainable claims is far more delicate than the pricing of attainable claims and typically requires a description of the preferences of the buyers and sellers. In the following we list some different approaches to pricing and hedging in incomplete markets.

Super-replication

One approach to pricing in incomplete markets is super-replication, see e.g. El Karoui and Quenez (1995). For a given contingent claim H , this approach essentially consists in finding the smallest number V_0^* , say, such that there exists a self-financing strategy $\tilde{\varphi}$ with $V_0(\tilde{\varphi}) = V_0^*$ and

$$V_T(\tilde{\varphi}) \geq H, \quad P\text{-a.s.}$$

By charging the price V_0^* and applying the strategy $\tilde{\varphi}$, the hedger can generate an amount which exceeds the needed amount H , P -a.s. Thus, the main advantage of

this approach is that it leaves no risk to the hedger, since, after an initial investment, no additional capital is needed in order to pay the amount H to the buyer of the contract.

A (marginal) utility approach

An alternative is to derive fair prices from some utility function describing the preferences of the buyers and sellers, see Davis (1997) and references therein. Using a marginal utility argument, Davis (1997) defines the fair price of a claim H as the price which makes investors indifferent between investing “a little of their funds” in the contract and not investing in this contract. More precisely, let u be a utility function, c the investor’s initial capital at time 0, p the price charged at time 0 per unit of some claim H , z the amount invested in H , and introduce

$$W(z, p, c) = \sup_{\vartheta} \mathbb{E} \left[u \left(c - z + \int_0^T \vartheta_u dX_u + \frac{z}{p} H \right) \right],$$

where the supremum is taken over all strategies ϑ from some suitable space of processes. The number $W(z, p, c)$ is the maximum obtainable expected utility for an investor with initial capital c who invests in z/p units of the risk H . The fair price $\tilde{u}(H; c)$ of H is then defined as the solution \tilde{p} to the equation

$$\frac{\partial}{\partial z} W(0, p, c) = 0,$$

provided that the relevant quantities exist.

Quadratic approaches

A third class of approaches for pricing and hedging in incomplete markets consists of the so-called quadratic methods, see e.g. Schweizer (2001a) for a survey. This class of approaches can be divided into (local) risk-minimization approaches, proposed by Föllmer and Sondermann (1986) for the case where X is a martingale and generalized to semimartingales by Schweizer (1988, 1991), and mean-variance hedging approaches, proposed by Bouleau and Lamberton (1989) and Duffie and Richardson (1991). With mean-variance hedging approaches, the main idea is essentially to “approximate” the claim H as closely as possible by the terminal value of a self-financing strategy using a quadratic criterion. More precisely, this amounts to finding a self-financing strategy $\varphi^* = (\vartheta^*, \eta^*)$ which minimizes

$$\mathbb{E} [(H - V_T(\varphi))^2] = \|H - V_T(\varphi)\|_{L^2(P)}^2 \tag{3.3}$$

over all self-financing strategies φ , i.e. a strategy which approximates H in the L^2 -sense. By (3.1), this strategy is completely determined by the pair $(V_0(\varphi^*), \vartheta^*)$, so that the solution to the problem of minimizing (3.3) is obtained in principle by projecting the random variable H in $L^2(P)$ on the subspace spanned by \mathbb{R} and random variables of the form $\int_0^T \vartheta dX$. The optimal initial capital $V_0(\varphi^*)$ is often called the *approximation price* for H , and the optimal strategy is the *mean-variance*

hedging strategy.

Let us now turn to the criterion of risk-minimization. For any (not necessarily self-financing) strategy $\varphi = (\vartheta, \eta)$ we define the *cost process* by

$$C_t(\varphi) = V_t(\varphi) - \int_0^t \vartheta_s dX_s. \quad (3.4)$$

This process keeps track of the hedger's accumulated costs associated with φ : At any time t , it is the current value $V_t(\varphi)$ of the strategy reduced by trading gains $\int_0^t \vartheta dX$. In particular, it follows by inserting (3.1) in (3.4) that the cost process of a self-financing strategy is P -a.s. constant. In contrast to (3.3), Föllmer and Sondermann (1986) proposed to drop the restriction to self-financing strategies but insisted on keeping the condition $V_T(\varphi) = H$. With their terminology, a strategy φ is now said to be *risk-minimizing* (for H) if $V_T(\varphi) = H$ and if it minimizes at any time t the conditional expected squared remaining costs

$$R_t(\varphi) = \mathbb{E} \left[(C_T(\varphi) - C_t(\varphi))^2 \mid \mathcal{F}_t \right].$$

This optimality criterion amounts to keeping the fluctuations in the cost process as small as possible under the condition $V_T(\varphi) = H$. In particular, Föllmer and Sondermann (1986) proved that the cost process of a risk-minimizing strategy is a martingale.

Quantile hedging and shortfall risk minimization

One possibly undesirable feature of the quadratic approaches is the fact that they punish losses and gains equally. An alternative is to use *quantile hedging*, see Föllmer and Leukert (1999), where the objective is to hedge the claim with a certain probability. Another alternative is the criterion of minimizing the expected shortfall risk, i.e. expected losses from hedging, which has been proposed by Föllmer and Leukert (2000) and Cvitanić (2000). They introduce a *loss function* $l : [0, \infty) \mapsto [0, \infty)$, which is taken to be an increasing convex function with $l(0) = 0$, and consider the problem of minimizing

$$\mathbb{E} \left[l \left((H - V_T(\varphi))^+ \right) \right], \quad (3.5)$$

over the class of self-financing hedging strategies. Typical loss functions are power functions, $l(x) = x^p$, $p \geq 1$, and in this case, (3.5) is related to minimizing the so-called lower partial moments.

4 Interplay between insurance and finance

This section mentions some specific areas of the interplay between finance and insurance.

4.1 Unit-linked insurance contracts

Unit-linked insurance contracts seem to have been introduced for the first time in the Netherlands in the early fifties; in the United States the first unit-linked insurance

contracts were offered around 1954, and in the United Kingdom unit-linked contracts appeared for the first time in 1957³. The contracts are also called *equity-linked* or *equity-based* insurance contracts, and in the United States they are known as *variable life* insurance contracts. A unit-linked life insurance contract differs from the traditional life insurance contracts described in Section 2.1 in that benefits (and sometimes also premiums) depend explicitly on the development of some stock index or the value of some (more or less) specified portfolio. This construction allows for great flexibility as compared with traditional life insurance products in that the policy-holder is offered the opportunity of deciding how his or her premiums are to be invested. Today, issuers of unit-linked life insurance contracts typically offer a variety of investment possibilities that include e.g. worldwide or country specific indices, and reference portfolios with specific investment profiles, e.g. investments in companies from certain branches or regions, or organizations with certain ethical codes.

Denote by S_t the value of the stock index at time t . In the following, we shall refer to the entire development of the stock by simply writing S . As in Section 2.1, consider a portfolio consisting of n policy-holders with remaining life times T_1, \dots, T_n . Assume for simplicity that they all buy the same form of unit-linked pure endowment contract at time 0 and that the life times are stochastically independent of the development on the financial market. The contracts specify the payment of some (non-negative) amount $f(S)$ to the policy-holder at time T if he or she is still alive at this time; f is a function which prescribes some dependence on the development of the stock price. Thus, the present value at time 0 of the insurance company's liability towards the n policy-holders is

$$H = \sum_{i=1}^n 1_{\{T_i > T\}} f(S) e^{-\int_0^T r_u du}, \quad (4.1)$$

where we have discounted the payment by the short rate of interest r . For example, the amount paid could be a function of the terminal value of the stock only, that is

$$f(S) = S_T, \quad (4.2)$$

or the terminal value guaranteed against falling short of some prefixed amount K ,

$$f(S) = \max(S_T, K). \quad (4.3)$$

The contract (4.2) is known as a *pure unit-linked* contract and (4.3) is called *unit-linked with guarantee* (the guaranteed benefit is K). However, f could also be a more complex function of the process S , for example a guaranteed annual return is given by

$$f(S) = K \cdot \prod_{j=1}^T \max \left(1 + \frac{S_j - S_{j-1}}{S_{j-1}}, 1 + \delta_j \right).$$

Here, the fraction $(S_j - S_{j-1})/S_{j-1}$ is the return in year j on the asset S and δ_j is the guaranteed return in year j . At time 0 the amount payable at time T is guaranteed

against falling short of $K \cdot \prod_1^T (1 + \delta_j)$, but the guarantee goes beyond this “worst case scenario”.

Unit-linked contracts have been analysed by actuaries since the late sixties, see e.g. Turner (1969), Kahn (1971) and Wilkie (1978); the two last mentioned give simulation studies for an insurance company administrating portfolios of unit-linked insurance contracts. Using modern theories of financial mathematics, Brennan and Schwartz (1979a, 1979b) proposed new valuation principles and investment strategies for unit-linked insurance contracts with so-called asset value guarantees (minimum guarantees). Their principles essentially consisted in combining traditional (law of large numbers) arguments from life insurance with the methods of Black and Scholes (1973) and Merton (1973). By appealing to the law of large numbers, Brennan and Schwartz (1979a, 1979b) first replaced the uncertain courses of the insured lives by their expected values, so that the actual insurance claims including mortality risk as well as financial risk were replaced by modified claims, which only contained financial uncertainty. More precisely, instead of considering the claim (4.1) they looked at

$$H' = n_{Tp_y} f(S) e^{-\int_0^T r_u du}. \quad (4.4)$$

(Recall the notation $_{Tp_y} = P(T_1 > T)$ introduced in Section 2.1.) These modified claims were then recognized as essentially being options (with a very long maturity, though) which could in principle be priced and hedged using the basic principles of (modern) financial mathematics due to Black and Scholes (1973) and Merton (1973). For the pure unit-linked contract (4.2), the claim (4.4) is proportional to the terminal value of the stock S_T and hence can be hedged by a buy-and-hold strategy which consists in buying n_{Tp_y} units of the stock at time 0 and holding these until T . Thus, in the case of no guarantee, the unique no-arbitrage price of H' is simply $n_{Tp_y} S_0$. Consequently, one possible fair premium for each policy-holder is $_{Tp_y} S_0$, the probability of survival to T times the value at time 0 of the stock index. Now consider the contract with benefit $f(S) = \max(S_T, K) = (S_T - K)^+ + K$. In this case pricing of (4.4) involves the pricing of a European call option. In the general case, we see that this principle suggest the premium $n_{Tp_y} V_0^f$, where V_0^f is the price at time 0 of the purely financial contract which pays $f(S)$ at time T . More recently, the problem of pricing unit-linked life insurance contracts (under constant interest rates) has been addressed by Delbaen (1986), Bacinello and Ortu (1993a) and Aase and Persson (1994), among others, who combined the martingale approach of Harrison and Kreps (1979) and Harrison and Pliska (1981) with law of large numbers arguments. Whereas all the above mentioned papers assumed a constant interest rate, Bacinello and Ortu (1993b), Nielsen and Sandmann (1995) and Bacinello and Persson (1998), among others, generalized existing results to the case of stochastic interest rates.

In contrast to earlier approaches, Aase and Persson (1994) worked with continuous survival probabilities (i.e. with death benefits that are payable immediately upon the death of the policy-holder and not at the end of the year as would be implied by discrete time survival probabilities) and suggested investment strategies for unit-linked insurance contracts by methods similar to the ones proposed by Brennan

and Schwartz (1979a, 1979b) for discrete time survival probabilities. In contrast to Brennan and Schwartz (1979a, 1979b), who considered a “large” portfolio of policy-holders and therefore worked with “deterministic mortality”, Aase and Persson (1994) considered a portfolio consisting of one policy-holder only. However, in all the above papers, the uncertain courses of the insured lives were replaced at an early point with the expected courses in order to allow an application of standard financial valuation techniques for complete markets. The resulting strategies therefore did not account for the mortality uncertainty within a portfolio of unit-linked life insurance contracts, and the approach thus leaves open the question of how to quantify and manage the combined actuarial and financial risk inherent in these contracts. In particular, it leaves open the question to which extent this combined risk can be hedged on the financial markets.

4.2 Other insurance derivatives

This section describes some further specific products that have appeared in practice and that combine traditional insurance risk and financial derivatives. The best known examples are probably catastrophe futures, catastrophe-linked bonds, financial stop-loss contracts and stop-loss contracts with a barrier. These new products are really genuine combinations of financial derivatives and insurance products, and they are known as *insurance derivatives*. The emergence of such products has been serving as a catalyst for breaking down borders between traditional reinsurance and finance and has opened up the possibility of rethinking fundamental principles of reinsurance and investment. This development presents a challenge to direct insurers and reinsurers as well as to financial institutions in general.

Catastrophe insurance (CAT) futures

In the 1980s and early 1990s, several severe catastrophes impaired the capacity of reinsurers offering traditional catastrophe covers, and this situation caused an increase in reinsurance premiums. In 1992 the so-called catastrophe insurance (CAT) futures and options on CAT futures were introduced. These instruments standardized catastrophe insurance risk and transformed it into tradeable securities, thus providing a new tool for insurers seeking cover against catastrophe risk. This securitization was modified in 1995, but the underlying idea essentially remained the same. For an introduction to CAT futures, see e.g. Cummins and Geman (1995) and references therein. An overview of securitization of catastrophe insurance risk and an analysis of some of the problems associated with securitization can be found in Tilley (1997).

The basic idea is the following. Consider losses occurring in a specific area and caused by certain well defined catastrophic events, e.g. hurricanes with a certain wind speed or earthquakes of a certain magnitude. Clearly, different insurers will be subject to different exposures from such risks as a consequence of differences in the composition of their insurance portfolios, and with traditional reinsurance contracts, each company would purchase their own insurance covers against risk. Assume now that a number of (suitably chosen) insurance companies report premiums and claims related to the pre-specified type of catastrophes (during certain pre-specified

periods) to some central office. Based on the reports, this office constructs a *loss index* $L = (L_t)_{0 \leq t \leq T}$ which is taken to be the underlying process for a futures price process. More precisely, this means that the index L is being reported regularly to the public and that a futures price process $F = (F_t)_{0 \leq t \leq T}$ is constructed by fixing the terminal value $F_T = \min(2, L_T/\kappa)$, where κ is the accumulated premiums for the reporting companies and T is some fixed finite time horizon. Insurance and reinsurance companies as well as other investors can now buy and sell this standardized catastrophe risk by purchasing and issuing options on this index on some stock exchange. For example, the *call spread* $H = (F_T - K_1)^+ - (F_T - K_2)^+$, $0 \leq K_1 \leq K_2 \leq 2$, provides cover for relative losses (i.e. the ratio of losses over premiums) in the interval $[K_1, K_2]$. The main advantage of this construction lies in the standardization and securitization of the catastrophic risk, which serves to transform the risk related to individual insurance companies into one (common) quantity. Thus, this transformed risk may be more attractive and conceivable to a group of investors which extends beyond traditional reinsurance companies, since it is relatively close in nature to existing financial derivatives. By attracting agents from a wider group of agents than just the traditional reinsurance companies, these instruments increased the financial capacity of the reinsurance market. On the other hand, the disadvantage for the direct insurers is that their own relative losses may differ considerably from the average relative losses of the reporting companies. Thus, for a given insurance company, the cover from the call spread on the CAT futures index will typically not correspond exactly to the actual loss experienced by this company.

Catastrophe-linked bonds

Individual insurance companies can also choose to securitize part of their insurance risk directly, for example by issuing bonds that are linked to insurance losses from certain insurance portfolios. One example of such an arrangement is the so-called *Winterthur Insurance Convertible Bond*, also called WinCAT bond. This bond, which was introduced by Winterthur in 1997, is described and analysed in Schmock (1999); see also Gisler and Frost (1999). With this three year bond, investors receive annual coupons as long as certain catastrophic events related to one of Winterthur's own insurance portfolios have not occurred. Thus, the investors receive a return from the bond which exceeds the market interest rate as long as no catastrophe has occurred and a lower return in the case of a catastrophe. The difference between the return under no catastrophe and the interest rate on the market was essentially a premium that Winterthur paid investors for "putting their money at risk"; similarly, the low return in connection with a catastrophic event essentially implied that the investors had covered part of Winterthur's losses. In Cox and Pedersen (2000) catastrophe bonds are priced within a discrete time model via some equilibrium considerations.

This type of product has the advantage over, for example options on the CAT futures index, that it provides a much more tailor-made cover for the issuer in that the trigger events that knock out the coupons are directly linked to the company's own insurance portfolio and not to some standardized index. The disadvantage is that there may be considerable costs associated with the selling of such bonds and

that the seller will have to convince buyers that they are only subject to a minimal moral hazard and credit risk.

Financial stop-loss contracts

Whereas CAT futures and Catastrophe-linked bonds are aimed at a larger group of investors, new reinsurance contracts that combine elements of insurance and financial derivatives have also been introduced by traditional reinsurers. In Swiss Re (1998) several new contracts are described under the title “Integrated Risk Management Solutions”. One example is the so-called financial stop-loss contract, which promises to pay at some fixed time T the amount

$$H = (U_T + Y_T - K)^+ \tag{4.5}$$

where U_T is the aggregate claim amount during $[0, T]$ on some insurance portfolio, Y_T is some financial loss and K is some retention limit. For $Y_T \equiv 0$ P -a.s., the contract is just a traditional stop-loss contract; however, the loss Y_T could for example be a put option on some underlying stock index S , that is $Y_T = (c - S_T)^+$ or it could simply be the loss associated with holding one unit of this index, that is, $Y_T = S_0 - S_T$. The financial stop-loss contracts provide a coverage not only for large losses due to fluctuations within the insurance portfolio (insurance risk) but also for adverse development of the financial markets (financial risk). In practice, reinsurance companies would typically sell spreads on the form $(U_T + Y_T - K_1)^+ - (U_T + Y_T - K_2)^+$, where $0 \leq K_1 \leq K_2$, which covers the $(K_1, K_2]$ layer of the losses $U_T + Y_T$.

The main idea behind the insurance contract (4.5) is that it provides cover for the insurer’s *total risk*, i.e. the combined insurance risk from the insurance portfolio and the financial risk from the financial portfolio. With a traditional stop-loss contract, the reinsurer would cover insurance losses exceeding the level K . However, the financial stop-loss contract is designed so that the cover is only paid provided that the insurance loss augmented by the financial loss exceeds this level. Thus, a large financial gain $-Y_T$ may compensate for large insurance losses, and in this situation, the buyer does not really need additional compensation from the reinsurer. This feature is illustrated by Figure 1.a, where the area above the solid line represents pairs (Y_T, U_T) of financial losses Y_T and accumulated insurance claims U_T that generate a payment from the reinsurer. The area between the solid line and the dashed line are pairs (Y_T, U_T) where (large) insurance claims U_T are partly compensated by financial gains $-Y_T$. The problem of pricing these contracts is a challenge to both actuaries and financial mathematicians. This fact is for example underscored by the following quotation from Swiss Re (1998, p. 15), “..., the risk-neutral valuation technique traditionally used for the pricing of financial derivatives cannot be applied directly but needs to be adjusted and complemented by actuarial methods”.

The contract (4.5) should be compared to the alternative of buying a traditional stop-loss contract with retention level K' paying $(U_T - K')^+$ and a traditional financial derivative, which pays $(Y_T - K'')^+$; the constants K' and K'' could for example be chosen such that $K' + K'' = K$. It follows already from the inequality

$$(U_T + Y_T - K)^+ \leq (U_T - K')^+ + (Y_T - K'')^+, \tag{4.6}$$

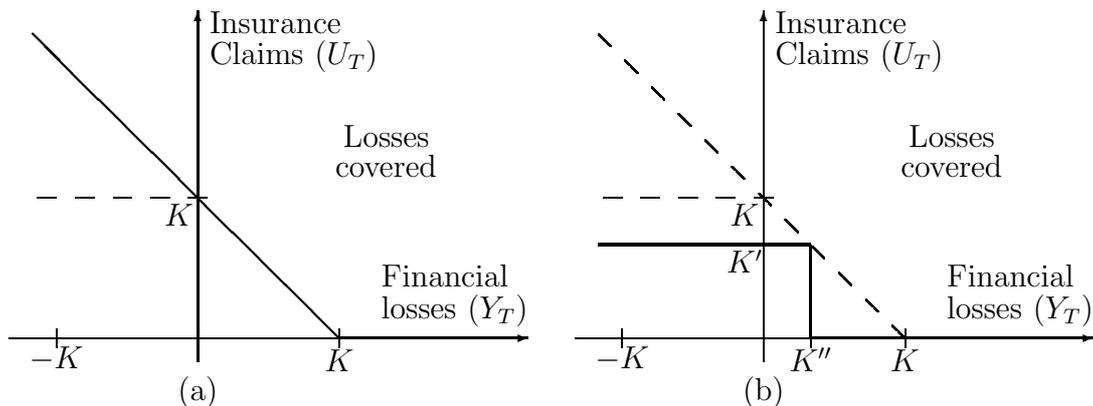


Figure 1: *Regions of cover under the financial stop-loss contract with retention K (figure (a)) and under the combination of a traditional stop-loss contract $(U_T - K')^+$ and a call option $(Y_T - K'')^+$ (figure (b)).*

which is satisfied provided that $K' + K'' \leq K$, that the cover from the financial stop-loss contract is dominated by combinations of a traditional stop-loss contract on U_T and a call option on Y_T . The region of cover under the stop-loss contract and the call option is depicted in Figure 1.b as the area above the solid lines. This figure shows that the region is indeed larger than the corresponding region under the financial stop-loss contract. In particular, it follows that the insurer will receive compensation from the reinsurer also in the situation where very large gains have arisen from investments. Thus, with the traditional instruments, the insurer has actually bought too much insurance cover; the financial stop-loss contract suits better the needs of the insurer.

Finally, we emphasize that the inequality (4.6) indeed indicates that the premium for the financial stop-loss contract should be dominated by the sum of the price on the financial market of $(Y_T - K'')^+$ and the reinsurers' premium for $(U_T - K')^+$. However, the difference may be relatively small since financial stop-loss contracts have only appeared recently and since they are only bought and sold in very limited amounts. Another important point is that, whereas the call option is sold on the financial market, the (financial and traditional) stop-loss contracts are agreements between a reinsurer and an insurer, and such contracts are typically not traded on stock exchanges. Therefore it is not in general possible to make statements like "by no-arbitrage arguments" etc. about insurance premiums; see also the discussion on the difference between actuarial and financial valuation principles in Embrechts (1996).

4.3 Combining theories for financial and actuarial valuation

One fundamental difference between the financial valuation techniques, or, more precisely, pricing by no-arbitrage, and the classical actuarial valuation principles reviewed above is that the financial valuation principles are formulated within a

framework which includes the possibility of trading certain assets, whereas several of the classical actuarial valuation principles are based on more or less ad hoc considerations involving the law of large numbers. While the financial valuation principles are based on *dynamic* trading, many decision problems in insurance, for example concerning the choice of optimal reinsurance plans and premiums, were traditionally analysed taking a *static* view. Several attempts have been made to bring together elements of the two theories, and this whole area is still very much “under construction”. We do not aim at giving a complete overview of this process but rather at focusing on some specific developments.

Dynamic reinsurance markets (From financial to actuarial valuation principles)

Several authors have studied dynamic reinsurance markets in a continuous time framework using no-arbitrage conditions, see for example Sondermann (1991), Delbaen and Haezendonck (1989) and de Waegenaere and Delbaen (1992). For an equilibrium analysis of dynamic reinsurance markets, see e.g. Aase (1993) and references therein. The main idea underlying the above mentioned papers is to allow for dynamic rebalancing of proportional reinsurance covers. They all assume that some process related to an *insurance risk process* (accumulated premiums minus claims) of some insurance business is tradeable and that positions can be rebalanced continuously. For example, this could mean that reinsurers can change at any time (continuously) the amount of insurance business that they have accepted. Thus, the insurance risk process can essentially be viewed as a traded security, and this already imposes no-arbitrage bounds on premiums for other (traditional) reinsurance contracts such as stop-loss contracts.

Let us review the main results obtained by Sondermann (1991) and Delbaen and Haezendonck (1989) in more detail. As in the previous section, let U_t be the accumulated claims during $[0, t]$ in some insurance business. Let furthermore $p = (p_t)_{0 \leq t \leq T}$ be a predictable process related to the premiums on this business, and define a new process X by

$$X_t = U_t + p_t. \tag{4.7}$$

Sondermann (1991) takes $-p_t$ to be the premiums paid during $[0, t]$, so that $-X_t$ is in fact identical to the insurance risk process. Thus, one can think of X_t as the value at time t of an account where claims are added and premiums subtracted as they incur. In particular, in the special case where premiums are paid continuously at a fixed rate κ , $p_t = -\kappa t$. Reinsurers can now participate in the risk by trading the asset X , i.e. by holding a position in the asset with price process X . Sondermann (1991) points out that in this setting of a dynamic market for proportional reinsurance contracts, traditional reinsurance contracts such as stop-loss contracts can be viewed as contingent claims and that these claims should be priced so that no arbitrage possibilities arise. Delbaen and Haezendonck (1989) take p_t to be the premium at which the direct insurer can sell the remaining risk $U_T - U_t$ on the reinsurance market. Thus, in their framework, X_t represents the insurer’s liabilities at time t . In the special case where the direct insurer receives continuously paid premiums at rate κ and provided that this premium is identical to the one charged by the reinsurers, we obtain that $p_t = \kappa(T - t)$, so that p_t in this situation differs from

Sondermann's choice only by the constant κT . Delbaen and Haezendonck (1989) assume that U is a compound Poisson process, i.e. $U_t = \sum_{i=1}^{N_t} Z_i$, where N is a Poisson process and Z_1, Z_2, \dots is a sequence of i.i.d. non-negative random variables which are independent of N . They then focus on the set of equivalent measures Q which are such that U is also a Q -compound Poisson process. For each such measure Q , a predictable premium process p is obtained by requiring that X be a Q -martingale. This procedure is partly motivated by no-arbitrage considerations (assuming in addition that all amounts have been discounted with the interest rate on the market), since this guarantees that no arbitrage possibilities arise from trading in X . In this way, Delbaen and Haezendonck (1989) recover several traditional actuarial valuation principles on a certain subspace of claims from no-arbitrage considerations, namely the expected value principle, the variance principle and the Esscher principle. A more detailed account of the results of Delbaen and Haezendonck (1989) is also given by Embrechts (1996).

In Steffensen (2000, 2001) general life insurance contracts are studied within a securitization framework which covers both classical and unit-linked life insurance contracts. More precisely, it is assumed that there exist certain traded assets whose price processes are affected by some underlying insurance risk, for example the number of deaths within a portfolio of insured lives. Within this setup, which also opens for a systematic treatment of bonus in life insurance, Steffensen (2000) defines the reserve as the market price of future payments and derives generalized versions of *Thiele's differential equation* under various assumptions about the structure of the payments.

From actuarial to financial valuation principles

Gerber and Shiu (1996) among others consider the situation where the logarithm of the stock price process is a Levy process, i.e. a process with independent and stationary increments. For example, this class of processes includes the geometric Brownian motion and the geometric (shifted) compound Poisson process. Within this setting, they demonstrate how the Esscher transform (2.6) can be used in the pricing of options. They give a very simple option pricing formula which involves Esscher transforms and which, for a European call option, indeed specializes to the well-known Black-Scholes formula in the case of a geometric Brownian motion. Furthermore, they demonstrate how this pricing formula can be derived via a simple utility indifference argument in the case of a power utility function $u(x) = \frac{x^{1-a}}{1-a}$ with parameter $a > 0$. This way they give a candidate for a martingale measure that could be used for pricing in incomplete markets also; they call the resulting martingale measure the *risk-neutral Esscher measure*. For further results on the relation between Esscher transforms, utility theory and equilibrium theory, see Bühlmann (1980, 1984) and references in Delbaen and Haezendonck (1989). A treatment of some of the mathematical aspects associated with Esscher transforms for stochastic processes can be found in Bühlmann, Delbaen, Embrechts and Shiryaev (1997).

In Schweizer (2001b), the starting points are the traditional standard deviation and variance principles, which are of the form (2.4). These principles are taken as

measures of riskiness, which assign to each claim a premium. It is then argued that the measures can equivalently be viewed as measures of preferences which operate on the insurer's terminal wealth by simply changing the sign on the loading factor. This way Schweizer (2001b) obtains functionals which to each outcome of the insurer's final wealth assign a number, and one can think of this number as the expected value of the insurer's utility of this wealth. For the standard deviation principle, the corresponding functional is given by

$$u(Y) = E[Y] - a(\text{Var}[Y])^{1/2}. \quad (4.8)$$

(Dana (1999) refers to a functional of this form as a *mean variance utility function*.) These new functionals are then embedded in a financial framework where the insurer can trade certain assets. Via an indifference argument, Schweizer (2001b) derives financial counterparts of the actuarial standard deviation and variance principles. More precisely, the fair premium π is defined as the unique solution to

$$\sup_{\varphi} u(c + \pi + V_T(\varphi) - H) = \sup_{\varphi} u(c + V_T(\varphi)), \quad (4.9)$$

where the suprema are taken over self-financing strategies with initial value 0 satisfying certain integrability conditions. Here, the term $c + \pi + V_T(\varphi) - H$ is the insurer's wealth at time T from selling the claim H at the price π : It is given by the initial capital c augmented by the premium π and trading gains $V_T(\varphi)$ and reduced by the claim H payable at time T . Similarly, $c + V_T(\varphi)$ is the insurer's wealth at T from not selling the claim and investing according to the strategy φ . The insurer is now said to be indifferent between selling H and not selling H , if the maximum obtainable utilities in the two scenarios are identical, that is, if (4.9) is satisfied. This leads to new financial valuation principles which resemble their actuarial counterparts in that they consist of an expectation plus some safety-loading. However, for the financial valuation principles, the expected value is now computed under a specific martingale measure \tilde{P} known as the *variance optimal martingale measure*. This martingale measure is the risk-neutral measure whose Radon-Nikodym derivative with respect to P has the smallest variance, i.e. it minimizes $\text{Var}[\frac{dQ}{dP}]$ among all risk-neutral measures Q . Moreover \tilde{P} has the special property that the Radon-Nikodym derivative can be represented as a constant plus a stochastic integral, that is

$$\frac{d\tilde{P}}{dP} = \tilde{Z}_0 + \int_0^T \tilde{\zeta}_t dX_t.$$

Furthermore, the loading factor is now a function of the variance of the so-called non-hedgeable part of the claim H which, in general, is smaller than the variance of H , and which can be quite difficult to determine. These new financial valuations are in accordance with no-arbitrage pricing for attainable claims, and thus, they provide alternative approaches for the valuation of options and other derivatives in incomplete markets.

5 Hedging and indifference pricing in insurance

In this section we mention some further results for insurance claims that combine financial and insurance risk. Section 5.1 reviews existing applications to insurance of the theory of risk-minimization with special emphasis on hedging (and pricing) of unit-linked insurance contracts, and Section 5.2 is related to the financial variance and standard deviation principles of Schweizer (2001b).

5.1 Hedging unit-linked insurance contracts

Risk-minimization

In Møller (1998) risk-minimizing hedging strategies were determined for a portfolio of unit-linked pure endowment contracts using the theory of risk-minimization due to Föllmer and Sondermann (1986). An introduction to the problem of pricing and hedging of unit-linked insurance contracts can also be found in Møller (2001a), where various approaches for hedging and pricing in incomplete markets are discussed in a discrete time model framework. This opens for a simple comparison of the techniques proposed by Brennan and Schwartz (1979a, 1979b) to the ones suggested by risk-minimization and super-replication, respectively. In contrast to the approaches of Brennan and Schwartz (1979a, 1979b) and Aase and Persson (1994), Møller (1998, 2001a) did not average away the mortality risk (the uncertainty associated with not knowing the number of survivors), but analysed the insurance contracts as contingent claims in an incomplete market. Consequently, the resulting strategies reflect, and react to, the financial risk as well as the insurance risk. In particular, it is clearly visible from these strategies how an insurer applying the risk-minimizing hedging strategy is currently adapting his portfolio of stocks and his deposit on the savings account to the actual development within the portfolio of insured lives.

As an example, consider unit-linked pure endowment contracts of the form (4.1) for n policy-holders aged y with i.i.d. remaining life times T_1, \dots, T_n , and assume that the amount payable upon survival to T , $f(S)$, is attainable in the sense that

$$f(S)e^{-\int_0^T r_u du} = H_0^f + \int_0^T \vartheta_u^f dX_u, \quad (5.1)$$

see Section 3 for more motivation. Thus, the discounted no-arbitrage price at time t of the claim $f(S)$ is given by

$$V_t^f = H_0^f + \int_0^t \vartheta_u^f dX_u.$$

Denote by $N_t = \sum_{i=1}^n 1_{\{T_i \leq t\}}$ the number of deaths up to time t , so that the current number of survivors at t is $(n - N_t)$. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ is defined as

$$\mathcal{F}_t = \sigma\{(N_u, S_u, B_u), u \leq t\},$$

where $B_t = \exp(\int_0^t r_u du)$. This has the usual interpretation: The insurance company is observing the process N as well as the price processes (S, B) . Under the natural assumption of independence between the remaining life times and the financial risk, the arguments used in Møller (1998) show the following:

Theorem 1 Assume that P is a martingale measure. The unique risk-minimizing strategy $\varphi^* = (\vartheta^*, \eta^*)$ for (4.1) is given by

$$\begin{aligned}\vartheta_t^* &= (n - N_{t-})_{T-t} p_{y+t} \vartheta_t^f, \\ \eta_t^* &= (n - N_t)_{T-t} p_{y+t} V_t^f - \vartheta_t^* X_t.\end{aligned}$$

This strategy prescribes holding a number of stocks at time t which is equal to the dynamic hedge ϑ_t^f for $f(S)$ multiplied by the current expected number of survivors just before time t , and the amount invested in the savings account η_t^* is chosen such that at any time t

$$V_t(\varphi^*) = (n - N_t)_{T-t} p_{y+t} V_t^f.$$

In particular, the value at time 0 is $V_0(\varphi^*) = n_{Tp_y} V_0^f$, which coincides with the price suggested by Brennan and Schwartz (1979a, 1979b): The expected number of survivors to T multiplied by the no-arbitrage price at time 0 of the amount $f(S)$ payable upon survival to T ; see also the discussion in Section 4.1 above. The strategy in the theorem is *not* self-financing and its cost process is given by

$$C_t(\varphi^*) = n_{Tp_y} V_0^f - \int_0^t V_u^f {}_{T-u} p_{y+u} dM_u,$$

where M is a martingale defined by

$$dM_t = dN_t - (n - N_{t-}) \mu_{y+t} dt,$$

and where μ_{y+t} is the mortality intensity (or the hazard rate), see Section 2.1. Using the results of Föllmer and Sondermann (1986), Møller (1998) also derived measures for the part of the total risk in the unit-linked contracts that cannot be hedged away by trading on financial markets only, the so-called *intrinsic risk*. It is given by

$$R_0(\varphi^*) = E[(C_T(\varphi^*) - C_0(\varphi^*))^2] = n_{Tp_y} \int_0^T E[(V_u^f)^2]_{T-u} p_{y+u} \mu_{y+u} du,$$

see Møller (1998, Theorem 4.2). Furthermore, it was shown that this intrinsic risk could actually be completely eliminated by including in addition a dynamic reinsurance market. More precisely, it was assumed that the insurer could trade continuously, in addition to the stock and the savings account, a third asset with a price process which was, at any time, equal to the prospective reserve associated with a pure endowment insurance with sum insured 1. In this way, the insurance risk was essentially transformed into a traded asset or a security. In the model considered there, this additional asset was indeed sufficient to restore completeness, leading to unique prices and self-financing investment strategies.

The theory of risk-minimization introduced by Föllmer and Sondermann (1986) focuses on the problem of hedging a contingent claim payable at a fixed time. However, insurance contracts often generate genuine payment streams where amounts are paid out over time. For example, with a so-called life annuity, payments are due yearly, say, from a certain time and as long as the policy-holder is still alive.

Similarly, life insurance contracts in general are often paid by periodic premiums, e.g. premiums paid at the beginning of each year as long as the policy-holder is still alive. In Møller (2001e), general payment streams are incorporated into the theory of risk-minimization, thus providing a framework which allows for the analysis of (insurance) payment processes. This modified framework is applied to the analysis of general unit-linked life insurance contracts, where the state of the policy is described by a Markov jump process with a finite state space. This generalizes previous results obtained in Møller (1998).

Mean-variance hedging

In the situation where P is not a martingale measure, that is, when the discounted stock price process $X = S/B$ is not a martingale under P , we can instead determine the mean-variance hedging strategy for H and the so-called approximation price, cf. Section 3. In the present situation, we have, under certain technical conditions on the process X , the following result, see Møller (2001b, Section 7):

Theorem 2 *The mean-variance hedging strategy $\tilde{\varphi} = (\tilde{\vartheta}, \tilde{\eta})$ for (4.1) is given by*

$$\begin{aligned}\tilde{\vartheta}_t &= (n - N_{t-})_{T-t} p_{y+t} \vartheta_t^f + \tilde{\zeta}_t \int_0^t V_u^f \tilde{Z}_u^{-1} {}_{T-u} p_{y+u} dM_u, \\ \tilde{\eta}_t &= V_0(\tilde{\varphi}) + \int_0^t \tilde{\vartheta}_u dX_u - \tilde{\vartheta}_t X_t,\end{aligned}$$

where the processes $\tilde{\zeta}$ and \tilde{Z} are determined by the variance optimal martingale measure. The approximation price is $V_0(\tilde{\varphi}) = n {}_{T} p_y V_0^f$.

The number of stocks $\tilde{\vartheta}_t$ held with the mean-variance hedging strategy consists of two terms. The first term is exactly the risk-minimizing strategy of Theorem 1. The second term is a correction term, which is driven by the martingale M introduced above, and which depends on the entire past development within the portfolio of insured lives. Moreover, this second term is related to the variance optimal martingale \tilde{P} , see the last paragraph of Section 4.3, via the terms $\tilde{\zeta}$ and $\tilde{Z}_t = \tilde{\mathbb{E}}[\frac{d\tilde{P}}{dP} | \mathcal{F}_t]$. For more details on the variance optimal martingale measure, see e.g. Schweizer (2001a). In addition, we see that the approximation price is equal to the price proposed by Brennan and Schwartz (1979a, 1979b).

Super-replication

For comparison we derive the super-replicating strategy for the unit-linked contract (4.1). With super-replication, the objective is essentially to determine the self-financing strategy among the ones whose terminal value dominate the claim H , i.e. $V_T(\varphi) \geq H$, which requires the smallest initial investment. As shown by El Karoui and Quenez (1995), this strategy is closely related to the process

$$\bar{V}_t = \operatorname{ess\,sup}_Q \mathbb{E}_Q[H | \mathcal{F}_t], \quad (5.2)$$

where the supremum is taken over all equivalent martingale measures. In the present situation we obtain:

Lemma 1 For the unit-linked pure endowment contract (4.1) the process (5.2) is given by $\bar{V}_t = (n - N_t) V_t^f$ and it admits the decomposition

$$\bar{V}_t = n V_0^f + \int_0^t (n - N_{u-}) \vartheta_u^f dX_u - \int_0^t V_u^f dN_u. \quad (5.3)$$

Idea of proof: (This result is similar to El Karoui and Quenez (1995, Example 3.4.2).) We first show that $\bar{V}_t = (n - N_t) V_t^f$. To see “ \leq ”, note that $N_T \geq N_t$ which implies that $(n - N_T) \leq (n - N_t)$. Thus, for any martingale measure Q

$$\begin{aligned} & \mathbb{E}_Q[(n - N_T) f(S) e^{-\int_0^T r_u du} | \mathcal{F}_t] \\ & \leq (n - N_t) \mathbb{E}_Q[f(S) e^{-\int_0^T r_u du} | \mathcal{F}_t] \\ & = (n - N_t) V_t^f. \end{aligned}$$

To see that $\bar{V}_t \geq (n - N_t) V_t^f$, consider for $h > -1$ the martingale measure $Q^{(h)}$ such that the remaining life times T_1, \dots, T_n are i.i.d. with mortality intensity $(1 + h)\mu_{y+t}$ and independent of (S, B) ; see Møller (1998, Section 2) for a construction of this measure. Thus by the independence between the two sources of risk

$$\mathbb{E}_{Q^{(h)}}[(n - N_T) f(S) e^{-\int_0^T r_u du} | \mathcal{F}_t] = (n - N_t) {}_{T-t}p_{y+t}^{(h)} V_t^f,$$

where the survival probability under $Q^{(h)}$ is given by

$${}_{T-t}p_{y+t}^{(h)} = Q^{(h)}(T_1 > T | T_1 > t) = \exp\left(- (1 + h) \int_t^T \mu_{y+u} du\right),$$

and where we have used that

$$\mathbb{E}_{Q^{(h)}}[f(S) e^{-\int_0^T r_u du} | \mathcal{F}_t] = V_t^f$$

for any $h > -1$. Since $\lim_{h \searrow -1} {}_{T-t}p_{y+t}^{(h)} = 1$, we obtain that $\bar{V}_t \geq (n - N_t) V_t^f$. The decomposition (5.3) finally follows by applying the product rule to $(n - N_t) V_t^f$. \square

Using the decomposition (5.3) with $t = T$, we see that

$$n V_0^f + \int_0^T (n - N_{u-}) \vartheta_u^f dX_u = H + \int_0^T V_u^f dN_u.$$

Here, the two terms on the left represent the value at T of a self-financing strategy with initial value $n V_0^f$ and with $(n - N_{u-}) \vartheta_u^f$ stocks held at time $0 \leq u \leq T$. This value exceeds H by the amount $\int_0^T V_u^f dN_u$, which is non-negative since $V_u^f \geq 0$ and since N is increasing. We can in fact currently withdraw the amount $\int_0^T V_u^f dN_u$ from the strategy and still ensure that the terminal value exceeds H . We summarize this result in the following:

Theorem 3 The super-replicating strategy $\hat{\varphi} = (\hat{\vartheta}, \hat{\eta})$ for (4.1) is determined by

$$\begin{aligned} \hat{\vartheta}_t &= (n - N_{t-}) \vartheta_t^f, \\ \hat{\eta}_t &= V_0(\hat{\varphi}) + \int_0^t \hat{\vartheta}_u dX_u - \hat{\vartheta}_t X_t - \int_0^t V_u^f dN_u, \end{aligned}$$

and $V_0(\hat{\varphi}) = n V_0^f$.

Thus the super-replicating strategy requires an initial investment at time 0 of the amount nV_0^f . Comparing with the results obtained in Theorem 1 and 2, we see that this corresponds to using a survival probability of 1! The number of stocks held at t is exactly the current number of survivors multiplied with the hedge ϑ_t^f for $f(S)$, which also differs from the risk-minimizing and mean-variance strategies in that no survival probability is involved. If a policy-holder dies during the infinitesimal interval $(t, t + dt]$, then $dN_t = 1$, which implies that the discounted deposit on the savings $\hat{\eta}_t$ is being reduced by the amount V_t^f , i.e. that the amount V_t^f can be withdrawn from the strategy.

5.2 On transformations of actuarial valuation principles

This section reviews some results on the financial variance and standard deviation principles of Schweizer (2001b) mentioned in Section 4.3. Instead of using the indifference principle applied there, we present two apparently ad-hoc ways of modifying the classical principles. These results are closely related to an alternative and more direct characterization of the financial standard deviation principle given in Møller (2001d), which does not involve an indifference argument. For this purpose, it suffices to consider a standard Black-Scholes market. There are two traded assets S and B with $S_0 = B_0 = 1$ and dynamics

$$\begin{aligned} dB_t &= rB_t dt, \\ dS_t &= \alpha S_t dt + \sigma S_t dW_t. \end{aligned}$$

These processes are defined on some probability space (Ω, \mathcal{F}, P) equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ where T is a fixed finite time horizon; W is a standard Brownian motion with respect to \mathbb{F} , and r, α and σ are known constants. Consider in addition some insurance (risk) process $U = (U_t)_{t \in [0, T]}$, which for example could be defined by $U_t = \sum_{i=1}^{N_t} Z_i$, where N is a homogeneous Poisson process with intensity λ and Z_1, Z_2, \dots is a sequence of i.i.d. random variables representing claim amounts. Alternatively, U_t could be the number of deaths up to time t within a portfolio of n insured lives. We assume for simplicity that U and W are independent under P (or, equivalently, that U and S are independent) and that $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^U$, i.e. the filtration is taken to be the (P -augmentation of the) natural filtration of (W, U) .

With the present construction, one can trade the two assets S, B via trading strategies that are adapted to the filtration generated by S and U . This means that investors can base investment strategies on observed prices as well as on the observed insurance claims. Note however that the process U is not related to any traded assets, so that this risk cannot be eliminated by trading on the market. In this setting, we will consider the problem of assigning premiums to insurance contracts that depend on both sources of risk. More precisely, this means that we consider contingent claims H payable at time T which are $\mathcal{F}_T^{U, W}$ -measurable. Non-trivial examples are a financial stop-loss contract or a unit-linked life insurance contract

described above.

Define an equivalent measure Q via

$$\frac{dQ}{dP} = L_T = \exp(-\nu W_T - \frac{1}{2}\nu^2 T),$$

where $\nu = \frac{\alpha-r}{\sigma}$. This measure has the following properties

- The discounted price process $X = S/B$ is a Q -martingale.
- U is not affected by the change of measure.
- S and U are independent under Q .

Thus, Q is an equivalent martingale measure, so that the model is free of arbitrage. To see that the last property is satisfied, it must be verified that for any $A \in \mathcal{F}_T^W$ and $B \in \mathcal{F}_T^U$ we have: $Q(A \cap B) = Q(A)Q(B)$. By using the definition of the measure Q and by exploiting the independence between the \mathcal{F}_T^W -measurable random variable L_T and \mathcal{F}_T^U we get

$$Q(A \cap B) = E_Q[1_{A \cap B}] = E[L_T 1_A 1_B] = E[L_T 1_A]E[1_B] = Q(A)Q(B).$$

(Here and throughout we use the notation “E” and “Var” for “ E_P ” and “ Var_P ”.) Finally, we point out that Q is just one possible martingale measure. Whereas the change of measure from P to Q does not affect the process U , one can also consider equivalent martingale measures which change the distribution of U , see e.g. Delbaen and Haezendonck (1989) for the situation where U is a compound Poisson process and Møller (1998) for the case where U counts the number of deaths within a portfolio of insured lives. Since this can be done without affecting the stock S , it follows that there are infinitely many martingale measures in the current model, i.e. the model is incomplete.

According to the classical actuarial standard deviation principle, the premium for a contract specifying a discounted payoff H at T is:

$$\tilde{u}(H) = E[H] + a(\text{Var}[H])^{1/2}, \quad (5.4)$$

cf. Section 2.2 (here we apply the principle on the discounted payoff, thus deviating slightly from tradition). Clearly it would not make sense to apply this principle directly to (say) a European call option with discounted payoff $H = e^{-rT}(S_T - K)^+$, since this contract is attainable and hence can be priced uniquely by no-arbitrage arguments alone. One idea is therefore to modify (or transform) the principle slightly, so as to get a principle that on the one hand still resembles the standard deviation principle and on the other hand is consistent with absence of arbitrage in the sense that the premium of an attainable claim equals the unique no-arbitrage price. We shall look at two simple ways of modifying the standard deviation principle directly.

Modified standard deviation principle 1

Consider the following modified premium principle

$$\pi_1(H) = E_Q[H] + a(\text{Var}[E_Q[H | \mathcal{F}_T^U]])^{1/2}. \quad (5.5)$$

It is not difficult to show that this principle has the properties:

1. For $\gamma \geq 0$, $\pi_1(\gamma H) = \gamma \pi_1(H)$, i.e. the principle allows for scaling of the claim. (This property is called *positive homogeneity* in the literature, cf. Goovaerts et al. (1984)).

2. For any $H \sim \mathcal{F}_T^W$,

$$\pi_1(H) = E_Q[H],$$

i.e. for any purely financial contract, the premium under (5.5) is equal to the unique no-arbitrage price.

3. For any $H \sim \mathcal{F}_T^{W,U}$,

$$\pi_1(H) = \tilde{u}(E_Q[H|\mathcal{F}_T^U]).$$

i.e. this modified standard deviation principle corresponds to applying the traditional standard deviation principle to the no-arbitrage price of H conditional on the insurance uncertainty \mathcal{F}_T^U .

Property 1 follows immediately from the definition (5.5). To see that Property 2 is satisfied, consider a claim H which only depends on the uncertainty from the financial market, so that H is \mathcal{F}_T^W -measurable. We can again exploit the independence between U and W under Q and well-known properties for conditional expected values to obtain that $E_Q[H|\mathcal{F}_T^U] = E_Q[H]$ for such H . Since the variance of a constant is 0, the loading term in (5.5) vanishes, and this shows that Property 2 is indeed satisfied. In particular, this ensures that the premium for a European call option on S coincides with its unique no-arbitrage price.

Finally, the last property follows by verifying that $E_Q[H] = EE_Q[H|\mathcal{F}_T^U]$, which in turn follows from the rule of iterated expectation under Q and the definition of Q :

$$\begin{aligned} E_Q[H] &= E_Q[E_Q[H|\mathcal{F}_T^U]] = E[L_T E_Q[H|\mathcal{F}_T^U]] \\ &= E[L_T] E[E_Q[H|\mathcal{F}_T^U]] = E[E_Q[H|\mathcal{F}_T^U]], \end{aligned}$$

where the third equality follows from the independence between L_T and U , and the last equality follows since $E[L_T] = 1$.

Modified standard deviation principle 2

As an alternative, consider the following modification of the classical standard-deviation principle:

$$\pi_2(H) = E_Q[H] + a (E[\text{Var}[H|\mathcal{F}_T^W]])^{1/2}. \quad (5.6)$$

Similarly to the principle π_1 , one can show that π_2 is positively homogeneous, i.e. it satisfies Property 1 above. Property 2 is also satisfied for π_2 , since for any $H \sim \mathcal{F}_T^W$ we have that $\text{Var}[H|\mathcal{F}_T^W] = 0$. The principle π_2 does not satisfy Property 3, but we can instead give the following intuitive characterization:

4. For any claim H , there exists a self-financing strategy φ with initial value 0 such that

$$\pi_2(H) = \tilde{u}(H - V_T(\varphi)),$$

where $V_T(\varphi)$ is the terminal value of the strategy φ .

Thus, π_2 simply amounts to applying the traditional standard deviation principle to the claim H reduced by the terminal value of a certain self-financing strategy which requires 0 initial investment.

To see that Property 4 is satisfied, consider the (artificial) claim

$$H' = E[H|\mathcal{F}_T^W] - E_Q[H].$$

Since by Møller (2001c, Proposition 6.12), $E[H|\mathcal{F}_T^W] = E_Q[H|\mathcal{F}_T^W]$, we find that $E_Q[H'] = 0$. Furthermore, since H' is \mathcal{F}_T^W -measurable, there exists a self-financing strategy φ which replicates H' , i.e. $V_T(\varphi) = H'$. Moreover it follows e.g. from the fact that $E_Q[H'] = 0$ and the results reviewed in Section 3 that this self-financing strategy requires no initial investment (initial capital 0). To see that Property 4 is satisfied, we only need to compute $E[H - V_T(\varphi)]$ and $\text{Var}[H - V_T(\varphi)]$ and check that these will correspond to the terms appearing in (5.6):

$$\begin{aligned} E[H - V_T(\varphi)] &= E[H - (E[H|\mathcal{F}_T^W] - E_Q[H])] = E_Q[H], \\ \text{Var}[H - V_T(\varphi)] &= \text{Var}[H - E[H|\mathcal{F}_T^W]] = E[\text{Var}[H|\mathcal{F}_T^W]], \end{aligned}$$

where the last equality follows by using standard rules for conditional variances. The idea of applying the original standard deviation principle to the claim reduced by the terminal value of a self-financing strategy with initial capital 0 is pursued further in Møller (2001d). More precisely, it is shown that one can give an equivalent definition of the financial standard deviation principle of Schweizer (2001b) by defining a premium principle via

$$\pi_3(H) = \inf_{\varphi} \tilde{u}(H - V_T(\varphi)),$$

where the infimum is taken over all self-financing strategies φ with initial value 0 that in addition satisfy some integrability conditions.

Indifference pricing under a change of filtration

In Møller (2001b, 2001c) the properties of the financial variance and standard deviation principles of Schweizer (2001b) are studied further. In particular, focus is on the dependence of the fair premiums on the amount of information available to the insurer, that is, on the choice of filtration. Via a comparison result for mean-variance hedging errors in different filtrations, a natural ordering of the fair premiums is obtained. More precisely, it is shown in Møller (2001c) that more actuarial information leads to lower premiums, and this difference is characterized further. The results allow for derivation of relatively simple upper and lower bounds for the fair premiums of reinsurance contracts under the assumption of independence between the traded assets and the insurance risk involved. An upper bound is obtained by allowing the hedger to adapt trading strategies to the information from the financial market only, and the lower bound corresponds to the artificial situation where the actuarial uncertainty \mathcal{F}_T^U is revealed immediately after the signing of the contract. These bounds are in fact closely related to the above mentioned ad-hoc modifications of the classical standard deviation principles; for comparison, we quote the result from Møller (2001c) here:

Theorem 4 Assume that $a^2 \geq \text{Var}[L_T]$. For the standard deviation principle, the upper bound for the fair premium is

$$\pi_{max}(H) = \mathbb{E}_Q[H] + a_1 \left(\mathbb{E}[\text{Var}[H | \mathcal{F}_T^W]] \right)^{1/2}, \quad (5.7)$$

and the lower bound is given by

$$\pi_{min}(H) = \mathbb{E}_Q[H] + a_2 \left(\text{Var}[\mathbb{E}_Q[H | \mathcal{F}_T^U]] \right)^{1/2}, \quad (5.8)$$

where

$$a_1 = a \sqrt{1 - \frac{\text{Var}[L_T]}{a^2}}, \quad \text{and} \quad a_2 = \frac{a_1}{\sqrt{\mathbb{E}[L_T^2]}}.$$

Note that in the present model, the upper and lower bounds in the theorem differ from the modified principles (5.5) and (5.6) only via the safety-loading parameters a_1 and a_2 . The fair premiums of the above theorem are only valid provided that the condition $a^2 \geq \text{Var}[L_T]$ is satisfied. Since $\text{Var}[L_T] = e^{\nu^2 T} - 1$, this means that the so-called *market price of risk* $\nu = \frac{\alpha - r}{\sigma}$ has to be small compared to the safety-loading parameter a . If $\alpha = r$ then $L_T = 1$, so that $P = Q$. In this special situation, $a_1 = a_2 = a$, so that the bounds are actually identical to the above mentioned modified principles. Some applications related to insurance of these results can be found in Møller (2001b), where fair premiums and optimal trading strategies are determined under various scenarios corresponding to different amounts of information, for example for unit-linked insurance contracts and financial stop-loss contracts.

Notes

¹i.e. *uncertain payoffs*. The notion of *risk* is used in several different contexts in both the actuarial and the financial literature; often it is simply used vaguely describing the fact that there is some uncertainty, for example in *mortality risk* known from insurance and *credit risk* known from finance. However, the notion also appears in various more specific concepts. Examples are *insurance risk process*, which is typically defined as the accumulated premiums minus claims in an insurance portfolio, and *risk-minimization*, which is a theory from mathematical finance that can be used for determining hedging strategies.

²Recall that two probability measures P and Q are said to be equivalent if they have the same null sets, i.e. if they assign probability 0 to the same events. This means that the probability of an event A is 0 under P if and only if the probability of A is 0 under Q , i.e. $\forall A \in \mathcal{F} : P(A) = 0 \Leftrightarrow Q(A) = 0$.

³See Turner (1971) for an overview of the early history of unit-linked life insurance products. For a treatment of some institutional aspects of unit-linked insurance contracts, see Squires (1986).

References

Aase, K. K. (1993). Premiums in a Dynamic Model of a Reinsurance Market, *Scandinavian Actuarial Journal*, 134–160.

- Aase, K. K. and Persson, S. A. (1994). Pricing of Unit-Linked Life Insurance Policies, *Scandinavian Actuarial Journal*, 26–52.
- Bachelier, L. (1900). Théorie de la Spéculation, *Annales Scientifiques de l'École Normale Supérieure* **17**, 21–86.
- Bacinello, A. R. and Ortu, F. (1993a). Pricing Equity-Linked Life Insurance with Endogenous Minimum Guarantees, *Insurance: Mathematics and Economics* **12**, 245–257.
- Bacinello, A. R. and Ortu, F. (1993b). Pricing Guaranteed Securities-Linked Life Insurance under Interest-Rate Risk, Proceedings of the 3rd AFIR Colloquium, Rome, Vol. I, 35–55.
- Bacinello, A. R. and Persson, S. A. (1998). Design and Pricing of Equity-Linked Life Insurance under Stochastic Interest Rates, Preprint, Institute of Finance and Management Science, The Norwegian School of Economics and Business Administration.
- Black, F. and Scholes, M. (1973). The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* **81**, 637–654.
- Bouleau, N. and Lamberton, D. (1989). Residual Risks and Hedging Strategies in Markovian Markets, *Stochastic Processes and their Applications* **33**, 131–150.
- Braun, H. (1937). *Urkunden und Materialien zur Geschichte der Lebensversicherung und der Lebensversicherungstechnik*, Veröffentlichungen des Deutschen Vereins für Versicherungs-Wissenschaft, Heft 58, Verlag von E.S. Mittler & Sohn, Berlin.
- Brennan, M. J. and Schwartz, E. S. (1979a). Alternative Investment Strategies for the Issuers of Equity-linked Life Insurance with an Asset Value Guarantee, *Journal of Business* **52**, 63–93.
- Brennan, M. J. and Schwartz, E. S. (1979b). *Pricing and Investment Strategies for Guaranteed Equity-Linked Life Insurance*, Monograph no. 7/The S. S. Huebner Foundation for Insurance Education, Wharton School, University of Pennsylvania, Philadelphia.
- Bühlmann, H. (1970). *Mathematical Methods in Risk Theory*, Springer-Verlag, Berlin.
- Bühlmann, H. (1980). An Economic Premium Principle, *ASTIN Bulletin* **11**, 52–60.
- Bühlmann, H. (1984). The General Economic Premium Principle, *ASTIN Bulletin* **14**, 13–21.
- Bühlmann, H., Delbaen, F., Embrechts, P. and Shiryaev, A. (1997). No-Arbitrage, Change of Measure and Conditional Esscher Transforms, *CWI Quarterly, Amsterdam* **9**(4), 291–317.
- Cox, J., Ross, S. and Rubinstein, M. (1979). Option Pricing: A Simplified Approach, *Journal of Financial Economics* **7**, 229–263.
- Cox, S. H. and Pedersen, H. W. (2000). Catastrophe Risk Bonds, *North American Actuarial Journal* **4**(4), 56–82.
- Cummins, J. D. and Geman, H. (1995). Pricing Catastrophe Insurance Futures and Call Spreads: An Arbitrage Approach, *Journal of Fixed Income* **4**, 46–57.

- Cvitanović, J. (2000). Minimizing Expected Loss of Hedging in Incomplete and Constrained Markets, *SIAM Journal on Control and Optimization* **38**, 1050–1066.
- Dana, R. A. (1999). Existence, Uniqueness and Determinacy of Equilibrium in C.A.P.M. with a Riskless Asset, *Journal of Mathematical Economics* **32**, 167–175.
- Davis, M. H. A. (1997). Option Pricing in Incomplete Markets, *in* M. A. H. Dempster and S. Pliska (eds), *Mathematics of Derivative Securities*, Cambridge University Press, 216–226.
- de Waegenaere, A. and Delbaen, F. (1992). A Dynamic Reinsurance Theory, *Insurance: Mathematics and Economics* **14**, 205–218.
- Delbaen, F. (1986). Equity Linked Policies, *Bulletin de l'Association Royale des Actuaire Belges*, 33–52.
- Delbaen, F. and Haezendonck, J. (1989). A Martingale Approach to Premium Calculation Principles in an Arbitrage Free Market, *Insurance: Mathematics and Economics* **8**, 269–277.
- Duffie, D. (1996). *Dynamic Asset Pricing Theory*, 2nd edn, Princeton University Press, Princeton, New Jersey.
- Duffie, D. and Richardson, H. R. (1991). Mean-Variance Hedging in Continuous Time, *Annals of Applied Probability* **1**, 1–15.
- El Karoui, N. and Quenez, M. C. (1995). Dynamic Programming and Pricing of Contingent Claims in an Incomplete Market, *SIAM Journal on Control and Optimization* **33**, 29–66.
- Embrechts, P. (1996). Actuarial versus Financial Pricing of Insurance, *Working paper no. 1996-17*, Wharton Financial Institutions Center.
- Föllmer, H. and Leukert, P. (1999). Quantile Hedging, *Finance and Stochastics* **3**, 251–273.
- Föllmer, H. and Leukert, P. (2000). Efficient Hedging: Cost Versus Shortfall Risk, *Finance and Stochastics* **4**, 117–146.
- Föllmer, H. and Sondermann, D. (1986). Hedging of Non-Redundant Contingent Claims, *in* W. Hildenbrand and A. Mas-Colell (eds), *Contributions to Mathematical Economics*, North-Holland, 205–223.
- Gerber, H. U. (1979). *An Introduction to Mathematical Risk Theory*, Monograph no. 8/The S. S. Huebner Foundation for Insurance Education, Wharton School, University of Pennsylvania, Philadelphia.
- Gerber, H. U. (1986). *Lebensversicherungsmathematik*, Springer-Verlag, Berlin.
- Gerber, H. U. and Pafumi, G. (1998). Utility Functions: From Risk Theory to Finance, *North American Actuarial Journal* **2**(3), 74–91.
- Gerber, H. U. and Shiu, E. S. W. (1996). Actuarial Bridges to Dynamic Hedging and Option Pricing, *Insurance: Mathematics and Economics* **18**, 183–218.

- Gisler, A. and Frost, P. (1999). An Addendum and a short Comment on the Paper: "Estimating the Value of the WinCAT Coupons of the Winterthur Insurance Convertible Bond: A Study of the Model Risk", *ASTIN Bulletin* **29**, 165–171.
- Goovaerts, M. J., De Vylder, F. and Haezendonck, J. (1984). *Insurance Premiums: Theory and Applications*, North-Holland.
- Harrison, J. M. and Kreps, D. M. (1979). Martingales and Arbitrage in Multiperiod Securities Markets, *Journal of Economic Theory* **20**, 381–408.
- Harrison, J. M. and Pliska, S. R. (1981). Martingales and Stochastic Integrals in the Theory of Continuous Trading, *Stochastic Processes and their Applications* **11**, 215–260.
- Heilmann, W. R. (1987). *Grundbegriffe der Risikotheorie*, Verlag Versicherungswirtschaft, Karlsruhe.
- Hull, J. C. (1997). *Options, Futures, and other Derivatives*, International Edition. Prentice-Hall.
- Kahn, P. M. (1971). Projections of Variable Life Insurance Operations, with discussion, *Transactions of the Society of Actuaries* **XXIII**, 335–366.
- Lamberton, D. and Lapeyre, B. (1996). *Introduction to Stochastic Calculus applied to Finance*, Chapman & Hall.
- Merton, R. C. (1973). Theory of Rational Option Pricing, *Bell Journal of Economics and Management Science* **4**, 141–183.
- Møller, T. (1998). Risk-Minimizing Hedging Strategies for Unit-Linked Life Insurance Contracts, *ASTIN Bulletin* **28**, 17–47.
- Møller, T. (2001a). Hedging Equity-Linked Life Insurance Contracts, *North American Actuarial Journal* **5**(2), 79–95.
- Møller, T. (2001b). Indifference Pricing of Insurance Contracts: Applications, *Working Paper no. 171*, Laboratory of Actuarial Mathematics, University of Copenhagen.
- Møller, T. (2001c). Indifference Pricing of Insurance Contracts: Theory, *Working Paper no. 170*, Laboratory of Actuarial Mathematics, University of Copenhagen.
- Møller, T. (2001d). On Transformations of Actuarial Valuation Principles, *Insurance: Mathematics and Economics* **28**, 281–303.
- Møller, T. (2001e). Risk-minimizing Hedging Strategies for Insurance Payment Processes, to appear in *Finance and Stochastics*.
- Nielsen, J. and Sandmann, K. (1995). Equity-Linked Life Insurance: A Model with Stochastic Interest Rates, *Insurance: Mathematics and Economics* **16**, 225–253.
- Norberg, R. (1999). A Theory of Bonus in Life Insurance, *Finance and Stochastics* **3**, 373–390.
- Norberg, R. (2000). *Basic Life Insurance Mathematics*, Lecture notes, Laboratory of Actuarial Mathematics, University of Copenhagen.

- Persson, S. A. (1998). Stochastic Interest in Life Insurance: The Principle of Equivalence Revisited, *Scandinavian Actuarial Journal*, 97–112.
- Ramlau-Hansen, H. (1991). Distribution of Surplus in Life Insurance, *ASTIN Bulletin* **21**, 57–71.
- Samuelson, P. A. (1965). Rational Theory of Warrant Pricing, *Industrial Management Review* **6**, 13–31.
- Schmock, U. (1999). Estimating the Value of the WinCAT Coupons of the Winterthur Insurance Convertible Bond: A Study of the Model Risk, *ASTIN Bulletin* **29**, 101–163.
- Schweizer, M. (1988). Hedging of Options in a General Semimartingale Model, Diss. ETHZ no. 8615, Zürich.
- Schweizer, M. (1991). Option Hedging for Semimartingales, *Stochastic Processes and their Applications* **37**, 339–363.
- Schweizer, M. (2001a). A Guided Tour through Quadratic Hedging Approaches, in E. Jouini, J. Cvitanić and M. Musiela (eds), *Option Pricing, Interest Rates and Risk Management*, Cambridge University Press, 538–574.
- Schweizer, M. (2001b). From Actuarial to Financial Valuation Principles, *Insurance: Mathematics and Economics* **28**, 31–47.
- Sondermann, D. (1991). Reinsurance in Arbitrage-free Markets, *Insurance: Mathematics and Economics* **10**, 191–202.
- Squires, R. J. (1986). *Unit Linked Business*, Life Assurance Monograph, Institute of Actuaries and Faculty of Actuaries.
- Steffensen, M. (2000). A No Arbitrage Approach to Thiele’s Differential Equation, *Insurance: Mathematics and Economics* **27**, 214–214.
- Steffensen, M. (2001). On Valuation and Control in Life and Pension Insurance, Ph.D. thesis, University of Copenhagen.
- Swiss Re (1998). Integrated Risk Management Solutions — Beyond Traditional Reinsurance and Financial Hedging, Internal Paper, Swiss Re New Markets, Zürich.
- Tilley, J. A. (1997). The Securitization of Catastrophic Property Risks, Proceedings of the 7th AFIR Colloquium and the 28th ASTIN Colloquium, Cairns, Joint Day Vol., 27–53.
- Turner, S. H. (1969). Asset Value Guarantees under Equity-Based Products, with discussion, *Transactions of the Society of Actuaries* **XXI**, 459–493.
- Turner, S. H. (1971). Equity-Based Life Insurance in the United Kingdom, with discussion, *Transactions of the Society of Actuaries* **XXIII**, 273–324.
- Wilkie, A. D. (1978). Maturity (and other) Guarantees under Unit Linked Policies, *Transactions of the Faculty of Actuaries* **36**, 27–41.