Probability Transforms with Elliptical Generators

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Introduction

- We introduce the notion of *elliptical transformations* for possible applications in constructing insurance premium principles.

- The idea is to distort the real probability measure of a risk X based on the relative ratio of a density generator of a member of the class of elliptical distributions to that of the Normal distribution.

- We examine premium principle implied by this elliptical transformation:
  - Wang premium principle;
  - Esscher premium principle;
  - Wang student-t distortion principle.

- For location-scale families, this leads to the standard deviation principle.

Risk-adjusted premiums

- Premium principle π is a mapping from Γ of real-valued r.v's defined on $(\Omega, \mathcal{F})$ to the set of reals, $\pi : \Gamma \longrightarrow R$, so that $\pi[X] \in R$, is the assigned premium.

- Risk-adjusted premiums are computed based on expectation w.r.t. Q:

$$\pi[X] = \mathsf{E}_Q[X] = \mathsf{E}\left[\Psi X\right]$$

  where Ψ is a positive r.v. (Radon-Nikodym derivative). See Gerber and Pafumi (1998).

- Ψ has the form $\Psi = \dfrac{h(X;\lambda)}{\mathsf{E}\left[h(X;\lambda)\right]}$ for some function h of the r.v. X and parameter λ.

- Heilman (1985) and Hürlimann (2004).

Premium principles

- Premium principles are well-discussed in:
  - Goovaerts, DeVijlder and Haezendonck (1984)
  - Kaas, van Heerwaarden and Goovaerts (1994)
  - Young (2004), Encyclopedia of Actuarial Science
  - Kaas, Goovaerts, Dhaene, and Denuit (2001)

Family of elliptical distributions

- X is said to be elliptical with parameters μ and σ^2 if char. function is $\mathsf{E}[\exp(itX)] = \exp(it\mu) \cdot \psi(t^2\sigma^2)$ for some scalar function ψ.

- Notation: $X \sim E(\mu, \sigma^2, \psi)$

- If the density exists, it has the form $f_X(x) = \frac{C}{\sigma} g\left[\left(\frac{x-\mu}{\sigma}\right)^2\right]$ for some non-negative function $g(\cdot)$ satisfying the condition $0 < \int_0^\infty z^{-1/2} g(z)\, dz < \infty$ and normalizing constant $C = \left[\int_0^\infty z^{-1/2} g(z)\, dz\right]^{-1}$.

- Any non-negative function $g(\cdot)$ can be used to define a one-dimensional density of an elliptical distribution.

- $g(\cdot)$ is called the *density generator*. One then sometimes writes $X \sim E(\mu, \sigma^2, g)$

Normal and Student-t distributions

- Normal distribution: $X \sim N\left(\mu, \sigma^2\right)$
  - density generator: $g_N(u) = \exp(-u/2)$
  - normalizing constant: $C = \dfrac{1}{\sqrt{2\pi}}$
- Student-t distribution:
  - density generator: $g(u) = \left(1 + \dfrac{u}{m}\right)^{-(m+1)/2}$, $m > 0$ an integer
  - normalizing constant: $C = \dfrac{\Gamma\left((m+1)/2\right)}{\sqrt{m\pi}\,\Gamma\left(m/2\right)}$

Other known elliptical distributions

Family	Density generators $g(u)$
Bessel	$g(u) = (u/b)^{a/2} K_a \left[(u/b)^{1/2} \right]$, $a > -1/2, b > 0$ where $K_a(\cdot)$ is the modified Bessel function of the 3rd kind
Cauchy	$g(u) = (1+u)^{-1}$
Exponential Power	$g(u) = \exp[-r(u)^s]$, $r, s > 0$
Laplace	$g(u) = \exp(-\lvert u \rvert)$
Logistic	$g(u) = \dfrac{\exp(-u)}{[1+\exp(-u)]^2}$

Elliptical transforms

- Let $g_Z(u)$ be density generator of spherical r.v. $Z \sim E(0, 1, g_Z)$.

- Ratio of density generators g_Z and g_N:

$$h_{g_Z}(X;\lambda) = \frac{g_Z\left[\left(\Phi^{-1}\left(\overline{F}_X(X)\right) + \lambda\right)^2\right]}{g_N\left[\left(\Phi^{-1}\left(\overline{F}_X(X)\right)\right)^2\right]}$$

  for some non-negative parameter $\lambda \geq 0$ and where $\Phi(\cdot)$ is the c.d.f of standard Normal and g_N is the density generator of Normal.

- Expectation of this ratio can be expressed as

$$\mathsf{E}\left[h_{g_Z}(X;\lambda)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_Z(z^2)\, dz.$$

Some illustrative examples

- **Example 3.1:** *Normal-to-Normal Generators*

$$h_{g_N}(X;\lambda) = \exp\left[-\lambda\left(\Phi^{-1}\left(\overline{F}_X(X)\right) + \tfrac{1}{2}\lambda\right)\right] = e^{-\lambda^2/2}\exp\left[-\lambda\Phi^{-1}\left(\overline{F}_X(X)\right)\right].$$

It is easy to see that in this case, $\mathsf{E}\left[h_{g_Z}(X;\lambda)\right] = 1.\blacksquare$

- **Example 3.2:** *Student-t-to-Normal Generators*

$$h_{g_Z}(X;\lambda) = \frac{\exp\left[\frac{1}{2}\left(\Phi^{-1}\left(\overline{F}_X(X)\right)\right)^2\right]}{\left[1 + \frac{1}{m}\left(\Phi^{-1}\left(\overline{F}_X(X)\right) + \lambda\right)^2\right]^{(m+1)/2}}.$$

Applying Theorem 1 to solve for the expectation, we have

$$\mathsf{E}\left[h_{g_Z}(X;\lambda)\right] = \sqrt{\frac{m}{2}}\frac{\Gamma(m/2)}{\Gamma((m+1)/2)}.$$

Now interestingly, the case where $m = 1$ leads us to the Cauchy-to-Normal generators:

$$\mathsf{E}\left[h_{g_Z}(X;\lambda)\right] = \sqrt{\frac{1}{2\pi}\frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1)}} = \sqrt{\frac{\pi}{2}}.$$

Also, in the limiting case where $m \longrightarrow \infty$, this leads us back to the Normal-to-Normal density generators. ■

- **Example 3.3:** *Exponential Power-to-Normal Generators*

$$h_{g_Z}(X;\lambda) = \exp\left\{-\left[r\left(\Phi^{-1}(\overline{F}_X(X)) + \lambda\right)^{2s} - \tfrac{1}{2}\left(\Phi^{-1}(\overline{F}_X(X))\right)^{2}\right]\right\}.$$

Applying Theorem 1 to solve for the expectation, we have

$$\mathsf{E}\left[h_{g_Z}(X;\lambda)\right] = \frac{1}{s\sqrt{2\pi}} r^{1-(1/2s)}\Gamma\left(\frac{1}{2s}\right).$$

The case where $r = 1/2$ and $s = 1$ leads us to the Normal distribution case. ■

Transformed distributions

- Define the transformed r.v. X^* to be one with a (transformed) density:

$$f_{X^*}(x) = C \times \frac{g_Z\left[\left(\Phi^{-1}\left(\overline{F}_X\left(X\right)\right) + \lambda\right)^2\right]}{g_N\left[\left(\Phi^{-1}\left(\overline{F}_X\left(X\right)\right)\right)^2\right]} \times f_X(x)$$

- By recognizing that $h_{g_Z}(X;\lambda) = \frac{g_Z\left[\left(\Phi^{-1}\left(\overline{F}_X(X)\right)+\lambda\right)^2\right]}{g_N\left[\left(\Phi^{-1}\left(\overline{F}_X(X)\right)\right)^2\right]}$, the normalizing constant $C = \frac{1}{\mathsf{E}\left[h_{g_Z}(X;\lambda)\right]}$.

- We can, as a matter of fact, find an expression for the d.f. of the (transformed) r.v. X^*:

$$\overline{F}_{X^*}(x) = \int_x^\infty f_{X^*}(v)\,dv = F_Z\left[\Phi^{-1}\left(\overline{F}_X(x)\right) + \lambda\right].$$

The work of Landsman (2004)

- Landsman (2004) [*elliptical tilting*] also proposed to use transformation of densities of elliptical r.v.'s: $\dfrac{g\left[\left((x-\mu)/\sigma\right)^2 - 2\lambda x\right]}{g\left[\left((x-\mu)/\sigma\right)^2\right]}$.
  - Generalizes Esscher transform for elliptical distributions.
  - Generalizes variance premium principle applied to elliptical distributions.

- Several differences between Landsman's elliptical tilting with what we propose:
  - Two different density generators
  - Translation of the distribution by introducing a shift parameter λ
  - Not limited to transforming elliptical
  - Variance versus standard deviation premium principle

Premium principle implied by the elliptical transformation

- Let X^* be the transformed random variable of X according to the elliptical transformation. Then the expectation of X^* is defined to be the premium principle implied by this transformation:

$$\pi[X] = \mathsf{E}(X^*) = \mathsf{E}\left[\frac{h_{g_Z}(X;\lambda)}{\mathsf{E}\left[h_{g_Z}(X;\lambda)\right]} \cdot X\right].$$

- Observe that we can also derive the premium (or expectation of the transformed distribution) using

$$\pi[X] = -\int_{-\infty}^{0} F_Z\left[\Phi^{-1}\left(\overline{F}_X(x)\right) + \lambda\right] dx + \int_0^{\infty} \overline{F}_Z\left[\Phi^{-1}\left(\overline{F}_X(x)\right) + \lambda\right] dx$$

which reduces to just $\int_0^{\infty} \overline{F}_Z\left[\Phi^{-1}\left(\overline{F}_X(x)\right) + \lambda\right] dx$ for r.v.'s with non-negative support.

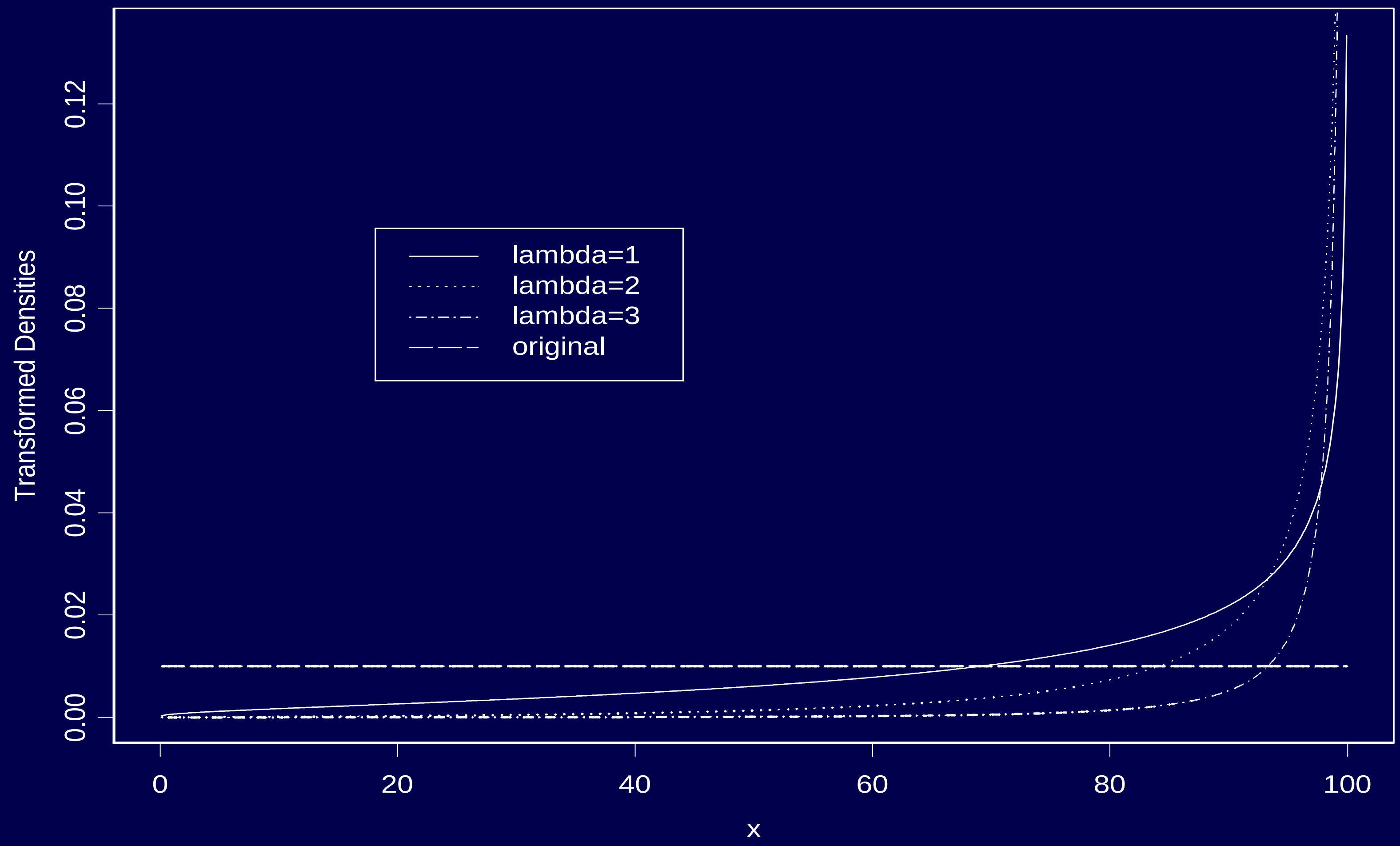

Figure 1. Elliptical transformation using the Normal density generator

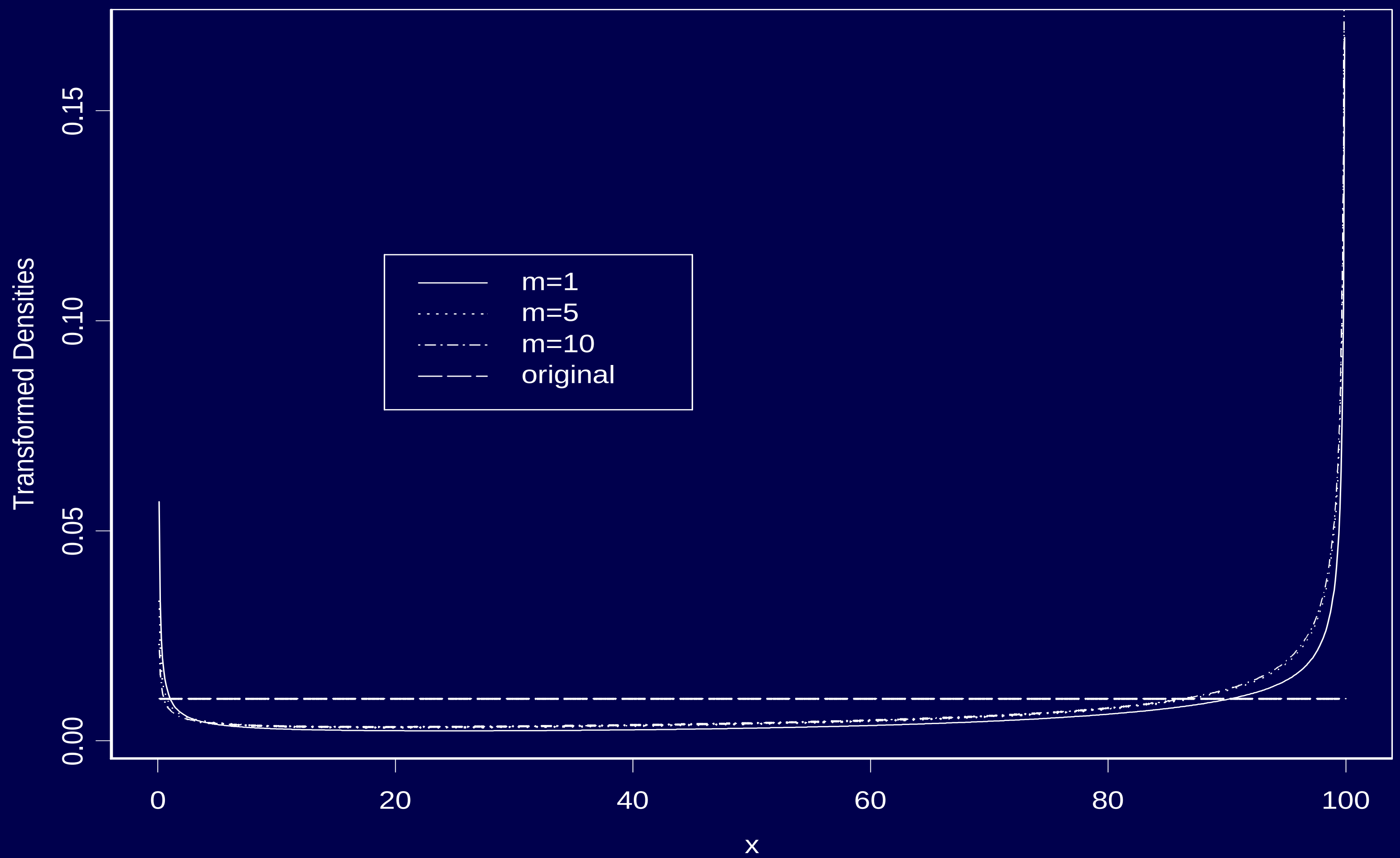

Figure 2. Elliptical transformation using the Student-t density generator

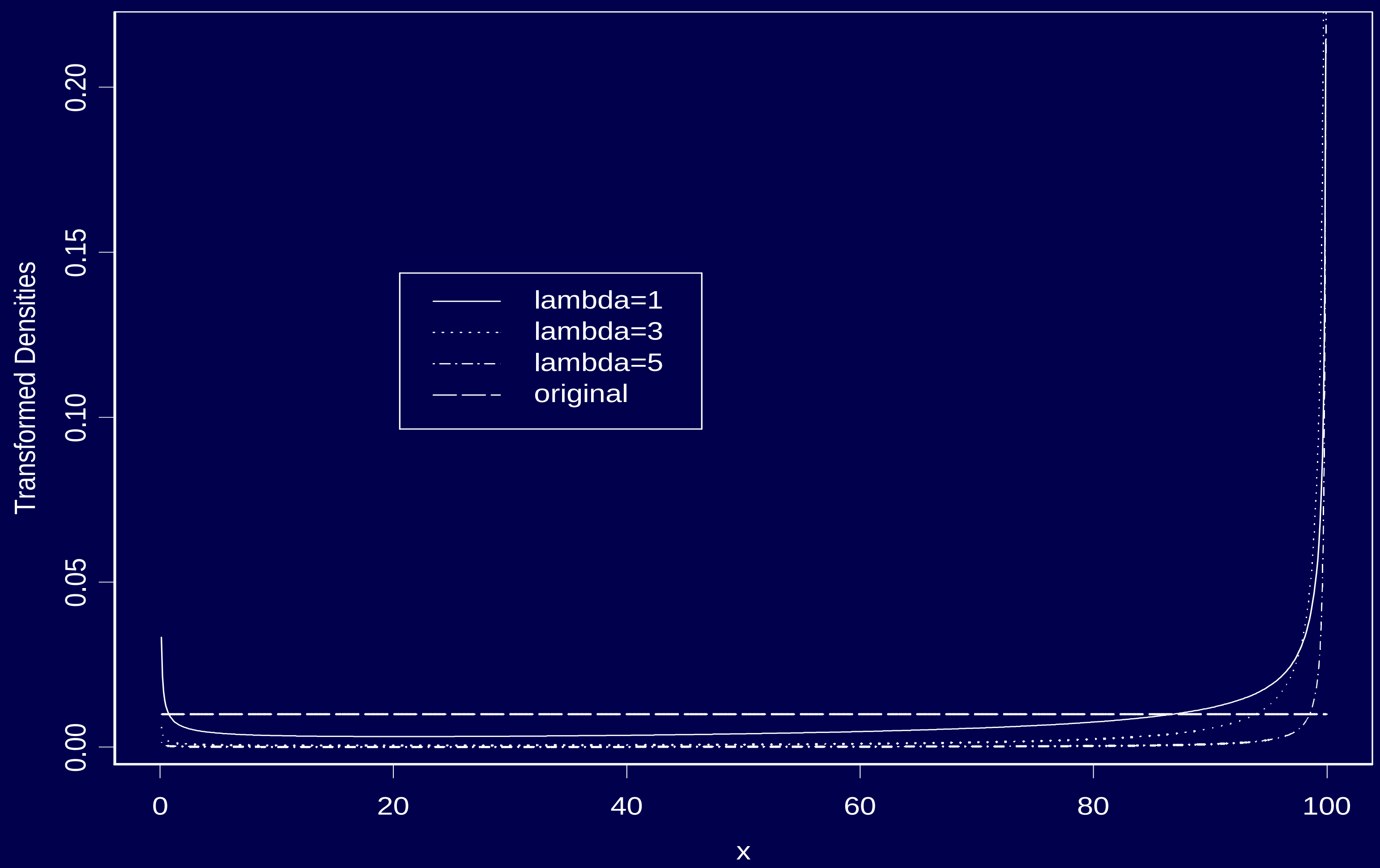

Figure 3. Using the Student-t density generator but varying lambda

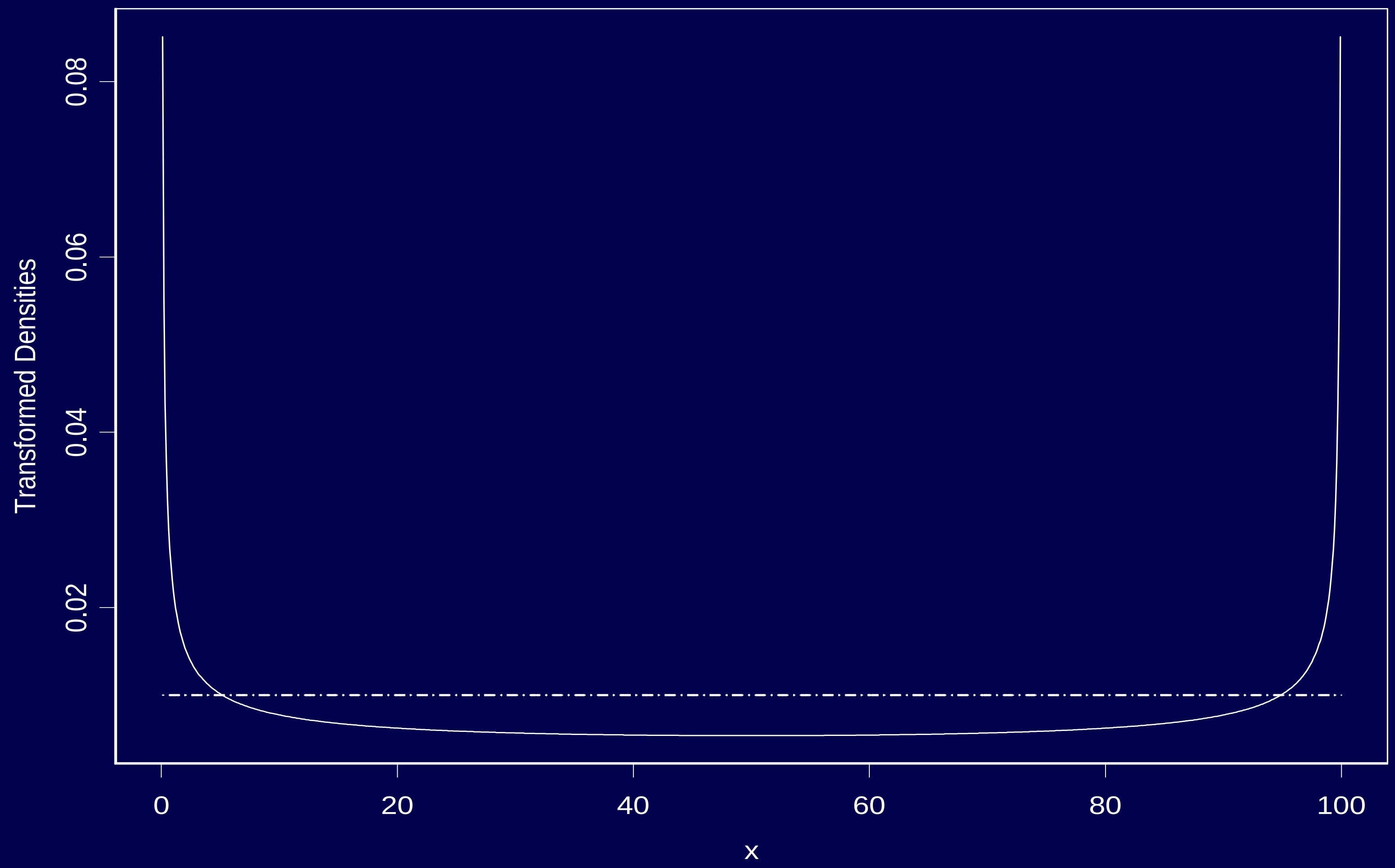

Figure 4. The case of lambda = 0

Recovering familiar premium principles

- **Example 3.4:** *Wang Premium Principle* Choose g_Z to be the density generator of a Normal:

$$\overline{F}_{X^*}(x) = \Phi\left[\Phi^{-1}\left(\overline{F}_X(x)\right) + \lambda\right].$$

See Wang (1996) and Wang (2000).

- **Example 3.5:** *Wang's Student-t Distortion Premium Principle* Choose g_Z to be the density generator of a Student-t distribution with m d.f., then we have as in Wang (2004)

$$\overline{F}_{X^*}(x) = Q\left[\Phi^{-1}\left(\overline{F}_X(x)\right) + \lambda\right],$$

where, following Wang's notation, $Q(\cdot)$ denotes the d.f. of a Student-t with m degrees of freedom. Wang actually has set $\lambda = 0$ and used $\overline{F}_{X^*}(x) = Q\left[\Phi^{-1}\left(\overline{F}_X(x)\right)\right]$ instead.

- **Example 3.6:** *Esscher Premium Principle* In the special case where X is $N\left(\mu,\sigma^2\right)$, the elliptical transformation leads to the Esscher transform:

$$\frac{h_{g_Z}(X;\lambda)}{\mathsf{E}\left[h_{g_Z}(X;\lambda)\right]} = \frac{e^{-\lambda^2/2}\exp\left[-\lambda\Phi^{-1}\left(\overline{F}_X(X)\right)\right]}{\mathsf{E}\left[e^{-\lambda^2/2}\exp\left[-\lambda\Phi^{-1}\left(\overline{F}_X(X)\right)\right]\right]} = \frac{\exp\left(\frac{\lambda}{\sigma}X\right)}{\mathsf{E}\left[\exp\left(\frac{\lambda}{\sigma}X\right)\right]}.$$

See Esscher (1932) and also more recent articles, for example, Gerber and Shiu (1994).

Location-scale families

- Let X belong to location-scale family so that $\overline{F}_X(x) = \overline{F}_{Z^*}\left(\frac{x-\mu}{\sigma}\right)$ for some $Z^* = \frac{X-\mu}{\sigma}$, independent of μ and σ. Then

$$\pi[X] = \mu + \mathsf{E}_Z\left[\overline{F}_{Z^*}^{-1}(\Phi(Z-\lambda))\right] \times \sigma.$$

- In case X is $N\left(\mu, \sigma^2\right)$ and we choose Z to be the standard Normal:

$$\mathsf{E}_Z\left[\overline{F}_{Z^*}^{-1}(\Phi(Z-\lambda))\right] = \mathsf{E}_Z(\lambda - Z) = \lambda - \mathsf{E}_Z(Z) = \lambda,$$

and the resulting premium principle leads to: $\pi[X] = \mu + \lambda\sigma$.

- Here we have $\lambda = \frac{\pi[X]-\mu}{\sigma}$, risk premium per unit of risk, σ.

Location-scale families

- This paper introduces notion of elliptical transformation leading to a premium principle which in some sense generalizes the familiar Wang transformation as well as the premium principle introduced by Wang (2004) using the Student-t distortion function.

- We also note that in the special case of transforming location-scale families, the resulting premium principle is the standard deviation premium principle.

- Transformation introduces a heavy penalty on the extreme right tails of the distribution but also encourages small losses by placing relatively reasonable weights on the extreme left tails of the distribution.

- How much this penalty depends on the choice of the density generator together with the parameter λ which in a sense gives a measure of aversion to the level of risk of the insurer. This parameter introduces a shift in the distribution of the risk.

- We also show that the elliptical transformation recovers many other familiar premium principles.
- Elliptical transforms may also be applied as a risk measure to compute economic capital.
- In the future, it will be an interesting work to examine some properties of this premium principle.

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