On a correlated aggregate claims model with thinning-dependence structure

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Outline

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2. Some properties of the proposed model
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The Risk Model

Suppose that an insurance company has n $(n \geq 2)$ dependent classes of business.

Stochastic sources (for example, fires, floods, traffic accidents, earthquakes) that may cause a claim in at least one of the n classes are classified into m groups.

We assume that: each event occurred at time t in the kth group may cause a claim in the jth class with probability $p_{kj}(t)$ for $k = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

$N^k(t)$ — the number of events of the kth group occurred up to time t.

$N_j^k(t)$ — the number of claims of the jth class up to time t generated from the events in group k,

$X_i^{(j)}$ — the amount of the ith claim in the jth class.

Then, the total amount, $S_j(t)$, of claims of the jth class up to time t can be written as

$$S_j(t) = \sum_{i=1}^{N_j(t)} X_i^{(j)} \ , \tag{1}$$

where

$$N_j(t) = N_j^1(t) + N_j^2(t) + \cdots + N_j^m(t) \qquad (2)$$

is the claim-number process of the jth class.

Therefore, the aggregate claims process of the company is given by

$$S(t) = \sum_{j=1}^{n} S_j(t) = \sum_{j=1}^{n} \sum_{i=1}^{N_j(t)} X_i^{(j)} \ , \qquad (3)$$

where $\{X_i^{(j)};\ i = 1, 2, \cdots\}$ is assumed to be a sequence of i.i.d. non-negative random variables with common distribution F_j for each j.

As usual, we assume that the n sequences

$\{X_i^{(1)};\ i = 1, 2, \cdots\}, \ldots, \{X_i^{(n)}; \imath = 1, 2, \cdots\}$

are mutually independent and are independent of all claim-number processes.

Write u as the initial capital and c as the constant rate of premium. Then, the surplus process of the insurance company is defined as

$$R(t) = u + ct - S(t) = u + ct - \sum_{j=1}^{n} \sum_{i=1}^{N_j(t)} X_i^{(j)} \ . \qquad (4)$$

In order to make the analysis of $R(t)$ mathematically tractable, we need the following assumptions:

(A1) The processes $N^1(t), \ldots, N^m(t)$ are independent Poisson processes with parameters $\lambda_1, \ldots, \lambda_m$,respectively. For $k \neq k'$, the two vectors of claim-number processes,
$(N^k(t), N_1^k(t), \ldots, N_n^k(t))$ and
$(N^{k'}(t), N_1^{k'}(t), \ldots, N_n^{k'}(t))$, are independent.

(A2) For each k $(k = 1, \ldots, m)$,$N_1^k(t), \ldots, N_n^k(t)$ are conditionally independent given$N^k(t)$.

Note that Assumptions (A1) and (A2) hold true if $N^1(t), \ldots, N^m(t)$ are independent Poisson processes and, for each pair of (k, j) $(k = 1, \ldots, m,\ j = 1, \ldots, n)$, whether the process of any event in $N^k(t)$ giving rise to a claim in the jth class or not is independent of all other events.

Assume that $p_{kj}(0^+) = \lim_{s \downarrow 0} p_{kj}(s)$ exists for all k, j.

Define $q_{kj}(t) = t^{-1} \int_0^t p_{kj}(s) ds$ with the convention that $q_{kj}(0) = p_{kj}(0^+)$.

Then, under (A1) and (A2),

$N_j^k(t)$ **is a non-homogeneous Poisson process with** $E[N_j^k(t)] = \lambda_k q_{kj}(t) t$ **for** $k = 1, \ldots, m$ **and** $j = 1, \ldots, n$**.**

Some properties of the proposed model

$M_j(r) = E[\exp\{rX_1^{(j)}\}]$ — the moment generating function of distribution $F_j, j = 1, \cdots, n$.

Let $t > 0$. Assume that the the moment generating function of $S(t)$ exists. Then

$$\begin{aligned} & E[\exp\{rS(t)\}] \\ = \quad & \exp\left\{ t \sum_{k=1}^{m} \lambda_k \left(\prod_{j=1}^{n} \Big(q_{kj}(t) M_j(r) + 1 - q_{kj}(t) \Big) - 1 \right) \right\} \\ \triangleq \quad & \exp\{tg(r,t)\} \ . \end{aligned} \tag{5}$$

If all $p_{kj}(t)$ are all constants $p_{kj}(t) \equiv p_{kj}$, then

$q_{kj}(t) = p_{kj}$ and $g(r, t) = g(r)$ does not depend on t.

We now present some moments for the claim numbers and the aggregate claims:

$$\mathrm{Var}\left(N_j^k(t)\right) = \lambda_k q_{kj}(t)t$$

for $j = 1, \cdots, n$, $k = 1, \cdots, m$, and a fixed $t > 0$;

$$\mathrm{Cov}\left(N_i^k(t), N_j^k(t)\right) = \lambda_k q_{ki}(t) q_{kj}(t)t;$$

$$\mathrm{Cov}(N_i(t), N_j(t)) = \sum_{k=1}^{m} \lambda_k q_{ki}(t) q_{kj}(t)t$$

for $1 \leq i \neq j \leq n$ and $k = 1, \cdots, m$.

Denote by μ_j and σ_j^2 the mean and the variance of the distribution F_j, respectively. Then

$$\operatorname{Cov}\left(S_i(t), S_j(t)\right) = \mu_i \mu_j \sum_{k=1}^{m} \lambda_k q_{ki}(t) q_{kj}(t) t \ ,$$

for $1 \leq i \neq j \leq n$ and $k = 1, \cdots, m$. Since $S_j(t)$ is a compound Poisson random variable with parameter $\sum_{l=1}^{m} \lambda_l q_{lj}(t) t$, it is easy to see that

$$\operatorname{Var}\left(S_j(t)\right) = \left(\mu_j^2 + \sigma_j^2\right) \sum_{k=1}^{m} \lambda_k q_{kj}(t) t \ ,$$

for $j = 1, \cdots, n$.

Example Assume that $n = 2$, $m = 3$, $p_{12} = p_{21} = 0$, $p_{31} = p_{32} = 1$,$p_{11} = p_{22} = 1$, then $R(t)$ of (4) becomes

$$R(t) = u + ct - \sum_{k=1}^{N^1(t)+N^3(t)} X_k^{(1)} - \sum_{k=1}^{N^2(t)+N^3(t)} X_k^{(2)}. \qquad (6)$$

The risk process (6) is the well known risk model with the so-called **common shock** with two classes of business, see, for example, Cossette and Marceau (2000). In fact, any more general risk model with common shock for $n > 2$ classes of business can be obtained from (4) by specializing its parameters.

Dependence and Lundberg exponent

Let $h_j(r) = M_j(r) - 1$ for $j = 1, \ldots, n$. Assume that there exists $r^j_\infty > 0$ for $j = 1, \ldots, n$ such that $\lim_{r \to r^j_\infty} h_j(r) \uparrow \infty$. We allow for the possibility that $r^j_\infty = \infty$. Moreover, we write $\min\{r^1_\infty, \ldots, r^n_\infty\}$ as r_∞. The ruin probability of risk process (4) is defined as $\Psi_D(u) = P(\inf_{t \geq 0} R(t) < 0)$. As usual, we assume that $E[R(t)] > 0$ for all $t > 0$ so that $\Psi_D(u) < 1$ for $u \geq 0$.

For simplicity, we assume from now on that all $p_{kj}(s)$ are constants, that is, $p_{kj}(s) \equiv p_{kj}$ for $j = 1, \cdots, n$ and $k = 1, \cdots, m$.

We first define the risk model with n independent classes of business.

Let $\overline{N_1(t)}, \ldots, \overline{N_n(t)}$ represent the independent versions of $N_1(t), \ldots, N_n(t)$. That is,

(i) $\overline{N_1(t)}, \ldots, \overline{N_n(t)}$ are mutually independent;

(ii) $\overline{N_i(t)}$ and $N_i(t)$ are identically distributed for each i.

Define

$$\overline{S_i(t)} = \sum_{k=1}^{\overline{N_i(t)}} X_k^{(i)} \tag{7}$$

$$S_I(t) = \sum_{i=1}^{n} \overline{S_i(t)} \tag{8}$$

and

$$R_I(t) = u + ct - S_I(t) \ . \tag{9}$$

So, $R_I(t)$ represents the surplus of an insurer with n independent lines of business at time t.

Furthermore, $E[S(t)] = E[S_I(t)]$ for all $t \geq 0$, that is, $R(t)$ of (4) and $R_I(t)$ of (9) have the same expected aggregate loss.

The ruin probability of risk model (9) is defined as
$\Psi_I(u) = P(\inf_{t\geq 0} R_I(t) < 0)$.

Define

$$g_I(r) = \sum_{k=1}^{m} \lambda_k \left(\sum_{j=1}^{n} p_{kj} h_j(r) \right) . \tag{10}$$

Then, we have

$$E[\exp\{rS_I(t)\}] = \exp\{tg_I(r)\}.$$

Note that the two equations, $cr = g(r)$ and $cr = g_I(r)$, have unique positive roots, R^D and R^I, respectively.

We call these roots the Lundberg exponents.

Using R^D and R^I, we have $\Psi_I(u) \le \exp\{-R^I u\}$ and $\Psi_D(u) \le \exp\{-R^D u\}$.

Proposition 4.1. *For $u \ge 0$, we have*

$$R^D \le R^I .$$

If there exist i and j with $i \ne j$ such that $p_{ki}p_{kj} > 0$ for some k, then

$$R^D < R^I \ .$$

Classification of the stochastic sources

The classification of the stochastic sources influences the degree of dependence among the claim-number processes.

An unrealistic classification of the stochastic sources may lead one to overestimate or to underestimate the underlying risks.

We now consider the impact of misclassification of the stochastic sources on the ruin probability.

Assume that the correct classification should involve m groups of stochastic sources.

Suppose that the company misclassified the sources into l groups with $l < m$.

We now investigate how such a misgrouping influences on the Lundberg exponent.

For notational convenience, we rewrite $R(t)$, $S(t)$, $g(r)$, $\Psi_D(u)$, $S_i(t)$ and R^D defined in the previous sections as $R^{(m)}(t)$, $S^{(m)}(t)$, $g^{(m)}(r)$, $\Psi_D^{(m)}(u)$, $S_i^{(m)}(t)$ and $R_D^{(m)}$, respectively, to indicate that the full model consists of m groups of stochastic sources.

We now consider a special simplified model in which the last $m-l+1$ groups are treated as a single group. Thus, the number of groups is reduced from m to l.

Write the surplus process of the simplified model as

$$R^{(l)}(t) = u + ct - \sum_{i=1}^{n} \sum_{k=1}^{N_i^{(l)}(t)} X_k^{(i)} \triangleq u + ct - S^{(l)}(t) \ , \quad (11)$$

where $N_i^{(l)}(t) = N_i^1(t) + \cdots + N_i^{l-1}(t) + N_i^{[l]}(t)$ and

$N_i^{[l]}(t)$ is the p_i'-thinning of

$N^{[l]}(t) = N^l(t) + \cdots + N^m(t)$

for $i = 1, \ldots, n$.

Apparently, $N^{[l]}(t)$ is a Poisson process with parameter

$\lambda^{[l]} = \lambda_l + \cdots + \lambda_m$.

Similar to $g^{(m)}(r)$, $\Psi_D^{(m)}(u)$, $S_i^{(m)}(t)$ and $R_D^{(m)}$, we define $g^{(l)}(r)$, $\Psi_D^{(l)}(u)$, $S_i^{(l)}(t)$ and $R_D^{(l)}$ for the simplified model . To make the comparison fair, we let

$p'_i = \sum_{k=l}^{m} \lambda_k p_{ki} / \lambda^{[l]}$ for $i = 1, \ldots, n$.

Thus, $N_i^{(l)}(t)$ $\left(S_i^{(l)}(t)\right)$ and $N_i^{(m)}(t)$ $\left(S_i^{(m)}(t)\right)$ are identically distributed for $i = 1, \ldots, n$, and hence $R^{(l)}(t)$ and $R^{(m)}(t)$ have the same expected aggregate loss. It is easy to show that

$$E\Big[\exp\big\{rS^{(l)}(t)\big\}\Big] = \exp\{tg^{(l)}(r)\}$$

$$= \quad \exp\left\{t\left(\sum_{k=1}^{l-1}\lambda_k\left(\prod_{i=1}^{n}\big(p_{ki}h_i(r)+1\big)-1\right)\right.\right.$$

$$\left.\left. + \ \lambda^{[l]}\left(\prod_{i=1}^{n}\big(p_i'h_i(r)+1\big)-1\right)\right)\right\} . \qquad (12)$$

From (5) and (12), we get

$$\Delta(r) \triangleq g^{(m)}(r) - g^{(l)}(r)$$

$$= \quad \sum_{k=l}^{m}\lambda_k\prod_{j=1}^{n}\big(p_{kj}h_j(r)+1\big) - \lambda^{[l]}\prod_{j=1}^{n}\big(p_j'h_j(r)+1\big) \ . \qquad (13)$$

Using (12) and (13) we obtain

Proposition 5.1. *For* $0 < r < r_\infty$, $\Delta(r) < (=, >)\ 0$ *implies that* $R_D^{(l)} < (=, >)\ R_D^{(m)}$.

From (13), we have the following lemma in which sufficient conditions for $\Delta(r) = 0$ are given.

Lemma 5.1. *Let* $i_1, \cdots, i_{n-1}$ *be* $n-1$ *different numbers in* $\{1, \cdots, n\}$. *Assume that* $p_{ki_q} = p_{i_q}$ *for* $l \leq k \leq m$ *and* $1 \leq q \leq n-1$. *Then,* $\Delta(r) \equiv 0$ *for* $0 < r < r_\infty$.

Numerical example

We now present a numerical example with $m = n = 2$. In the example, we set $l = 1$ for the simplified model. Note that the surplus process of the simplified model is $R^{(1)}(t)$ and that $R^{(2)}(t) = R(t)$ is the full surplus process.

Here, the claim-amount random variables $X_i^{(1)}$ and $X_i^{(2)}$ are exponentially distributed with means $\mu_1 = 4$ and $\mu_2 = 2$, respectively.

Also, we set $\lambda_1 = 4$, $\lambda_2 = 7$, $p_{11} = 0.8$, $p_{12} = 0.3$, $p_{21} = 0.2$ and $p_{22} = 0.6$.

Therefore, $\lambda^{[1]} = \lambda_1 + \lambda_2 = 11$,

$p'_1 = (\lambda_1 p_{11} + \lambda_2 p_{21})/\lambda^{[1]} = 4.6/11$,

$p'_2 = (\lambda_1 p_{12} + \lambda_2 p_{22})/\lambda^{[1]} = 5.4/11$.

Define

$$\theta = \frac{c - \sum_{j=1}^{2} \mu_j \sum_{k=1}^{2} \lambda_k p_{kj}}{\sum_{j=1}^{2} \mu_j \sum_{k=1}^{2} \lambda_k p_{kj}} ,$$

which is known as the relative security loading(see Gerber (1979, pp. 111)). Let $\theta = 0.1$ so that $c = 32.12$.

Since $R_I(t)$,$R^{(1)}(t)$ and $R^{(2)}(t)$ are all compound Poisson risk models with claim-amount distributions being mixtures of exponential distributions, one can follow Gerber (1979) to obtain explicit expressions for ruin probabilities $\Psi_I(u)$,$\Psi^{(1)}(u)$ and $\Psi^{(2)}(u)$.

The numerical values of $\Psi_I(u)$,$\Psi^{(1)}(u)$ and $\Psi^{(2)}(u)$ for various values of u are shown in Table 1. From the table 1, we see that $\Psi^{(1)}(u) > \Psi^{(2)}(u) > \Psi_I(u)$ for all u and that the impact of the dependence among the claim-number processes on the ruin probability is prominent.

Table 1. Numerical illustration of ruin probabilities

u	$\Psi_I(u)$	$\Psi^{(2)}(u)$	$\Psi^{(1)}(u)$
10	0.688368	0.711868	0.781291
30	0.393031	0.439163	0.489395
50	0.266092	0.270937	0.306469
100	0.056745	0.080998	0.095105
150	0.014242	0.024215	0.029513
200	0.003524	0.007239	0.009159
300	0.000225	0.000647	0.000882
400	0.000014	0.000057	0.000085

Table 2. Comparisons of ruin probabilities

u	$\frac{\Psi^{(1)}(u)}{\Psi_I(u)}$	$\frac{\Psi^{(2)}(u)}{\Psi_I(u)}$	$\frac{\Psi^{(1)}(u)}{\Psi^{(2)}(u)}$
10	1.143	1.042	1.098
30	1.245	1.117	1.114
50	1.356	1.198	1.131
100	1.676	1.427	1.174
150	2.072	1.700	1.219
200	2.562	2.025	1.265
300	3.917	2.873	1.363
400	5.988	4.077	1.469

The Table 2 indicates that the ratios get larger as u increases. In this example, $R^{(2)}(t)$ is the surplus process of the full model. It is clear that the use of $R^{(1)}(t)$ overestimates the underlying risk while the use of $R_I(t)$ underestimates the underlying risk. Hence, correct classification of the stochastic sources is an important issue in model (4).

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