Normalized Exponential Tilting: Pricing and Measuring Multivariate Risks

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Abstract: This paper discusses exponential tilting of the probability density function of an underlying risk X, with respect to some reference risk Y. By introducing a normalization procedure on the reference risk Y, it shows that normalized exponential tilting is equivalent to applying Wang transform distortion to the cumulative distribution of X, and is an extension of CAPM to risks with general-shaped distributions. The second part of the paper deals with multivariate cases. It shows how multivariate normalized exponential tilting is related to multivariate probability distortions. It provides efficient routines for computing risk-adjusted multivariate probability distributions, and gives examples of pricing contingent claims on multiple risks.

Introduction:

Adjustment of probability measure is a common theme in pricing and valuation of risks. The need for changing multivariate probability measures often arises in pricing of contingent claims on multiple underlying assets or liabilities, in measuring portfolio risks, and when allocating total company risk capital to various business units.

Part 1 of the paper reviews two methods of adjustment of probability measure for a given risk (random variable) X: i) applying an *exponential tilting* (with respect to a reference variable Y) to the probability density function of X; ii) applying a *distortion* directly to the cumulative distribution function of X. The paper then introduces a normalization procedure via percentile mapping that converts the reference Y to a standard normal variable. It is shown that *normalized exponential tilting* of the probability density function of X is equivalent to applying the Wang transform distortion to the cumulative distribution of X, and is an extension of the Capital Asset Pricing Model to risks with general-shaped distributions.

Part 2 of the paper extends both the normalized exponential tilting and probability distortion to multivariate cases, and establishes an important link between these two parallel approaches in the multivariate case. It gives efficient routines for computing the risk-adjusted multivariate probability distribution, and provides examples in pricing contingent claims on multiple risks, and in measuring portfolio risks.

PART 1. NORMALIZED EXPONENTIAL TILTING & DISTORTION --- THE UNIVARIATE CASE

1.1 Normalized Exponential Tilting

We shall consider *risks* that are random variables in some probability space (Ω, P). For any random variable *X*, we let F_X represent its cumulative distribution function (CDF). We let f_X represent the probability density function (p.d.f) of *X* (in the discrete case, the p.d.f. is also referred to as the probability function).

Consider two risks *X* and *Y*. We say that *X* is absolutely continuous w.r.t. *Y*, if $f_Y(x) > 0$ for all points *x* with $f_X(x) > 0$.

Definition 1. Assume that *X* is absolutely continuously w.r.t *Y*.
We define the exponential tilting of *X* with respect to reference *Y* as follows:

$$f_X^*(x) = f_X(x) \cdot \frac{E[\exp(\lambda Y) \mid X = x]}{E[\exp(\lambda Y)]}, \qquad \text{(eq-1)}$$

where f_X and f_X^* represent the p.d.f. for *X*, before and after the exponential tilting, respectively. The λ in (eq-1) is a real-valued parameter controlling the magnitude of risk-adjustment.

With the exponential tilting in (eq-1), the ratio

$$RN(x) = \frac{f_X^*(x)}{f_X(x)} = \frac{E[\exp(\lambda Y) \mid X = x]}{E[\exp(\lambda Y)]},$$

gives the Radon-Nikodym derivative of f_X^* w.r.t. f_X.

In the context of an economic model for optimal risk exchange, Buhlmann (1980) derived that the Pareto-optimal equilibrium price for a risk *X* can be represented as the expected value of the risk-adjusted distribution as in (eq-1), whereas the reference *Y* represents portfolio aggregate risk.

In the special case of *Y*=*X*, (eq-1) defines an exponential tilting of *X* w.r.t. itself:

$$f_X^*(x) = f_X(x) \cdot \frac{\exp(\lambda x)}{E\left[\exp(\lambda X)\right]}.$$

The relation is also widely known as the Esscher transform. Gerber and Shiu (1994) have successfully applied the Esscher transform in pricing options.

With the exponential tilting in (eq-1), we do not have a consistent interpretation of the λ parameter, except for in the special case when *Y* is a normal (Gaussian) variable. Indeed, if we keep the value of λ fixed, the scale and shape of the reference variable *Y* can make a huge difference in the results of exponential tilting.

In order to get a consistent interpretation of λ, here we propose a normalization procedure to be performed on the reference variable *Y*. We define the inversion of the CDF of *Y* as $F_Y^{-1}(p) = \inf\left\{y \middle| F_Y(y) \geq p\right\}$. Let *Z* be a standard normal variable such that $Y = F_Y^{-1}\left(\Phi(Z)\right)$, where Φ is the CDF of normal(0,1). We shall refer to Z as a normalized variable of *Y*, and next we shall use *Z* to replace *Y* in (eq-1).

Definition 2. Let *Z* be a normalized variable of the reference *Y*. We define a **normalized exponential tilting** of *X* with respect to reference *Y* (or say, w.r.t. reference *Z*) as follows:

$$f_X^*(x) = f_X(x) \cdot \frac{E\left[\exp(\lambda Z) \mid X = x\right]}{E\left[\exp(\lambda Z)\right]}. \qquad \text{(eq-2)}$$

To keep the notations straight, we summarize the above as follows:

- Before: Exponential tilting of *X* w.r.t *Y*
- Normalization: Transform *Y* into a standard normal variable *Z*:
- After: Normalized exponential tilting of *X* w.r.t. *Z.*

Note that here Z essentially replaces Y as a *normalized* reference variable.

The rationales and benefits of introducing the above normalization procedure will be given in section 1.3.

1.2 Probability Distortions

Now it is time to introduce probability distortions, as another approach that parallels to the exponential tilting method.

Definition 3. Let g:[0, 1]→[0, 1] be a differentiable function with g(0)=0 and g(1)=1. Given the CDF $F(x)$ for a random variable X, the transformed CDF

$$F_X^*(x) = g\left(F(x)\right),$$

defines a risk-adjusted probability measure.

The probability distortion in definition 3 implies the following Radon-Nikodym derivative:

- In the discrete case where X takes on values $\{\cdots, \quad x_{i-1}, \quad x_i, \quad \cdots\}$:

$$RN_g(x_i) = \frac{f_X^*(x_i)}{f_X(x_i)} = \frac{g\left(F_X(x_i)\right) - g\left(F_X(x_{i-1})\right)}{F_X(x_i) - F_X(x_{i-1})}$$

- In the continuous case where X has a positive probability density at x:

$$RN_g(x) = g'\left(F_X(x)\right).$$

As shown in Wang (2000, 2002), the following specific form of distortion is referred to as the Wang transform:

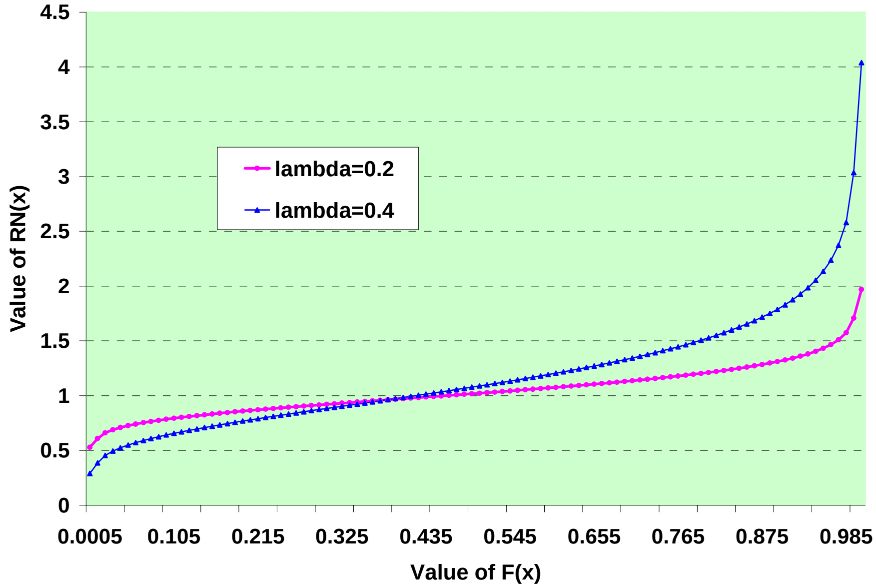
$$F_X^*(x) = g(F_X(x)) = \Phi\left[\Phi^{-1}(F_X(x)) - \lambda\right]. \qquad \text{(eq-3)}$$

When X is a continuous variable, we have

$$RN_g(x) = g'\left(F_X(x)\right) = \exp\left(\lambda \cdot \Phi^{-1}(F_X(x))\right) \cdot \exp\left(-\lambda^2/2\right).$$

A plot of the Radon-Nikodym derivative is given in Figure 1.

Figure 1. Radon-Nikodym Derivative Implied By The Wang Transform



Both normal and lognormal distributions are preserved under the Wang transform in (eq-3):

- If F has a Normal(μ,σ^2) distribution, F^* is also a normal distribution with $\mu^* = \mu - \lambda\sigma$ and $\sigma^* = \sigma$.
- If F has a log-normal(μ,σ^2) distribution such that $\ln(X) \sim$ Normal(μ,σ^2), F^* is another log-normal distribution with $\mu^* = \mu - \lambda\sigma$ and $\sigma^* = \sigma$.

1.3 Link Between Exponential Tilting and Distortion – Univariate Case

Theorem 1: Assume that X and Y have bivariate normal copula with a correlation coefficient of $\rho_{X,Y}$. The normalized exponential tilting (eq-2) is equivalent to applying the following Wang transform distortion:

$$F_X^*(x) = g(F_X(x)) = \Phi\left[\Phi^{-1}(F_X(x)) - \beta\right], \quad \text{with } \beta = \rho_{X,Y} \cdot \lambda .$$

A proof can be found in Wang (2003).

Theorem 1 establishes an important link between normalized exponential tilting and the Wang transform distortion. This result is a generalization of the Capital Asset Pricing Modeling, which reveals the meaning of the λ parameter.

Consider a stock index R_M (represent the market portfolio) whose prospective end-of-period return has a normal (μ_M, σ_M^2) distribution with mean μ_M and standard deviation σ_M. The discounted end-of-period stock price

$$S_M(1) = S_M(0) \cdot \exp(R_M) \cdot \exp(-r)$$

has a log-normal $\left(\mu_M - r - \frac{\sigma_M^2}{2}, \quad \sigma_M^2 \right)$ distribution, and r is the risk-free rate of return.

If we apply normalized exponential tilting on $X = R_M$ with the reference Y being the stock return R_M, the risk-adjusted distribution for the stock index return has a normal distribution with $E^*[R_M] = \mu_M - \lambda_M \cdot \sigma_M$.

Alternatively, if we apply normalized exponential tilting on the stock price X= $S_M(1)$ with the reference $Y = S_M(1)$ being the stock price, the risk-adjusted distribution for the stock price is log-normal $\left(\mu_M - \lambda\sigma_M - r - \frac{\sigma_M^2}{2}, \quad \sigma_M^2 \right)$.

To force the risk-adjusted expected return equal the risk free rate, r, we get,

$$\lambda_M = \frac{E[R_M] - r}{\sigma_M} = \frac{\mu_M - r}{\sigma_M}.$$

To force the risk-adjusted expected value of the discounted stock price $S_M(1)$ to equal the current stock price $S_M(0)$, we get the same result[1]:

$$\lambda_M = \frac{E[R_M] - r}{\sigma_M} = \frac{\mu_M - r}{\sigma_M}.$$

For the stock index, the risk adjustment parameter λ is exactly the market price of risk (or the Sharpe ratio). This special case helps us to assign a definite meaning to the parameter λ, as an extension of the market price of risk (or Sharpe ratio) to risks with general-shaped distributions.

Now we consider an asset i on a one-period time horizon. Let $X=R_i$ be the return for asset i. Let $\rho_{i,M}$ be the correlation coefficient between R_i and R_M. Applying normalized

[1] However, for the exponential tilting in (eq-1) without the normalization procedure, using the stock price as reference Y would not yield another lognormal risk-adjusted stock price distribution.

exponential tilting on $X=R_i$ with the reference Y being *either* the return R_M *or* the stock index price $S_M(1)$, Theorem 1 states that

$$\lambda_i = \rho_{i,M} \cdot \lambda_M, \quad \text{or equivalently}$$

$$\frac{E[R_i]-r}{\sigma_i} = \rho_{i,M} \cdot \frac{E[R_M]-r}{\sigma_M}.$$

This is exactly the CAPM result for the expected return of stock i in relation to the market portfolio. Thus, Theorem 1 extends CAPM to the case that $\{R_i, R_M\}$ follow a normal copula, a more general case than the multivariate normal framework.

1.4 Valuation of Contingent Claims

If $X=h(Y)$ be a (monotone) function of the variable Y, we say that X is a (monotone) contingent claim of the underlying risk Y. The market price of risk for a monotone contingent claim $X=h(Y)$ is the same as that for the underlying risk Y.

Theorem 2. When valuing a monotone contingent claim $X=h(Y)$, the following are equivalent:

1) Apply normalized exponential tilting of Y w.r.t. Y, and calculate the expected value of $X=h(Y)$ under the risk-adjusted distribution of the underlying risk Y.
2) Apply normalized exponential tilting of X w.r.t. Y, and calculate the expected value of X under the risk-adjusted distribution of X.
3) Apply Wang transform to the CDF of Y, and calculate the expected value of $X=h(Y)$ under the risk-adjusted distribution for the underlying risk Y.
4) Applying Wang transform to the CDF of X, and calculate the expected value of X under the risk-adjusted distribution of X.

Consider the special case that $X = \max\{0, \quad Y-K\}\exp(-r)$, where Y represents the end-of-period stock price variable which has a lognormal distribution. X is the payoff of a European call option on Y with a strike price K. When valuing this contingent claim using the normalized exponential tilting with λ being the stock's market price of risk, we recover the Black-Scholes formula for European call options (also see Wang, 2000).

In the remaining of this paper we shall extend the normalized exponential tilting to multivariate cases.

PART 2. NORMALIZED EXPONENTIAL TILTING & DISTORTION --- THE MULTIVARIATE CASE

2.1 Normalized Multivariate Exponential Tilting

First we extend the exponential tilting concept to the multivariate case.

Definition 4.
Consider *n* variables $\{X_1, X_2, ..., X_n\}$ and *k* references $\{Y_1, Y_2, ..., Y_k\}$. The exponential tilting of $\{X_1, X_2, ..., X_n\}$ with respect to references $\{Y_1, Y_2, ..., Y_k\}$ is defined by the following p.d.f.:

$$f^*(x_1, x_2, \cdots, x_n) = c \cdot f(x_1, x_2, \cdots, x_n) \cdot \prod_{j=1}^{k} E\left[\exp(\lambda_j Y_j) \;\middle|\, X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n\right],$$

(eq-4)

where $\{\lambda_1, \lambda_2, ..., \lambda_k\}$ are real-valued parameters that control the magnitude of risk-adjustment, and *c* is a normalizing coefficient.

In this definition we leave much flexibility in the choice of the references $\{Y_1, Y_2, ..., Y_k\}$. For instance, one can choose the references $\{Y_1, Y_2, ..., Y_k\}$ to be the risks $\{X_1, X_2, ..., X_n\}$ themselves, the company aggregate, or the industry aggregate. Sometimes we can choose $\{Y_1, Y_2, ..., Y_k\}$ as the underlying risks for the contingent claims $\{X_1, X_2, ..., X_n\}$.

In order to get a meaningful interpretation (as well as cross-contract comparison) of the parameters $\{\lambda_1, \lambda_2, ..., \lambda_k\}$, we need to apply the normalization procedure to all references $\{Y_1, Y_2, ..., Y_k\}$.

Definition 5.
Assume that there exist standard normal variables $\{Z_1, Z_2, ..., Z_k\}$, such that

$$Y_1 = F_{Y_1}^{-1}(\Phi(Z_1)), \quad Y_2 = F_{Y_2}^{-1}(\Phi(Z_2)), \quad \cdots, \quad Y_k = F_{Y_k}^{-1}(\Phi(Z_k))$$

We define the normalized exponential tilting of $\{X_1, X_2, ..., X_n\}$ with respect to references $\{Y_1, Y_2, ..., Y_k\}$ by the following:

$$f^*(x_1, x_2, \cdots, x_n) = c \cdot f(x_1, x_2, \cdots, x_n) \cdot \prod_{j=1}^{k} E\left[\exp(\lambda_j Z_j) \;\middle|\, X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n\right].$$

(eq-5)

2.2 Multivariate Distortions

Now we extend the distortion method to multivariate cases.

Consider multivariate risks $\{X_1, X_2, \cdots, X_n\}$ that have marginal CDFs

$\{F_{X_1}(x_1), F_{X_2}(x_2), \cdots, F_{X_n}(x_n)\}$, respectively.

Assume that $\{X_1, X_2, \cdots, X_n\}$ have a joint CDF specified by

$$F_{X_1, X_2, \cdots, X_n}(x_1, x_2, \cdots, x_n) = C\big(F_{X_1}(x_1), \quad F_{X_2}(x_2), \quad \cdots, \quad F_{X_n}(x_n)\big),$$

where $C(\ldots)$ is a multivariate uniform distribution (or a copula function, e.g., Embrechts et al 2002).

Definition 6.

We define *separate* distortions $\{g_1, g_2, \cdots, g_n\}$ such that the resulting multivariate distribution has the following marginal distributions:

$$F_{X_j}^*(x_j) = g_j[F_{X_j}(x_j)],$$

and the same correlation structure in term of copula:

$$F_{X_1, X_2, \cdots, X_n}^*(x_1, x_2, \cdots, x_n) = C\big(F_{X_1}^*(x_1), \quad F_{X_2}^*(x_2), \quad \cdots, \quad F_{X_n}^*(x_n)\big).$$

Definition 7.

We define *joint* distortions $\{g_1, g_2, \cdots, g_n\}$ in terms of the joint p.d.f.:

$$f_{X_1, X_2, \cdots, X_n}^*(x_1, x_2, \cdots, x_n) = RN_{g_1, g_2, \cdots, g_n}(x_1, x_2, \cdots, x_n) \cdot f_{X_1, X_2, \cdots, X_n}(x_1, x_2, \cdots, x_n),$$

where the Radon-Nikodym derivative is given by:

- In the discrete case for the point $\vec{\mathrm{x}}_k = (x_{1;i}, x_{2;i}, \cdots, x_{n;i})$ we have

$$RN_{g_1, g_2, \cdots, g_n}(x_{1;i}, x_{2;i}, \cdots, x_{n;i}) = c \cdot \prod_{j=1}^{n} \frac{g_j\big(F_{X_j}(x_{j;i})\big) - g_j\big(F_{X_j}(x_{j;i-1})\big)}{F_{X_j}(x_{j;i}) - F_{X_j}(x_{j;i-1})}$$

- In the continuous case for the point $\vec{\mathrm{x}} = (x_1, x_2, \cdots, x_n)$ we have

$$RN_{g_1, g_2, \cdots, g_n}(x_1, x_2, \cdots, x_n) = c \cdot \prod_{j=1}^{n} g_j'\big(F_{X_j}(x_j)\big).$$

Theorem 3. When $\{X_1, X_2, \cdots, X_n\}$ have uncorrelated marginal distributions, both the separate distortions and the joint distortions yield the same adjusted multivariate probability distribution with uncorrelated marginal distributions

$$f^*(x_1, x_2, \cdots, x_n) = \prod_{j=1}^{n} f^*_{X_j}(x_j),$$

$$F^*_{X_j}(x_j) = g_j[F_{X_j}(x_j)], \text{ with } j = 1,2,\ldots,n.$$

Remark:

1) This result may have implications in the aggregation of risks. For instance, if insurance risks and market risks are assumed to be uncorrelated, then we can apply separate adjustments.
2) When $\{X_1, X_2\}$ are correlated, the separate distortions and the joint distortions can yield different results. Joint distortions reflect the inter-relation between X_1 and X_2 in the probability adjustment, while separate distortions do not. Consider the special case that $X_1 = X_2$, and $g_1 = g_2$ being the Wang transform with parameter λ. The joint distortions $\{g_1, g_2\}$ is equivalent to applying a single Wang transform to X_1 with parameter 2λ, while the separate distortions $\{g_1, g_2\}$ is equivalent to applying Wang transform to X_1 with parameter λ.

When $g_j(u) = \Phi\left[\Phi^{-1}(u) + \lambda_j\right]$, for $j=1, 2, ..., n$, we shall refer to the separate distortions in definition 6 as the *separate Wang transforms* with parameters $\{\lambda_1, \lambda_2, ..., \lambda_n\}$; and we shall refer to the joint distortions in definition 7 as the *joint Wang transforms* with parameters $\{\lambda_1, \lambda_2, ..., \lambda_n\}$.

2.3 Link between Exponential Tilting and Distortion – Multivariate Case

Theorem 4. Assume that the n variables and the k references

$$\{X_1, X_2, \cdots, X_n; Y_1, Y_2, \cdots, Y_k\}$$

follow a normal copula. The multivariate normalized exponential tilting (eq-5) is equivalent to applying *separate* distortions to X_i with:

$$g_i(u) = \Phi\left[\Phi^{-1}(u) + \beta_i\right], \text{ and } \beta_i = \sum_{j=1}^{k} \rho_{X_i, Y_j} \cdot \lambda_j, \text{ (for } i=1, 2, ..., n)$$

The correlation matrix between $\{X_1, X_2, \cdots, X_n\}$ is unchanged after the normalized exponential tilting:

$$\Sigma^* = \Sigma = \begin{pmatrix} 1 & \rho_{X_1,X_2} & \cdots & \rho_{X_1,X_n} \\ \rho_{X_1,X_2} & 1 & \cdots & \rho_{X_2,X_n} \\ \vdots & \vdots & \cdots & \vdots \\ \rho_{X_1,X_n} & \rho_{X_2,X_n} & \cdots & 1 \end{pmatrix}.$$

Example 1. Assume that the risks $\{X_1, X_2\}$ have a bivariate normal(0,1) with correlation coefficients:

$$\Sigma = \begin{pmatrix} 1 & \rho_{X_1,X_2} \\ \rho_{X_1,X_2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix}.$$

According to Theorem 4, under bivariate normalized exponential tilting (eq-5) with references $Y_1 = X_1$ and $Y_2 = X_2$, the adjusted joint distribution for $\{X_1^*, X_2^*\}$ is also bivariate normal with correlation coefficients:

$$\Sigma^* = \Sigma = \begin{pmatrix} 1 & \rho_{X_1,X_2} \\ \rho_{X_1,X_2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix}.$$

For illustration, we choose $\lambda_1 = 0.3$ and $\lambda_2 = 0.2$. The adjusted marginal distributions are equivalent to applying *separate* Wang transforms $F_{X_j}^*(x) = \Phi\left[\Phi^{-1}(F_{X_j}(x)) - \beta_j\right]$, for j =1, 2, with

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 & \rho_{X_1,X_2} \\ \rho_{X_1,X_2} & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \rho_{X_1,X_2}\lambda_2 \\ \rho_{X_1,X_2}\lambda_1 + \lambda_2 \end{pmatrix} = \begin{pmatrix} 0.42 \\ 0.38 \end{pmatrix}.$$

Figure 2. Scatter plot bivariate variables $\{X_1, X_2\}$, and Radon-Nikodym derivatives $RN(x_1, x_2)$

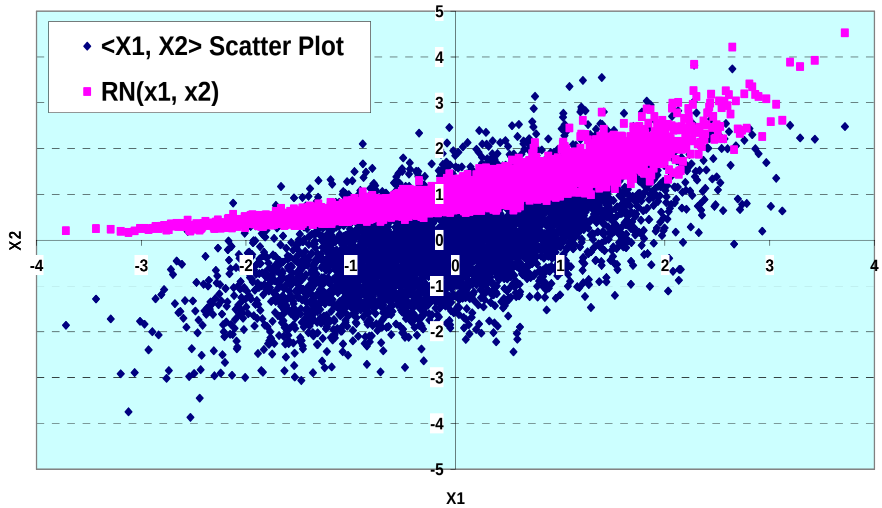


Figure 2 shows a scatter plot of $\{X_1, X_2\}$ and their corresponding Radon-Nikodym derivatives. The right-most blue diamond is a scatter plot of (x_1=3.195, x_2=2.505). The right-most red square gives its value of Radon-Nikodym derivative *RN*(3.195, 2.505)=3.884. One can see that the Radon-Nikodym derivatives increase exponentially when the point (x_1, x_2) moves from the lower left quadrant to the upper right quadrant.

2.4 Valuing Contingent Claims on Multiple Underlying Risks

Consider contingent claims on multiple underlying variables:

$$X_i = h_i(Y_1, Y_2, \cdots, Y_k),\ i=1, 2, \ldots, n.$$

When valuing the contingent claim $X_i = h_i(Y_1, Y_2, \cdots, Y_k)$, the market price of risk should be specified through the underlying risks $\{Y_1, Y_2, \cdots, Y_k\}$.

Theoretically, we should first adjust the multivariate probability measure for the underlying risks, and then valuing contingent claims as expected payoff under the risk-adjusted probability measure. Accordingly, we should first apply normalized exponential tilting of $\{Y_1, Y_2, \cdots, Y_k\}$ w.r.t. themselves, and calculate the expected value of $X_i = h_i(Y_1, Y_2, \cdots, Y_k)$ under the risk-adjusted distribution of the underlying risks $\{Y_1, Y_2, \cdots, Y_k\}$.

Theorem 5. If we let X_j=Y_j be the underlying risks themselves, for j=1, 2, ..., k, The multivariate normalized exponential tilting (eq-5) of $\{Y_1, Y_2, \cdots, Y_k\}$ w.r.t. themselves is equivalent to joint Wang transforms with parameters $\{\lambda_1, \lambda_2, ..., \lambda_k\}$.

This result has implications in pricing contingent claims on multiple underlying risks $\{Y_1, Y_2, \cdots, Y_k\}$.

> **Example 2.** Applications in Pricing Contingent Claims.
> Suppose that the underlying risks, (Y_1, Y_2), have the following bivariate empirical distribution. Note that (Y_1, Y_2) are correlated with a linear correlation coefficient of 0.38. However, their correlation structure does not follow a normal copula.

Scenario	Y_1	Y_2
1	-	1,954.08
2	-	2,239.22
3	-	2,974.21
4	-	3,275.38
5	-	3,351.93
6	-	6,526.96
7	-	9,542.63
8	-	13,999.95
9	-	14,279.63
10	-	14,519.32
11	-	16,179.92
12	-	19,134.14
13	-	35,071.98
14	-	57,591.43
15	-	62,967.38
16	-	82,638.17
17	-	248,909.05
18	638.80	3,331.31
19	1,533.11	2,047.14
20	5,110.36	1,159.07
21	6,387.95	2,152.74
22	6,387.95	8,940.58
23	8,943.13	4,949.35
24	11,498.32	-
25	15,331.09	-
26	27,279.12	-
27	35,772.54	5,634.79
28	93,264.12	24,115.73
29	102,207.25	6,287.06
30	191,638.60	34,096.74
31	246,010.43	232,641.59
32	511,036.26	39,161.24
33	511,036.26	150,301.30
34	662,650.50	73,140.35

- Contract #1 has a contingent payoff in the amount of Y_1 in excess of 200,000. That is, the payoff $X_1 = \max\{Y_1 - 200000, \quad 0\}$.
- Contract #2 has a contingent payoff of 50% of the amount of Y_2. That is, X_2=0.5Y_2
- Contract #3 has a contingent payoff in the amount of Y_1 in excess of 200,000, plus 50% of the amount of Y_2. Technically, Contract 3 is simply the combination of Contract #1 and Contract #2: X_3= X_1+X_2.

Without risk-adjustment, the expected payoffs for Contract #1, #2, #3 are $33,257, $17,399, $50,656, respectively.

Suppose that the market price of risk for the underlying risks Y_1 and Y_2 are λ_1=0.3 and λ_2=0.2, respectively. We derive a risk-adjusted distribution by applying normalized exponential tilting of Y_1 and Y_2 with respect to themselves, using λ_1=0.3 and λ_2=0.2.

Theorem 5 facilitates a numerical method for calculating the risk adjusted probabilities for each of the 34 scenarios. Based on the adjusted probabilities for each scenario, we calculated the prices for Contract #1, #2, #3 being $68,240, $24,847, and $93,087 respectively.

	Expected Payoff of Contract #1	Expected Payoff of Contract #2	Expected Payoff of Contract #3
No Risk Adjustment	$ 33,257	$ 17,399	$ 50,656
With Risk Adjustment	$ 68,240	$ 24,847	$ 93,087
Loading	105%	43%	84%

Note that the obtained prices are additive. Indeed, the only way to ensure price additivity is by a change of bivariate probability measure.

Example 3. Numerical Techniques Involving Discrete Distributions.

Consider the following bivariate distribution (that does not follow a normal copula).

	X_2=1	X_2=2	X_2=3	X_2=4	X_2=5
X_1=1	*0.20*	*0.07*	*0.06*	*0.05*	*0.04*
X_1=2	*0.06*	*0.05*	*0.04*	*0.03*	*0.03*
X_1=3	*0.05*	*0.04*	*0.03*	*0.03*	*0.02*
X_1=4	*0.03*	*0.03*	*0.02*	*0.02*	*0.01*
X_1=5	*0.03*	*0.02*	*0.01*	*0.02*	*0.01*

We want to compute the adjusted joint distribution for the multivariate normalized exponential tilting of (X_1, X_2), with reference to themselves, and with λ_1=0.3 and λ_2=0.2.

We first apply the Wang transform to X_1 with λ_1=0.3.

$X_1 = x_1$	$f(x_1)$	$F(x_1)$	$F^*(x_1)$	$f^*(x_1)$
1	0.42	0.42	0.30787	0.30787
2	0.21	0.63	0.51271	0.20483
3	0.17	0.80	0.70596	0.19325
4	0.11	0.91	0.85101	0.14505
5	0.09	1.00	1.00000	0.14899

We then apply the Wang transform to X_2 with λ_2=0.2.

$X_2 = x_2$	$f(x_2)$	$F(x_2)$	$F^*(x_2)$	$f^*(x_2)$
1	0.37	0.37	0.29741	0.29741
2	0.21	0.58	0.50076	0.20334
3	0.16	0.74	0.67124	0.17049
4	0.15	0.89	0.84768	0.17644
5	0.11	1.00	1.00000	0.15232

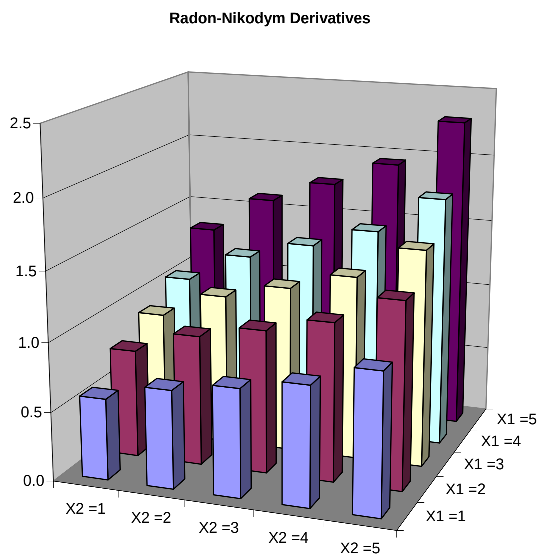
According to Theorem 4, the bivariate Radon-Nikodym derivatives are:

$$RN_g(x_1, x_2) = \frac{f^*_{X_1,X_2}(x_1, x_2)}{f_{X_1,X_2}(x_1, x_2)} = c \cdot \frac{f^*_{X_1}(x_1)}{f_{X_2}(x_1)} \cdot \frac{f^*_{X_2}(x_2)}{f_{X_2}(x_2)}.$$

The final risk-adjusted joint probability (density) function is:

	X_2=1	X_2=2	X_2=3	X_2=4	X_2=5
X_1=1	0.1178	0.0497	0.0469	0.0431	0.0406
X_1=2	0.0470	0.0472	0.0416	0.0344	0.0405
X_1=3	0.0457	0.0440	0.0363	0.0401	0.0315
X_1=4	0.0318	0.0383	0.0281	0.0310	0.0183
X_1=5	0.0399	0.0321	0.0176	0.0389	0.0229

Figure 3. The Bivariate Radon-Nikodym Derivatives



As shown in Figure 3, the Radon-Nikodym derivative increases to its highest value at $\{X_1{=}5,\ X_2{=}5\}$, indicating the largest relative risk adjustment at the joint tail of the bivariate variables.

2.5 Portfolio Risk Measures

Here we mention briefly that multivariate distortions can be employed to define portfolio risk measures.

Consider a portfolio of risks $\{X_1, X_2, \cdots, X_n\}$ with CDFs

$$\{F_{X_1}(x_1), F_{X_2}(x_2), \cdots, F_{X_n}(x_n)\}, \text{ respectively.}$$

Assume that $\{X_1, X_2, \cdots, X_n\}$ have a joint CDF specified by

$$F_{X_1,X_2,\cdots,X_n}(x_1, x_2, \cdots, x_n) = C\big(F_{X_1}(x_1), \quad F_{X_2}(x_2), \quad \cdots, \quad F_{X_n}(x_n)\big),$$

where $C(\ldots)$ is a multivariate uniform distribution (or a copula function).

Let $\{X_1^*, X_2^*, \cdots, X_n^*\}$ have a multivariate distribution produced by *separate* distortions $\{g_1, g_2, \cdots, g_n\}$ on $\{X_1, X_2, \cdots, X_n\}$as in definition 6. That is,

$$F_{X_1^*,X_2^*,\cdots,X_n^*}(x_1, x_2, \cdots, x_n) = C\big(g_1[F_{X_1}(x_1)], \quad g_2[F_{X_2}(x_2)], \quad \cdots, \quad g_n[F_{X_n}(x_n)]\big)$$

with marginal distributions:

$$F_{X_j^*}(x_j) = g_j[F_{X_j}(x_j)].$$

Let $W = X_1^* + X_2^* + \cdots + X_n^*$ represent the sum of random variables $\{X_1^*, X_2^*, \cdots, X_n^*\}$. Any risk measure on the aggregate variable W defines a portfolio risk measure for $\{X_1, X_2, \cdots, X_n\}$. We can even apply another distortion function h on the CDF of W, and define the portfolio risk measure of $\{X_1, X_2, \cdots, X_n\}$ as the expected value of W under the risk-adjusted distribution:

$$PRM = \int_{-\infty}^{0} \left(h(F_W(x)) - 1\right) dx + \int_{0}^{\infty} h(F_W(x)) dx .$$

Such a portfolio risk measure has applications in measuring portfolio risks, in portfolio optimization, and in allocating risk capitals. That is an important subject and deserves a separate discussion.

Conclusion:

We have introduced the concept of normalized exponential tilting, and established an important link with probability distortions, first for the univariate case, then for the multivariate case. Normalized exponential titling provides a general framework for pricing risks with respect to some reference risks, or for valuing contingent claims on some underlying risks. The paper also provides efficient numerical routines for adjusting multivariate probability distributions.

In a sequel paper, we shall explore further normalized exponential tilting of multivariate risks, with practical considerations such as adjustment for parameter uncertainty and interactions among reference risks.

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