On a correlated aggregate claims model with thinning-dependence structure

[a]Guojing Wang, [b]Kam C. Yuen*

[a]*Department of Mathematics, Suzhou University,*
Suzhou 215006, P.R. China

[b]*Department of Statistics and Actuarial Science, University of Hong Kong,*
Pokfulam Road, Hong Kong

Abstract

In this paper, we consider a risk model with n $(n \geq 2)$ dependent classes of insurance business. Stochastic sources related to claim occurrences of the n classes are classified into m groups. It is assumed that each event in the kth $(k = 1, 2, \cdots, m)$ group may cause a claim in the jth class $(j = 1, 2, \cdots, n)$ with probability p_{kj}. Within this framework, there exists certain correlation between the n claim-number processes due to the so-called thinning-dependence structure. We examine basic properties of the proposed risk process and study upper bounds for the ruin probability under certain assumptions. We also investigate the impact of the thinning-dependence structure as well as that of the classification of the stochastic sources on the ruin probability.

Keywords: Correlated aggregate claims; Lundberg exponent; Ruin probability; Stochastic sources; Thinning-dependence structure

* Corresponding author: Tel: (852) 2859-1915; Fax: (852) 2858-9041; e-mail: kcyuen@hku.hk
Classification code: 0167-6687

1. Introduction

In recent years, the study of risk processes with correlated classes of business has been one of main topics in the actuarial literature. A frequently-used approach to modelling dependent classes of business is to assume that there exists a common component in each of the associated claim-number processes. This assumption leads to the so-called risk model with common shock which has been studied by many authors, for example, see Ambagaspitiya (1998, 1999, 2003), Cossette and Marceau (2000), Wang (1998), and Yuen, Guo and Wu (2002). Another one proposed by Yuen and Wang (2002) is the continuous-time risk model with thinning dependence in which claims in one class may induce claims in other classes with certain probabilities. A typical example is that a severe car accident may cause not only the loss of the damaged car but also the medical expenses of the injured driver and passengers. Motivated by the work of Yuen and Wang (2002), Wu and Yuen (2003) studied the thinning relation in the discrete-time case. In this paper, we further investigate the thinning-dependence structure in a more general framework.

Suppose that an insurance company has n ($n \geq 2$) dependent classes of business. Stochastic sources that may cause a claim in at least one of the n classes are classified into m groups. It is assumed that each event occurred at time t in the kth group may cause a claim in the jth class with probability $p_{kj}(t)$ for $k = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, and that for each j, there exists at least some k such that $p_{kj}(t) > 0$. Denote by $N^k(t)$ the number of events from the kth group occurred up to time t. Let

$N_j^k(t)$ be the number of claims of the jth class up to time t generated from the events in group k, and $X_i^{(j)}$ be the amount of the ith claim in the jth class. Then, the total amount of claims for the jth class up to time t can be expressed as

$$S_j(t) = \sum_{i=1}^{N_j(t)} X_i^{(j)} ,$$

where $N_j(t) = N_j^1(t) + N_j^2(t) + \cdots + N_j^m(t)$ is the claim-number process of the jth class. Therefore, the aggregate claims process of the company is given by

$$S(t) = \sum_{j=1}^{n} S_j(t) = \sum_{j=1}^{n} \sum_{i=1}^{N_j(t)} X_i^{(j)} \tag{1.1}$$

where $\{X_i^{(j)};\ i = 1, 2, \cdots\}$ is assumed to be a sequence of an i.i.d. non-negative random variables with common distribution F_j for each j. As usual, we assume that the n sequences $\{X_i^{(1)};\ i = 1, 2, \cdots\}, \ldots, \{X_i^{(n)};\ i = 1, 2, \cdots\}$ are mutually independent and are independent of all claim-number processes.

Write u as the initial capital and c as the constant rate of premium. Then, the surplus process of the insurance company is defined as

$$R(t) = u + ct - S(t) = u + ct - \sum_{j=1}^{n} \sum_{i=1}^{N_j(t)} X_i^{(j)} . \tag{1.2}$$

In order to make the analysis of $R(t)$ mathematically tractable, we need the following assumptions:

(A1) The processes $N^1(t), \ldots, N^m(t)$ are independent Poisson processes with parameters $\lambda_1, \ldots, \lambda_m$, respectively, and $(N^k(t), N_1^k(t), \ldots, N_n^k(t))$ and $(N^{k'}(t), N_1^{k'}(t), \ldots, N_n^{k'}(t))$ are independent for $k \neq k'$;

(A2) For each k ($k = 1, \ldots, m$), $N_1^k(t), \ldots, N_n^k(t)$ are conditionally independent given $N^k(t)$.

Note that Assumptions (A1) and (A2) hold true if $N^1(t), \ldots, N^m(t)$ are independent Poisson processes and, for each pair of $(k, j)(k = 1, \cdots, m, j = 1, \cdots, n)$, whether the process of any event in $N^k(t)$ giving rise to a claim in the jth class or not is independent of all other events.

Assume that $p_{kj}(0^+) = \lim_{s \downarrow 0} p_{kj}(s)$ exists for all k and j. Define $q_{kj}(t) = t^{-1} \int_0^t p_{kj}(s) ds$ with the convention that $q_{kj}(0) = p_{kj}(0^+)$. Then under (A1) and (A2), an application of Proposition 2.3.2 of Ross (1996) gives that $N_j^k(t)$ is a non-homogeneous Poisson process with $E[N_j^k(t)] = \lambda_k q_{kj}(t) t$ for $k = 1, \ldots, m$ and $j = 1, \ldots, n$.

Risk model (1.2) not only possesses the thinning-dependence structure but also embraces the risk model with common shock. It can be shown that (1.2) follows a compound Poisson risk model under certain assumptions (see Section 2). Another advantage of the risk model is that its structure and dependence can be used to investigate how less knowledge of detailed information that may give rise to claims impacts on the risk of the insurer (see Section 5, 6). In light of these nice features, we study various aspects of the proposed model in this paper. Section 2 derives some properties of (1.2). Some upper bounds for ruin probability are given in Section 3. Section 4 examines the impact of dependence on the ruin probability while Section 5 investigates how the classification of the stochastic sources affects the ruin probability. An numerical example is presented in Section 6.

2. Properties of the risk model

We first compute $E[\exp\{rS(t)\}]$, the moment generating function of $S(t)$, for any fixed $t > 0$. For $n_{kj} \leq n_k$ $(k = 1, \ldots, m,\ j = 1, \cdots, n)$, we obtain from Assumptions (A1) and (A2)

$$\begin{aligned}
& P(N^k(t) = n_k,\ N_j^k(t) = n_{kj};\ k = 1, \ldots, m,\ j = 1, \ldots, n) \\
= & \prod_{k=1}^{m} P(N^k(t) = n_k,\ N_j^k(t) = n_{kj};\ j = 1, \ldots, n) \\
= & \prod_{k=1}^{m} P(N^k(t) = n_k) P(N_j^k(t) = n_{kj};\ j = 1, \ldots, n | N^k(t) = n_k) \\
= & \prod_{k=1}^{m} \left(\frac{\exp\{-\lambda_k t\}(\lambda_k t)^{n_k}}{n_k!} \prod_{j=1}^{n} P(N_{kj}(t) = n_{kj} | N^k(t) = n_k) \right) . \qquad (2.1)
\end{aligned}$$

Following the proof of Proposition 2.3.2 of Ross (1996, pp.69), we obtain

$$P(N_j^k(t) = n_{kj} | N^k(t) = n_k) = C_{n_k}^{n_{kj}} q_{kj}^{n_{kj}}(t)(1 - q_{kj}(t))^{n_k - n_{kj}} . \qquad (2.2)$$

Assuming the existence of $E[\exp\{rS(t)\}]$, it follows from (2.1), (2.2), (A1) and (A2) that

$$\begin{aligned}
E[\exp\{rS(t)\}] &= E\left[\exp\left\{ r \sum_{j=1}^{n} \sum_{k=1}^{N_j(t)} X_k^{(j)} \right\}\right] \\
&= \sum_{n_1, \ldots, n_m = 0}^{\infty} \sum_{n_{11}, \ldots, n_{1n} = 0}^{n_1} \cdots \sum_{n_{m1}, \ldots, n_{mn} = 0}^{n_m} E\left[\exp\left\{ r \sum_{j=1}^{n} \sum_{k=1}^{n_{1j} + n_{2j} + \cdots n_{mj}} X_k^{(j)} \right\}\right] \\
&\quad \times P(N^k(t) = n_k,\ N_j^k(t) = n_{kj};\ k = 1, \ldots, m,\ j = 1, \ldots, n) \\
&= \sum_{n_1, \ldots, n_m = 0}^{\infty} \sum_{n_{11}, \ldots, n_{1n} = 0}^{n_1} \cdots \sum_{n_{m1}, \ldots, n_{mn} = 0}^{n_m} \left(\prod_{j=1}^{n} M_j(r)^{n_{1j} + n_{2j} + \cdots + n_{mj}} \right) \\
&\quad \times \prod_{k=1}^{m} \left(\frac{\exp\{-\lambda_k t\}(\lambda_k t)^{n_k}}{n_k!} \prod_{j=1}^{n} C_{n_k}^{n_{kj}} q_{kj}^{n_{kj}}(t)(1 - q_{kj}(t))^{n_k - n_{kj}} \right) ,
\end{aligned}$$

where $M_j(r) = E[\exp\{rX_1^{(j)}\}]$ for $j = 1, \ldots, n$. Since

$$\prod_{j=1}^{n} M_j(r)^{n_{1j}+n_{2j}+\cdots+n_{mj}} = \prod_{k=1}^{m} \left(\prod_{j=1}^{n} M_j(r)^{n_{kj}} \right) ,$$

the moment generating function becomes

$$\begin{aligned} E[\exp\{rS(t)\}] &= \sum_{n_1,\ldots,n_m=0}^{\infty} \left(\prod_{k=1}^{m} \frac{\exp\{-\lambda_k t\}(\lambda_k t)^{n_k}}{n_k!} \right) \\ &\quad \times \left(\sum_{n_{11},\ldots,n_{1n}=0}^{n_1} \cdots \sum_{n_{m1},\ldots,n_{mn}=0}^{n_m} \left(\prod_{k=1}^{m} \prod_{j=1}^{n} C_{n_k}^{n_{kj}} (q_{kj}(t) M_j(r))^{n_{kj}} \right.\right. \\ &\quad \left.\left. \times\ (1 - q_{kj}(t))^{n_k - n_{kj}} \right) \right) \\ &= \sum_{n_1,\ldots,n_m=0}^{\infty} \left(\prod_{k=1}^{m} \left(e^{-\lambda_k t} \frac{(\lambda_k t)^{n_k}}{n_k!} \sum_{n_{k1},\ldots,n_{kn}=0}^{n_k} \left(\prod_{j=1}^{n} C_{n_k}^{n_{kj}} (q_{kj}(t) M_j(r))^{n_{kj}} \right.\right.\right. \\ &\quad \left.\left.\left. \times\ (1 - q_{kj}(t))^{n_k - n_{kj}} \right) \right) \right) . \end{aligned}$$

Rearranging terms yields

$$\begin{aligned} E[\exp\{rS(t)\}] &= \sum_{n_1,\ldots,n_m=0}^{\infty} \left(\prod_{k=1}^{m} \frac{\exp\{-\lambda_k t\}(\lambda_k t)^{n_k}}{n_k!} \prod_{j=1}^{n} \left(\sum_{n_{kj}=0}^{n_k} \left(C_{n_k}^{n_{kj}} \Big(q_{kj}(t) M_j(r) \Big)^{n_{kj}} \right.\right.\right. \\ &\quad \left.\left.\left. \times \Big(1 - q_{kj}(t)\Big)^{n_k - n_{kj}} \right) \right) \right) \\ &= \sum_{n_1,\ldots,n_m=0}^{\infty} \left(\prod_{k=1}^{m} \left(\frac{\exp\{-\lambda_k t\}(\lambda_k t)^{n_k}}{n_k!} \right.\right. \\ &\quad \left.\left. \times \prod_{j=1}^{n} \Big(q_{kj}(t) M_j(r) + 1 - q_{kj}(t) \Big)^{n_k} \right) \right) \\ &= \prod_{k=1}^{m} \left(\sum_{n_k=0}^{\infty} \left(\frac{\exp\{-\lambda_k t\}(\lambda_k t)^{n_k}}{n_k!} \prod_{j=1}^{n} \Big(q_{kj}(t) M_j(r) + 1 - q_{kj}(t) \Big)^{n_k} \right) \right) \\ &= \exp\left\{ t \sum_{k=1}^{m} \lambda_k \left(\prod_{j=1}^{n} \Big(q_{kj}(t) M_j(r) + 1 - q_{kj}(t) \Big) - 1 \right) \right\} \\ &\triangleq \exp\{t g(r,t)\} . \end{aligned} \tag{2.3}$$

For $t > s \geq 0$, put $q_{kj}(s,t) = (t-s)^{-1} \int_s^t p_{kj}(s) ds$. Similar to the derivation of (2.3), one can verify that

$$\begin{aligned} E[\exp\{r[S(t)-S(s)]\}] &= \exp\left\{(t-s)\sum_{k=1}^{m}\lambda_k\left(\prod_{j=1}^{n}\Big(q_{kj}(s,t)(M_j(r)-1)+1\Big)-1\right)\right\} \\ &\triangleq \exp\{(t-s)g(r,s,t)\}\ . \end{aligned} \tag{2.4}$$

The following lemma gives an exponential martingale.

Lemma 2.1. *For $t \geq 0$, let*

$$\mathcal{M}_u(t) = \frac{\exp\{-rR(t)\}}{\exp\{-crt + tg(r,t)\}} \qquad \textit{and} \qquad \mathcal{F}_t = \sigma(R(s), s \leq t).$$

Assume that at least $n-1$ functions of $p_{m1}(s)$, $p_{m2}(s), \ldots, p_{mn}(s)$ are constants for each $m \geq 1$. Then, $\{\mathcal{M}_u(t), t \geq 0\}$ is an $\mathcal{F}_t$-martingale.

Proof: By definitions and the assumptions of independence we see that risk process (1.2) has independent increments. Under the assumptions of the lemma we have $(t-s)g(r,s,t) = tg(r,t) - sg(r,s)$. This, together with (2.3) and (2.4), proves the lemma. $\square$

We now derive moments for the claim-numbers and the aggregate claims. Since $N_j^k(t)$ is a non homogeneous Poisson process with intensity $\lambda_k p_{kj}(t)$, we have

$$\mathrm{Var}\left(N_j^k(t)\right) = \lambda_k q_{kj}(t)t, t > 0,$$

for $j = 1, \cdots, n, k = 1, \cdots, m$, and a fixed $t > 0$. Furthermore, for any fixed $t > 0$, we have the following results. Applying (2.2) one obtains that

$$\mathrm{Cov}\left(N_i^k(t), N_j^k(t)\right) = \lambda_k q_{ki}(t)q_{kj}(t)t, t > 0,$$

for $1 \leq i \neq j \leq n$ and $k = 1, \cdots, m$. Thus, we have

$$\begin{aligned}
& \text{Cov}(N_i(t), N_j(t)) \\
= & \text{Cov}\left(\sum_{k=1}^{m} N_i^k(t), \sum_{k=1}^{m} N_j^i(t)\right) \\
= & \sum_{k=1}^{m} \text{Cov}\left(N_i^k(t), N_j^k(t)\right) \\
= & \sum_{k=1}^{m} \lambda_k q_{ki}(t) q_{kj}(t) t \ ,
\end{aligned} \tag{2.5}$$

for $1 \leq i \neq j \leq n$, and $k = 1, \cdots, m$. Denote by μ_j and σ_j^2 the mean and the variance of the distribution F_j, respectively. Using the above results, one can derive from the the independence assumptions that

$$\begin{aligned}
\text{Cov}\left(S_i(t), S_j(t)\right) &= \mu_i \mu_j \text{Cov}(N_i(t), N_j(t)) \\
&= \mu_i \mu_j \sum_{k=1}^{m} \lambda_k q_{ki}(t) q_{kj}(t) t, t > 0,
\end{aligned}$$

for $1 \leq i \neq j \leq n$, and $k = 1, \cdots, m$. Note that $S_j(t)$ is a compound Poisson variable with parameter $\sum_{l=1}^{m} \lambda_l p_{lj}(t) t$, it is easy to see that

$$\text{Var}\left(S_j(t)\right) = \left(\mu_j^2 + \sigma_j^2\right) E\left[N_j(t)\right] = \left(\mu_j^2 + \sigma_j^2\right) \sum_{k=1}^{m} \lambda_k p_{kj}(t) t, t > 0.$$

The following example shows that (1.2) follows a compound Poisson risk model under certain assumptions.

Example 2.1. We consider a special case of (1.2) in which all the functions $p_{kj}(s)$ are constants, that is, $p_{kj}(s) \equiv p_{kj}$ for all $j = 1, \cdots, n$ and $k = 1, \cdots, m$. In this case, $q_{kj}(t) \equiv p_{kj}$, and hence (2.3) becomes

$$E[\exp\{rS(t)\}] = \exp\left\{t \sum_{k=1}^{m} \lambda_k \left(\prod_{j=1}^{n} \left(p_{kj} M_j(r) + 1 - p_{kj}\right) - 1 \right)\right\}$$

$$\triangleq \exp\{tg(r)\} , \tag{2.6}$$

which implies that $S(t)$ is a compound Poisson process with intensity

$$\lambda = \lambda_1\left(1 - \prod_{j=1}^{n}(1-p_{1j})\right) + \cdots + \lambda_m\left(1 - \prod_{j=1}^{n}(1-p_{mj})\right) .$$

For $n = 2$ and $m = 3$,

$$S(t) = \sum_{i=1}^{N_1^1(t)+N_1^2(t)+N_1^3(t)} X_i^{(1)} + \sum_{i=1}^{N_2^1(t)+N_2^2(t)+N_2^3(t)} X_i^{(2)}$$

is a compound Poisson process with the intensity λ and the distribution function G given by

$$\begin{aligned}
\lambda &= \lambda_1(1-(1-p_{11})(1-p_{12})) + \lambda_2(1-(1-p_{21})(1-p_{22})) + \\
&\quad \lambda_3(1-(1-p_{31})(1-p_{32})) \\
&= \lambda_1(p_{11}+p_{12}-p_{11}p_{12}) + \lambda_2(p_{21}+p_{22}-p_{21}p_{22}) + \\
&\quad \lambda_3(p_{31}+p_{32}-p_{31}p_{32})
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
G(x) &= \frac{\lambda_1 p_{11}(1-p_{12}) + \lambda_2 p_{21}(1-p_{22}) + \lambda_3 p_{31}(1-p_{32})}{\lambda} F_1(x) \\
&\quad + \frac{\lambda_1 p_{12}(1-p_{11}) + \lambda_2 p_{22}(1-p_{21}) + \lambda_3 p_{32}(1-p_{31})}{\lambda} F_2(x) \\
&\quad + \frac{\lambda_1 p_{11}p_{12} + \lambda_2 p_{21}p_{22} + \lambda_3 p_{31}p_{32}}{\lambda} F_1 \star F_2(x) ,
\end{aligned} \tag{2.8}$$

where $F_1 \star F_2$ indicates the convolution of F_1 and F_2.

Remark 2.1. Risk process (1.2) is more general than the ones proposed by Cossette and Marceau (2000) and by Yuen and Wang (2002). If $m = n$ and $p_{kk} = 1$ for

$k = 1, \ldots, n$, then $R(t)$ of (1.2) is the risk model of Yuen and Wang (2001) with the so-called "thinning-dependence structure". If $n = 2$, $m = 3$, $p_{12} = p_{21} = 0$, $p_{31} = p_{32} = 1$, $p_{11} = p_{22} = 1$, then $R(t)$ of (1.2) is the risk model with common shock for two dependent classes of business discussed in Cossette and Marceau (2000). Note that more general risk models with common shock for $n > 2$ are still special cases of (1.2).

3. Some bounds for ruin probability

Let $h_i(r) = M_i(r) - 1$ for $i = 1, \cdots, n$. Then $h_i(0) = 0$ and $h_i(r) > 0$ for $r > 0$. Assume that there exists $r_\infty^i > 0$ for $i = 1, \cdots, n$ such that $\lim_{r \to r_\infty^i} h_i(r) = \infty$. We allow for the possibility that $r_\infty^i = \infty$. Define $r_\infty = \min\{r_\infty^1, \ldots, r_\infty^n\}$.

The ruin probability of risk process (1.2) is defined as $\Psi_D(u) = P(\inf_{t \geq 0} R(t) < 0)$. As usual, we assume that $E[R(t)] > 0$ for all $t > 0$ so that we have $\Psi_D(u) < 1$ for $u \geq 0$. Since $N_j^k(t)$ and $N^k(t)$ are either homogeneous or non-homogeneous Poisson processes, we may assume without loss of generality that the filtration $\{F_t, t \geq 0\}$ is right continuous. Under the assumptions of Lemma 2.1, we follow the martingale approach of Gerber (1973) and Grandell (1991, pp.92-95) to obtain

$$\Psi_D(u) \leq Q(\gamma - \varepsilon) e^{-(\gamma - \varepsilon)u} \,, \tag{3.1}$$

where $Q(r) = \sup_{t \geq 0} \exp\{g(r,t) - rc\}$, $\gamma = \sup\{r : Q(r) < \infty\}$ and $\varepsilon > 0$ is an arbitrary number such that $0 < \varepsilon < \gamma \leq r_\infty$.

If all $p_{kj}(s)$ are non-decreasing functions of s, then $g(r,t) \leq g(r, \infty)$. In this case, we have $\exp\{g(r,t)-rc\} \le \exp\{g(r,\infty)-rc\}$. Denote by γ_∞ the unique positive root of the equation $g(r,\infty)-rc=0$. Similar to (3.1), the assumption of Lemma 2.1 gives

$$\Psi_D(u) \le e^{-\gamma_\infty u} \ . \tag{3.2}$$

On the other hand, if all $p_{kj}(s)$ are non-increasing functions of s, then $g(r,t) \le g(r,0)$. Thus, we have $\exp\{g(r,t)-rc\} \le \exp\{g(r,0)-rc\}$. Write the unique positive root of the equation $g(r,0)-rc=0$ as γ_0. Similarly, we have

$$\Psi_D(u) \le e^{-\gamma_0 u} \ . \tag{3.3}$$

under the assumptions of Lemma 2.1. Along the same line, if all $p_{kj}(s)$ are invariant with respect to $s \ge 0$, we have

$$\Psi_D(u) \le e^{-R^D u} \ , \tag{3.4}$$

where R^D denotes the unique positive solution of the equation $g(r)-cr=0$.

Remark 3.1: We may call γ, γ_∞ γ_0 and R^D the Lundberg exponents. In general, it is hard to determine the explicit value, but is probably easy to determine that of γ_0, γ_∞ and R^D.

4. Dependence and Lundberg exponent

In this section, we study how the dependence among the claim number processes in (1.2) influences on the ruin probability.

For simplicity, we assume from now on that all $p_{kj}(s)$ are constants, that is, $p_{kj}(s) \equiv p_{kj}$ for $k=1,\cdots,m$ and $j=1,\cdots,n$. Let $N_1(t)^\perp,\ldots,N_n(t)^\perp$ represent the independent version of $N_1(t), \ldots, N_n(t)$. That is, (i) $N_1(t)^\perp, \ldots, N_n(t)^\perp$ are mutually independent, and (ii) $N_1(t)^\perp$ and $N_i(t)$ are identically distributed for each i. It is obvious that $S_i(t)^\perp = \sum_{k=1}^{N_i(t)^\perp} X_k^{(i)}$ for $i = 1, \ldots, n$. Put

$$S_I(t) = \sum_{i=1}^{n} S_i(t)^\perp \tag{4.1}$$

and

$$R_I(t) = u + ct - S_I(t) \ . \tag{4.2}$$

So, $R_I(t)$ represents the surplus of an insurer with n independent lines of business at time t. Furthermore, $R(t)$ of (1.2) and $R_I(t)$ of (4.2) have the same expected aggregate loss.

The ruin probability of the risk model (4.2) is defined as $\Psi_I(u) = P(\inf_{t\geq 0} R_I(t) < 0)$. Again, it is assume that $c - \sum_{i=1}^{n}\sum_{k=1}^{m} \mu_i \lambda_k p_{ki} > 0$ so that $\Psi_I(u) < 1$ for $u \geq 0$. Define

$$g_I(r) = \sum_{j=1}^{n}\left(\sum_{k=1}^{m} \lambda_k p_{kj} h_j(r)\right) = \sum_{k=1}^{m} \lambda_k \left(\sum_{j=1}^{n} p_{kj} h_j(r)\right) \ .$$

Then, for any fixed t, the moment generating function of $S_I(t)$ of (4.1) is given by

$$E[\exp\{rS_I(t)\}] = \exp\{tg_I(r)\} \ . \tag{4.3}$$

Note that $g_I(r)$ and $g(r)$ are both finite and convex on $(0, r_\infty)$ and that both $g_I(r)$ and $g(r)$ tend to ∞ as $r \uparrow r_\infty$. These imply that the two equations, $cr = g(r)$ and $cr = g_I(r)$, have unique positive roots, R^D and R^I, respectively.We call these roots the Lundberg exponents. Using R^D and R^I, We have $\Psi_I(u) \leq \exp\{-R^I u\}$ and $\Psi_D(u) \leq \exp\{-R^D u\}$.

Proposition 4.1. *Let $u \geq 0$, we have*

$$R^D \leq R^I \ . \tag{4.4}$$

If there exist i and j with $i \neq j$ such that $p_{ki}p_{kj} > 0$, then the strict inequality in (4.4) holds true.

Proof: It is obvious that

$$\sum_{j=1}^{n} p_{kj}h_j(r) \leq \prod_{j=1}^{n} (p_{kj}h_j(r) + 1) - 1, \tag{4.5}$$

for $k = 1, \ldots, m$ and $r > 0$,which implies that

$$\begin{aligned} g_I(r) &= \sum_{k=1}^{m} \lambda_k \left(\sum_{j=1}^{n} p_{kj}h_j(r) \right) \\ &\leq \sum_{k=1}^{m} \lambda_k \left(\prod_{j=1}^{n} (p_{kj}h_j(r) + 1) - 1 \right) = g(r) \ . \end{aligned} \tag{4.6}$$

Therefore, $g(r)$ will meet cr before $g_I(r)$. It follows that $R^D \leq R^I$. If there exist i and j with $i \neq j$ such that $p_{ki}p_{kj} > 0$, then the strict inequalities in (4.5) and (4.6) hold true for $r > 0$. Hence, $R^D < R^I$. □

Remark 4.1: The assumption that $p_{ki}p_{kj} > 0$ indicates that risk process (1.2) contains correlated classes of business. Proposition 4.1 also states that the Lundberg exponent related to the model with independent classes is larger than that with correlated classes.

The finite-time ruin probabilities $\Psi_I(u, t)$ and $\Psi_D(u, t)$ are defined by $\Psi_I(u, t) = P(\inf_{0 \leq s \leq t} R_I(s) < 0)$ and $\Psi_D(u, t) = P(\inf_{0 \leq s \leq t} R(s) < 0)$. Put $f_I(r) = r - yg_I(r) + yrc$ and $f_D(r) = r - yg(r) + yrc$. The finite-time Lundberg exponents R_y^I and R_y^D are defined by $R_y^I = \sup_{r \geq R^I} f_I(r)$ and $R_y^D = \sup_{r \geq R^D} f_D(r)$. By (2) of Grandell (1991, p.136) we have that $\Psi_I(u, yu) \leq \exp\{-R_y^I u\}$ and $\Psi_D(u, yu) \leq \exp\{-R_y^D u\}$.

Proposition 4.2. *Assume that there exist i and j with $i \neq j$ such that $p_{ki}p_{kj} > 0$ for some k, then $R_y^D < R_y^I$*

Proof: Under the assumption of the proposition, the strict inequality in (4.6) holds. It follows that

$$f_D(r) < f_I(r), \quad 0 < r < r_\infty \ . \tag{4.7}$$

By Proposition 4.1, we have $R^D < R^I$. It is obvious that

$$R_y^D = \max\left\{ \sup_{R^D \leq r \leq R^I} f_D(r), \sup_{r \geq R^I} f_D(r) \right\} . \tag{4.8}$$

From (4.7), we get

$$R_y^I \geq \sup_{r \geq R^I} f_D(r) \ .$$

Since

$$f_I(r) > r > f_D(r), \quad R^D \leq r \leq R^I \ , \tag{4.9}$$

we obtain

$$\sup_{R^D \leq r \leq R^I} f_D(r) \leq f_I(R^I) = R^I \leq R_y^I \ . \tag{4.10}$$

By (4.8)-(4.10), we have $R_y^D \leq R_y^I$. Note that $f_D(r)$ is concave and tends to $-\infty$ as $r \to \infty$. By the continuity of $f_D(r)$ we conclude that there exists r_0 such that $R^D \leq r_0 < r_\infty$ and $R_y^D = f_D(r_0)$. If $r_0 \leq R^I$, we see from (4.9) and (4.10) that

$$R_y^I \geq R^I > f_D(r_0) = R_y^D \ . \tag{4.11}$$

On the other hand, if $r_0 \geq R^I$, then it follow from (4.7) that

$$R_y^I \geq f_I(r_0) > f_D(r_0) = R_y^D \ . \tag{4.12}$$

Hence, (4.11) and (4.12) yield $R^D < R^I$. □

5. Classification of the stochastic sources

It is obvious that the degree of dependence among the claim number processes not only depends on the value p_{kj} but also the classification of the stochastic sources. Hence, an unrealistic classification of the stochastic sources may lead one to overestimate or to underestimate the underlying risks. In view of this, it is instructive to study the impact of misclassification of the stochastic sources on the ruin probability.

Assume that the correct classification for an insurance company should involve m groups of stochastic sources. Suppose that the company misclassified the sources into l groups only with $l < m$. In this section, we investigate how such a misgrouping influences on the Lundberg exponent. For notational convenience, we rewrite $R(t)$, $S(t)$, $g(r)$, $\Psi_D(u)$, $S_i(t)$ and R^D defined in the previous sections as $R^{(m)}(t)$, $S^{(m)}(t)$, $g^{(m)}(r)$, $\Psi_D^{(m)}(u)$, $S_i^{(m)}(t)$ and $R_D^{(m)}$ respectively, to indicate that the full model consists of m groups of stochastic sources.

We now consider the first simplified model in which the last $m - l + 1$ groups are treated as a single group. Thus, the number of groups is reduced from m to l. We write the surplus process of the first simplified model as

$$R^{(l)}(t) = u + ct - \sum_{i=1}^{n} \sum_{k=1}^{N_i^{(l)}(t)} X_k^{(i)} \triangleq u + ct - S^{(l)}(t) \ ,$$

where $N_i^{(l)}(t) = N_i^1(t) + \cdots + N_i^{l-1}(t) + N_i^{[l]}(t)$ and $N_i^{[l]}(t)$ is the p_i'-thinning of $N^{[l]}(t) = N^l(t) + \cdots + N^m(t)$ for $i = 1, \ldots, n$. Apparently, $N^{[l]}(t)$ is a Poisson process with parameter $\lambda^{[l]} = \lambda_l + \cdots + \lambda_m$. Then, similar to $g^{(m)}(r)$, $\Psi_D^{(m)}(u)$, $S_i^{(m)}(t)$ and $R_D^{(m)}$, we have $g^{(l)}(r)$, $\Psi_D^{(l)}(u)$, $S_i^{(l)}(t)$ and $R_D^{(l)}$ for the simplified model. To make the comparison be fair, we let $p_i' = \sum_{k=l}^{m} \lambda_k p_{ki}/\lambda^{[l]}$ for $i = 1, \cdots, n$. Thus, $N_i^{(l)}(t)(S_i^{(l)}(t))$ and $N_i^{(m)}(t)(S_i^{(m)}(t))$ are identically distributed for $i = 1, \cdots, n$, and hence $R^{(l)}(t)$ and $R^{(m)}(t)$ have the same expected aggregate loss.

It is easy to show that

$$\begin{aligned} E[\exp\{rS^{(l)}(t)\}] &= \exp\{tg^{(l)}(r)\} \\ &= \exp\Bigg\{t\Bigg[\sum_{k=1}^{l-1} \lambda_k \Bigg[\prod_{i=1}^{n} (p_{ki}h_i(r) + 1) - 1\Bigg] \\ &\qquad + \lambda^{[l]}[\prod_{i=1}^{n} (p_i'h_i(r) + 1) - 1]\Bigg]\Bigg\} \ . \end{aligned} \tag{5.1}$$

From (2.6) and (5.1), we get

$$\Delta(r) \triangleq g^{(m)}(r) - g^{(l)}(r) = \sum_{k=l}^{m} \lambda_k \prod_{j=1}^{n} (p_{kj}h_j(r) + 1) - \lambda^{[l]} \prod_{j=1}^{n} (p_j'h_j(r) + 1) \ . \tag{5.2}$$

Thus, using (5.1) and (5.2) and following the steps in the proofs of Proposition 4.1, we obtain

Proposition 5.1. *For* $0 < r < r_\infty$, $\Delta(r) < (=,>)0$, *implies that* $R_D^{(l)} < (=,>) R_D^{(m)}$.

From (5.2), we have the following lemma in which sufficient conditions for $\Delta(r) = 0$ are given.

Lemma 5.1. *Let $i_1, \ldots, i_{n-1}$ be $n-1$ different numbers in $\{1, \ldots, n\}$. Assume that $p_{ki_q} = p_{i_q}$ for $l \leq k \leq m$ and $1 \leq q \leq n-1$. Then, $\Delta(r) \equiv 0$ for $0 < r < r_\infty$.*

When $n = 2$, we have

$$\begin{aligned}\Delta(r) &= \left\{\sum_{k=l}^{m} \lambda_k p_{k1} p_{k2} - \frac{1}{\lambda^{[l]}} \prod_{j=1}^{2} \left(\sum_{k=l}^{m} \lambda_k p_{kj}\right)\right\} h_1(r) h_2(r) \\ &\triangleq \Delta_1 h_1(r) h_2(r) ,\end{aligned}$$

from which we see that $\Delta(r) \equiv 0$ for $0 < r < r_\infty$ if and only if $\Delta_1 = 0$. Moreover, if $p_{l1} = \cdots = p_{m1}$ or $p_{l2} = \cdots = p_{m2}$, then $\Delta_1 = 0$. Thus, we have $R^{(l)} = R^{(m)}$.

We next consider the second simplified model. Suppose that two disjoint subsets of the m groups were treated as two individual groups. One of them consists of q of the original m groups while the other comprises q' of the remaining $m-q$ groups. It is obvious that $q + q' \leq m$ and that the second simplified model classifies the stochastic sources into $m - q - q' + 2$ groups only. In this situation, there exist $q + q'$ different numbers $i_1, \ldots, i_q,\ l_1, \ldots, l_{q'} \in \{1, \ldots, m\}$ such that $N^{[q]}(t) = N^{i_1}(t) + \cdots + N^{i_q}(t)$, $N^{[q']}(t) = N^{l_1}(t) + \cdots + N^{l_{q'}}(t)$ are two Poisson processes with parameters $\lambda^{(q)} = \lambda_{i_1} + \cdots + \lambda_{i_q}$ and $\lambda^{(q')} = \lambda_{l_1} + \cdots + \lambda_{l_{q'}}$, respectively, and that

$$R^{(m-q-q'+2)}(t) = u + ct - \sum_{i=1}^{n} \sum_{k=1}^{N_i^{(m-q-q'+2)}(t)} X_k^{(i)} \triangleq u + ct - S^{(m-q-q'+2)}(t) , \tag{5.3}$$

where

$$N_i^{(m-q-q'+2)}(t) = \sum_{1\le k\le m,\ k\notin A} N_i^k(t) + N_i^{[q]}(t) + N_i^{[q']}(t),$$

$N_i^{[q]}(t)(N_i^{[q']}(t))$ is the $p_i^{(q)}$ $(p_i^{(q')})$-thinning of $N^{[q]}(t)(N^{[q']}(t))$, and $A = A_q \cup A_{q'}$ with $A_q = \{i_1, \ldots, i_q\}$ and $A_{q'} = \{l_1, \ldots, l_{q'}\}$. Parallel to $g^{(l)}(r)$, $\Psi_D^{(l)}(u)$, $S_i^{(l)}(t)$ and $R_D^{(l)}$ in the first simplified model, we have $g^{(m-q-q'+2)}(r)$, $\Psi_D^{(m-q-q'+2)}(u)$, $S_i^{(m-q-q'+2)}(t)$ and $R_D^{(m-q-q'+2)}$ for (5.3).

Again, to perform fair comparison, we set

$$p_i^{(q)} = \frac{\sum_{k\in A_q} \lambda_k p_{ki}}{\lambda^{(q)}} \qquad \text{and} \qquad p_i^{(q')} = \frac{\sum_{k\in A_{q'}} \lambda_k p_{ki}}{\lambda^{(q')}}$$

Therefore, $N_i^{(m-q-q'+2)}(t)(S_i^{(m-q-q'+2)}(t))$ of the second simplified model and $N_i^m(t)(S_i^m(t))$ of the full model are identically distributed for $i = 1, \cdots, n$. Hence, $R^{(m)}(t)$ and $R^{(m-q-q'+2)}(t)$ have the same expected aggregate loss. Similar to (2.6) we have

$$\begin{aligned}
E[\exp\{rS^{(m-q-q'+2)}(t)\}] &= \exp\Bigg\{t\Bigg[\sum_{1\le k\le m, k\notin A} \lambda_k\Bigg[\prod_{j=1}^{n}(p_{kj}h_j(r)+1)-1\Bigg] \\
&\quad + \lambda^{(q)}\Bigg[\prod_{j=1}^{n}(p_j^{(q)}h_j(r)+1)-1\Bigg] \\
&\quad + \lambda^{(q')}\Bigg[\prod_{j=1}^{n}(p_j^{(q')}h_j(r)+1)-1\Bigg]\Bigg]\Bigg\} \\
&\triangleq \exp\{tg^{(m-q-q'+2)}(r)\}\ . \qquad (5.4)
\end{aligned}$$

From (2.6) and (5.4), we obtain

$$\begin{aligned}
\Delta'(r) &\triangleq g^{(m)}(r) - g^{(m-q-q'+2)}(r) \\
&= \sum_{k\in A} \lambda_k \prod_{j=1}^{n}(p_{kj}h_j(r)+1) - \lambda^{(q)} \prod_{j=1}^{n}(p_j^{(q)}h_j(r)+1)
\end{aligned}$$

$$- \lambda^{(q')} \prod_{j=1}^{n} (p_j^{(q')} h_j(r) + 1) \ . \tag{5.5}$$

In line with Proposition 51. and Lemma 5.1, one can verify the following results.

Proposition 5.2. *For $0 < r < r_\infty$, $\Delta'(r) < (=, >)0$ implies that $R_D^{(m-q-q'+2)} < (=, >)R_D^{(m)}$.*

Lemma 5.2. *Assume that $p_{kj} = p_j$ for $k \in A_q$ and $p_{kj} = p_j^0$ for $k \in A_{q'}$, $j = 1, \cdots, n$. Then $\Delta'(r) = 0$ for $0 < r < r_\infty$.*

For $n = 2$, $m = 4$, $A_q = \{1, 2\}$ and $A_{q'} = \{3, 4\}$,it can be shown that

$$\begin{aligned}
\Delta'(r) = g^{(4)}(r) - g^{(2)}(r) \ &= \ \left\{ \left[\sum_{k=1}^{2} \lambda_k p_{k1} p_{k2} - \frac{1}{\lambda_1 + \lambda_2} \prod_{j=1}^{2} \left(\sum_{k=1}^{2} \lambda_k p_{kj} \right) \right] \right. \\
&\quad \left. + \left[\sum_{k=3}^{4} \lambda_k p_{k1} p_{k2} - \frac{1}{\lambda_3 + \lambda_4} \prod_{j=1}^{2} \left(\sum_{k=3}^{4} \lambda_k p_{kj} \right) \right] \right\} h_1(r) h_2(r) \\
&\triangleq \ \Delta_2 h_1(r) h_2(r) \ .
\end{aligned}$$

Thus, $\Delta_2 = 0$ implies that $\Delta'(r) = 0$ for $0 < r < r_\infty$. Under one of the following four conditions, (i) $p_{11} = p_{21}$ and $p_{31} = p_{41}$, (ii) $p_{11} = p_{21}$ and $p_{32} = p_{42}$, (iii) $p_{12} = p_{22}$ and $p_{31} = p_{41}$, (iv) $p_{12} = p_{22}$ and $p_{32} = p_{42}$, we have $\Delta_2 = 0$, and therefore $R_D^{(2)} = R_D^{(4)}$.

6. Numerical example

In this section, we present a numerical example with $m = n = 2$ which is a special case of the two simplified model discussed in the previous section. In the example, we set $l = 1$ for the first simplified model of Section 5 and $q = q' = 1$ for the second simplified model of Section 5. Note that the surplus process of the first simplified model is $R^{(1)}(t)$ and that $R^{(2)}(t)$ of the second simplified model and $R(t)$ of the full model are the same surplus process.

Define

$$\theta = \frac{c - \sum_{j=1}^{2} \mu_j \sum_{k=1}^{2} \lambda_k p_{kj}}{\sum_{j=1}^{2} \mu_j \sum_{k=1}^{2} \lambda_k p_{kj}},$$

where θ is known as the relative security loading (see (1.1) of Gerber (1979, pp.111)). Here, the claim-amount random variables $X_i^{(1)}$ and $X_i^{(2)}$ are exponentially distributed with means $\mu_1 = 4$ and $\mu_2 = 2$, respectively. Also, we set $\lambda_1 = 4$, $\lambda_2 = 7$, $p_{11} = 0.8$, $p_{12} = 0.3$, $p_{21} = 0.2$ and $p_{22} = 0.6$. Therefore, $\lambda^{([1]} = \lambda_1 + \lambda_2 = 11$, $\rho'_1 = (\lambda_1 p_{11} + \lambda_2 p_{21})/\lambda^{[1]} = 4.6/11$ and $\rho'_2 = (\lambda_1 p_{12} + \lambda_2 p_{22})/\lambda^{[1]} = 5.4/11$. Furthermore, from (2.5), we compute the correlation coefficients

$\rho_I(N_1(t)^{\perp}, N_2(t)^{\perp}) = 0,$

$\rho^{(1)}(N_1^{(1)}(t), N_2^{(1)}(t)) = \frac{\lambda^{([1])} p'_1 p'_2}{\left((\lambda^{[1]} p'_1)(\lambda^{[1]} p'_2)\right)^{1/2}} = 0.453089$

and

$\rho^{(2)}(N_1^{(2)}(t), N_2^{(2)}(t)) = \frac{\sum_{l=1}^{2} \lambda_l p_{l1} p_{l2}}{\left((\sum_{l=1}^{2} \lambda_l p_{l1})(\sum_{l=1}^{2} \lambda_l p_{l2})\right)^{1/2}} = 0.361158.$

Let $\theta = 0.1$ so that $c = 32.12$. Since $R_I(t)$, $R^{(1)}(t)$ and $R^{(2)}(t)$ are all compound Poisson risk models with claim amount distributions being mixtures of exponential distributions, one can follow Gerber (1979) to obtain explicit expressions for ruin probabilities $\Psi_I(u)$, $\Psi^{(1)}(u)$ and $\Psi^{(2)}(u)$. The numerical values of $\Psi_I(u)$, $\Psi^{(1)}(u)$ and $\Psi^{(2)}(u)$ for various values of u are shown in table 1. From the table, we see that $\Psi^{(1)}(u) > \Psi^{(2)}(u) > \Psi_I(u)$ for all $u > 0$ and that the impact of the dependence among the claim number processes on the ruin probability is prominent. The last three columns indicates that the ratios get larger as u increases. In this example,

$R^{(2)}(t)$ is the surplus process of the full model. It is clear that the use of $R^{(1)}(t)$ overestimates the underlying risk while the use of $R_I(t)$ underestimates the underlying risk. Hence, correct classification of the stochastic sources is an important issue in model (1.2)

Table 1

u	$\Psi_I(u)$	$\Psi^{(1)}(u)$	$\Psi^{(2)}(u)$	$\frac{\Psi^{(1)}(u)}{\Psi_I(u)}$	$\frac{\Psi^{(2)}(u)}{\Psi_I(u)}$	$\frac{\Psi^{(1)}(u)}{\Psi^{(2)}(u)}$
10	0.688368	0.781291	0.711868	1.143	1.042	1.098
30	0.393031	0.489395	0.439163	1.245	1.117	1.114
50	0.266092	0.306469	0.270937	1.356	1.198	1.131
100	0.056745	0.095105	0.080998	1.676	1.427	1.174
150	0.014242	0.029513	0.024215	2.072	1.700	1.219
200	0.003524	0.009159	0.007239	2.562	2.025	1.265
250	0.000897	0.002824	0.002164	3.168	2.412	1.313
300	0.000225	0.000882	0.000647	3.917	2.873	1.363
350	0.000056	0.000274	0.000193	4.843	3.423	1.451
400	0.000014	0.000085	0.000057	5.988	4.077	1.469
450	3.56×10^{-6}	0.000026	0.000017	7.404	4.856	1.525
500	8.94×10^{-7}	8.18×10^{-6}	5.17×10^{-6}	9.155	5.784	1.583
550	2.24×10^{-7}	2.54×10^{-6}	1.55×10^{-6}	11.319	6.890	1.643
600	5.63×10^{-8}	7.88×10^{-7}	4.02×10^{-7}	13.996	8.206	1.706

Comparison of ruin probabilities

Acknowledgements

Supports for Kam C. Yuen from a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. HKU 7202/99H) and for Guojing Wang from NNSF (grant No. 10271087) of China are acknowledged.

References

Ambagaspitiya, R.S., 1998. On the distribution of a sum of correlated aggregate claims. Insurance: Mathematics and Economics 23, 15-19.

Ambagaspitiya, R.S., 1999. On the distribution of two classes of correlated aggregate claims. Insurance: Mathematics and Economics 24, 301-308.

Ambagaspitiya, R.S., 2003. Aggregate survival probability of a portfolio with dependent subportfolios. Insurance: Mathematics and Economics 32, 431-443.

Cossette, H., Marceau, E., 2000. The discrete-time model with correlated classes of business. Insurance: Mathematics and Economics 26, 133-149.

Gerber, H.U., 1973. Martingales in risk theory. Mutteilungen der Schweizer Vereinigung der Versicherungs mathematiker, 205-216.

Gerber, H.U., 1979. An Introduction to Mathematical Risk Theory. S. S. Heubner Foundation monograph series 8, Philadelphia.

Grandell, J., 1991. Aspects of Risk Theory. Springer-Verlag, New York.

Ross, S.M., 1996. Stochastic Processes. Wiley, New York.

Wang, S., 1998. Aggregation of correlated risk portfolios: models and algorithms. Proceedings of the Casualty Actuarial Society, 848–939.

Wu, X., Yuen, K.C., 2003. A discrete-time risk model with interaction between classes of business. Insurance: Mathematics and Economics 33, 117-133.

Yuen, K.C., Guo, J.Y., Wu, X., 2002. On a correlated aggregate claims model with Poisson and Erlang risk processes. Insurance: Mathematics and Economics 31, 205-214.

Yuen, K.C., Wang. G., 2002. Comparing two models with dependent classes of business. ARCH, Society of Actuaries, 22 pages.