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Normalized Exponential Tilting: Pricing and Measuring Multivariate Risks

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Agenda

- The Univariate Case
  - Exponential Tilting
  - Normalized Exponential Tilting
  - Probability Distortion
- How to Extend to the Multivariate Case?
- Portfolio Risk Measures (no time to get into detail)

Why is it an important problem?

- Multivariate risk modelling & Portfolio Risk Measure directly impact corporate decisions
  - Copulas & Correlation Models
  - Combining market risk, credit risk, insurance risk, and operational risk
  - Capital allocation
- Let us turn to a new page of risk theory:

  multivariate risk adjustment

Exponential Tilting of X w.r.t. Y

$$f_X^*(x) = f_X(x) \cdot \frac{E\left[\exp(\lambda Y) \mid X = x\right]}{E\left[\exp(\lambda Y)\right]}$$

X: Bernoulli(1/5)

Y: Normal(0, 1),

How to do exponential tilting?

Case 1: X and Y independent

Case 2: X and Y are co-monotone

Exponential Tilting

Buhlmann Economic Model (AB 1980)

When X=Y, we get Esscher transform

λ controls the magnitude of risk-adjustment.

A big problem? No consistent interpretation of λ

Normalized Exponential Tilting

Let Z be normal(0,1) such that $Y = \Phi^{-1}(Z)$, where Φ is the CDF of normal(0,1).

Before: Exponential tilting of X w.r.t Y

Now: Transform Y into normal(0,1) variable Z

After: exponential tilting of X w.r.t. Z.

$$f_X^*(x) = f_X(x) \cdot \frac{E\left[\exp(\lambda Y) \mid X = x\right]}{E\left[\exp(\lambda Y)\right]}$$

$$Y --> Z$$

$$f_X^*(x) = f_X(x) \cdot \frac{E\left[\exp(\lambda Z) \mid X = x\right]}{E\left[\exp(\lambda Z)\right]}$$

We got consistent interpretation of λ:

Sharpe Ratio,

Market price of risk

Normalized Exponential Tilting
➔ Probability Distortion

Theorem 1: If the correlation between *X* and *Y* follows a normal copula, the *normalized* exponential tilting is equivalent to applying the Wang transform

$$F_X^*(x) = \Phi\left[\Phi^{-1}(F_X(x)) - \beta\right]$$

$$\beta = \rho_{X,Y} \cdot \lambda$$

Univariate Distortion: Radon-Nikodym Derivative

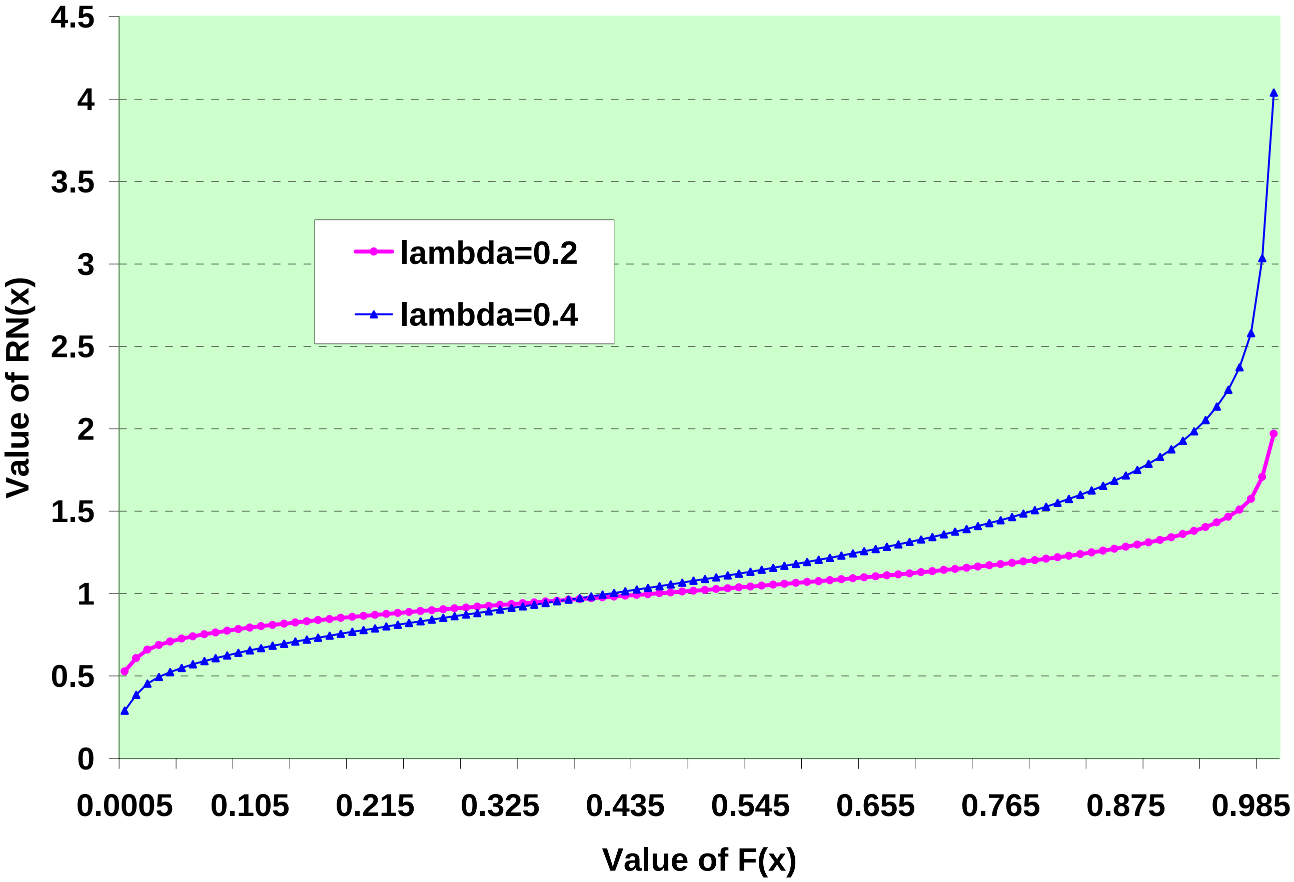
$$F_X^*(x) = g(F_X(x))$$

The Radon-Nikodym Derivative:

$$RN_g(x) = f^*(x) / f(x) = g'\left(F_X(x)\right)$$

Univariate Normalized Exponential Tilting ➔ Wang transform

Radon-Nikodym derivatives



Multivariate Exponential Tilting

The exponential tilting of $\{X_1, X_2, ..., X_n\}$ with respect to references $\{Y_1, Y_2, ..., Y_k\}$ is defined by

$$f^*(x_1,\cdots,x_n) = c\cdot f(x_1,\cdots,x_n)\cdot\prod_{j=1}^{k} E\left[\exp(\lambda_j Y_j)\ \middle|\ X_1 = x_1,\cdots,X_n = x_n\right]$$

The references $\{Y_1,..., Y_k\}$ can be

- ✓ business unit, company, and industry aggregates,
- ✓ the underlying risks for contingent claims $\{X_1 ..., X_n\}$,
- ✓ the risks $\{X_1,..., X_n\}$ themselves

Multivariate Normalized Exponential Tilting

To get consistent interpretation of $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, we normalize $\{Y_1, Y_2, \ldots, Y_k\}$ to $\{Z_1, Z_2, \ldots, Z_k\}$.

$$Z_j = \Phi^{-1}\left(F_{Y_j}(Y_j)\right),$$

$$f^*(x_1, \cdots, x_n) = c \cdot f(x_1, \cdots, x_n) \cdot \prod_{j=1}^{k} E\left[\exp(\lambda_j Z_j) \;\middle|\; X_1 = x_1, \cdots, X_n = x_n\right]$$

Example 1. The effect of Correlation!

Assume that the risks $\{X_1, X_2\}$ have a bivariate normal(0,1) with correlation coefficients:

$$\Sigma = \begin{pmatrix} 1 & \rho_{X_1,X_2} \\ \rho_{X_1,X_2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix}.$$

The bivariate normalized exponential tilting with references $Y_1 = X_1$ and $Y_2 = X_2$, $\lambda_1 = 0.3$ and $\lambda_2 = 0.2$, Is equivalent to applying *separate* Wang transforms $F^*_{X_j}(x) = \Phi\left[\Phi^{-1}(F_{X_j}(x)) - \beta_j\right]$, for j =1, 2, with

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 & \rho_{X_1,X_2} \\ \rho_{X_1,X_2} & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \rho_{X_1,X_2}\lambda_2 \\ \rho_{X_1,X_2}\lambda_1 + \lambda_2 \end{pmatrix} = \begin{pmatrix} 0.42 \\ 0.38 \end{pmatrix}$$

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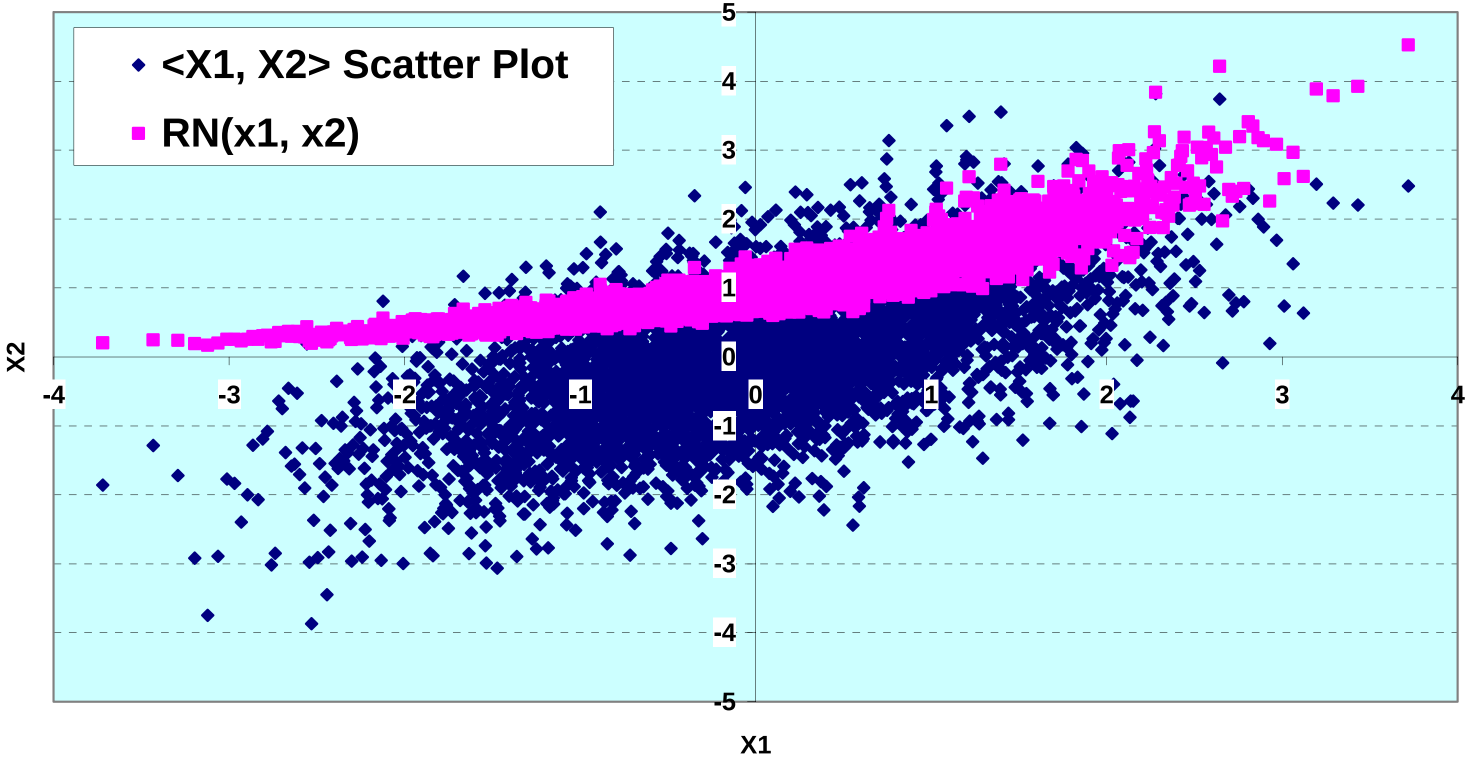
Bivariate Normalized Exponential Tilting: Radon-Nikodym derivatives



Define Joint Distortion

The Radon-Nikodym Derivative:

$$f^*_{X_1,\cdots,X_n}(x_1,\cdots,x_n) = RN_{g_1,\cdots,g_n}(x_1,\cdots,x_n)\cdot f_{X_1,\cdots,X_n}(x_1,\cdots,x_n)$$

$$RN_{g_1,\cdots,g_n}(x_1,\cdots,x_n) = c\cdot\prod_{j=1}^{n} g_j'\left(F_{X_j}(x_j)\right)$$

Main Theorem:

Multivariate Normalized Exponential Tilting of {X1, …, Xn} w.r.t. themselves

➔ Joint Distortions by Wang transforms

Example 2: Bivariate Normalized Exponential Tilting

- Given the following bivariate p.f., how to perform normalized exponential tilting with λ_1=0.3 and λ_2=0.2?

	X_2=1	X_2=2	X_2=3	X_2=4	X_2=5
X_1=1	*0.20*	*0.07*	*0.06*	*0.05*	*0.04*
X_1=2	*0.06*	*0.05*	*0.04*	*0.03*	*0.03*
X_1=3	*0.05*	*0.04*	*0.03*	*0.03*	*0.02*
X_1=4	*0.03*	*0.03*	*0.02*	*0.02*	*0.01*
X_1=5	*0.03*	*0.02*	*0.01*	*0.02*	*0.01*

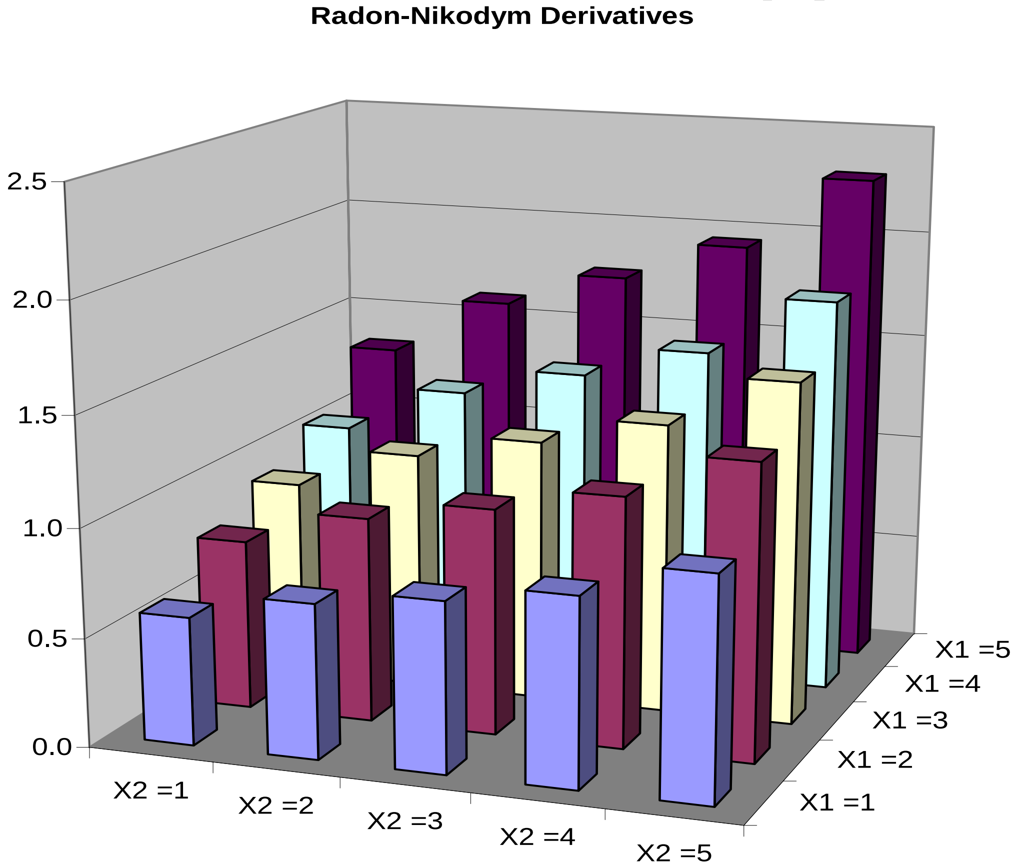
- First apply the Wang transform to X_1 with λ_1=0.3.
- Then apply the Wang transform to X_2 with λ_2=0.2

$X_1 = x_1$	$f(x_1)$	$F(x_1)$	$F^*(x_1)$	$f^*(x_1)$
1	0.42	0.42	0.30787	0.30787
2	0.21	0.63	0.51271	0.20483
3	0.17	0.80	0.70596	0.19325
4	0.11	0.91	0.85101	0.14505
5	0.09	1.00	1.00000	0.14899

$X_2 = x_2$	$f(x_2)$	$F(x_2)$	$F^*(x_2)$	$f^*(x_2)$
1	0.37	0.37	0.29741	0.29741
2	0.21	0.58	0.50076	0.20334
3	0.16	0.74	0.67124	0.17049
4	0.15	0.89	0.84768	0.17644
5	0.11	1.00	1.00000	0.15232

The bivariate Radon-Nikodym derivatives

$$RN_g(x_1,x_2)=\frac{f^*_{X_1,X_2}(x_1,x_2)}{f_{X_1,X_2}(x_1,x_2)}=c\cdot\frac{f^*_{X_1}(x_1)}{f_{X_2}(x_1)}\cdot\frac{f^*_{X_2}(x_2)}{f_{X_2}(x_2)}$$



Scenario	Loss	LAE
1	-	1,954.08
2	-	2,239.22
3	-	2,974.21
4	-	3,275.38
5	-	3,351.93
6	-	6,526.96
7	-	9,542.63
8	-	13,999.95
9	-	14,279.63
10	-	14,519.32
11	-	16,179.92
12	-	19,134.14
13	-	35,071.98
14	-	57,591.43
15	-	62,967.38
16	-	82,638.17
17	-	248,909.05
18	638.80	3,331.31
19	1,533.11	2,047.14
20	5,110.36	1,159.07
21	6,387.95	2,152.74
22	6,387.95	8,940.58
23	8,943.13	4,949.35
24	11,498.32	-
25	15,331.09	-
26	27,279.12	-
27	35,772.54	5,634.79
28	93,264.12	24,115.73
29	102,207.25	6,287.06
30	191,638.60	34,096.74
31	246,010.43	232,641.59
32	511,036.26	39,161.24
33	511,036.26	150,301.30
34	662,650.50	73,140.35

Example 3: Pricing of Bivariate Contingent Claims

Contract #1 has a payoff X_1 = *loss* in excess of 200,000

Contract #2 has a payoff X_2 = 50% * *LAE*

Contract #3:

$$X_3 = X_1 + X_2$$

Additive Pricing is Achieved

	X_1	X_2	X_1+X_2
No Risk Adjustment	$ 33,257	$ 17,399	$ 50,656
With Risk Adjustment	$ 68,240	$ 24,847	$ 93,087
Loading	105%	43%	84%

A Sequel Paper:

The Radon-Nikodym Derivative for a measure change from Bivariate

Standard Normal to Bivariate Student-t with d.f.=9.