Two binomial methods for evaluating the aggregate claims distribution in De Pril's individual risk model

Sundt, Bjørn
Vital Forsikring ASA
P.O.Box 250, N-1326 Lysaker, Norway
Phone: (+47) 67 83 44 71
Fax: (+47) 67 83 42 60
E-mail: bjoern.sundt@vital.no

Vernic, Raluca
Department of Mathematics and Informatics
'Ovidius' University of Constanta
124, Bd. Mamaia, 8700 Constanta, Romania
Fax: (+40) 241 618372
E-mail: rvernic@univ-ovidius.ro

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Abstract

In this paper, we deduce two new methods for evaluating the aggregate claims distribution in De Pril's individual risk model and compare them with some other methods by counting the number of dot operations (multiplications and divisions). Finally we consider one of the new methods in the individual life model.

Keywords: De Pril's individual risk model, aggregate claims distribution, recursions.

1 Introduction

1A. In the present paper, we shall represent probability distributions on the non-negative integers by their probability function, and we shall therefore normally mean the probability function when we refer to a distribution.

De Pril (1989) introduced a two-way individual risk model. We consider a portfolio of independent insurance policies during a specified period. It is assumed that a policy can have at most one claim during the period. The claim amounts are positive and integer-valued. In cell (i, j) for $i = 1, 2, \dots, I$ and $j = 1, 2, \dots, J$, the probability that a claim occurs, is π_j and the claim amount distribution given that a claim has occurred, is h_i. In this cell, there are n_{ij} policies. We are interested in the aggregate claims distribution f of the whole portfolio. De Pril (1989) presented two exact methods for recursive evaluation of f and discussed three approximations. Dhaene & Vandebroek (1995) presented another exact method that could be more efficient than De Pril's exact methods.

In the present paper, we shall introduce two other exact methods and compare them with the methods of De Pril (1989) and Dhaene & Vandebroek (1995).

The first one is based on developing a recursive method for evaluating the probability function f_i of the subportfolio of the policies with probability function h_i. Finally, we evaluate the convolution $f = *_{i=1}^{I} f_i$. For the evaluation of f_i, we utilise that the number of claims in cell (i, j) is binomially distributed. Thus, f_i is a compound distribution where the counting distribution is a convolution of binomial distributions and the severity distribution is h_i, and we apply methodology from Sundt (1992) to deduce a recursion for f_i.

In the second method, we introduce some features from the first method into Dhaene-Vandebroek's method.

Because of the relation between the new methods and the binomial distribution, we shall refer to them as binomial methods.

1B. After having introduced some notation in Section 2, we deduce the first binomial method in Section 3. In this connection, we also recapitulate some results from Sundt (1992). In Section 4, we describe some of the earlier methods for evaluation of f under the present conditions. Section 5 introduces the second binomial method. In Section 6, the methods will be compared by counting the number of dot operations (multiplication and division). Although this criterion is not perfect, it can give an idea of the efficiency of the methods. However, to not put too much into it and not drowning in details, we restrict to the case when the support of all the h_is is the set of positive integers, and compare the number of dot operations for the methods in an asymptotic setting. In this situation, we show that the winner is normally the first binomial method when there is a high number of non-empty cells for each i, and Dhaene-Vandebroek's method when there is a low number of non-empty cells for each i. It is also shown that sometimes it can be efficient to combine Dhaeme-Vandebroek's method with one of the binomial methods by using the binomial method for is with many non-empty cells and Dhaene-Vandebroek's method for is with few non-empty cells. Finally, in Section 7, we discuss application of the first binomial method in the individual life model where each severity distribution is concentrated in one point.

2 Notation

Let $\mathcal{P}_0$ denote the class of distributions on the non-negative integers with a positive probability at zero and $\mathcal{P}_+$ the class of distributions on the positive integers.

We denote by $p \vee h$ a compound distribution with counting distribution $p \in \mathcal{P}_0$ and severity distribution $h \in \mathcal{P}_+$, that is, $p \vee h = \sum_{n=0}^{\infty} p(n) h^{n*}$. As $h^{n*}(x) = 0$ when $n > x$, we have

$$(p \vee h)(x) = \sum_{n=0}^{x} p(n) h^{n*}(x). \qquad (x = 0, 1, 2, \ldots)$$

In subsection 1A, we have already introduced some notation for our De Pril model. Let us now introduce some more. To see how the methods are affected by a finite range of the h_is, we assume that h_i has range $\{1, 2, \ldots, m_i\}$, where m_i could either be some positive integer or infinity. We let g_{ij} denote the aggregate claims distribution for a policy in cell (i, j), that is, $g_{ij} = p_j \vee h_i$, where p_j is the claim number distribution of the policy, that is, the Bernoulli distribution given by $p_j(1) = 1 - p_j(0) = \pi_j$. Then we have

$$g_{ij}(x) = \begin{cases} 1 - \pi_j & (x = 0) \\ \pi_j h_i(x). & (x = 1, 2, \ldots, m_i) \end{cases} \tag{2.1}$$

The aggregate claims distribution of all the policies in cell (i, j) is $f_{ij} = g_{ij}^{n_{ij}*}$. Thus, we have $f_i = *_{j=1}^{J} f_{ij} = *_{j=1}^{J} g_{ij}^{n_{ij}*}$. Continuing on this, we obtain

$$f = *_{i=1}^{I} f_i = *_{i=1}^{I} *_{j=1}^{J} f_{ij} = *_{i=1}^{I} *_{j=1}^{J} g_{ij}^{n_{ij}*} = *_{i=1}^{I} *_{j=1}^{J} (p_j \vee h_i)^{n_{ij}*} =$$
$$*_{i=1}^{I} \left(\left(*_{j=1}^{J} p_j^{n_{ij}*} \right) \vee h_i \right) = *_{i=1}^{I} \left(\left(*_{j=1}^{J} q_{ij} \right) \vee h_i \right) = *_{i=1}^{I} (q_i \vee h_i),$$

where we have introduced the distribution of the number of claims in cell (i,j), $q_{ij} = p_j^{n_{ij}*}$, and the distribution of the number of claims from the union of these cells for fixed i, $q_i = *_{j=1}^J q_{ij}$.

For $i = 1, 2, \dots, I$, we let J_i be the number of non-empty cells among the cells $(i,1), (i,2), \dots, (i,J)$ and $n_i = \sum_{j=1}^J n_{ij}$ the total number of policies in these cells, and we let $J_\bullet = \sum_{i=1}^I J_i$ be the total number of non-empty cells.

We let $[z]$ denote the greatest integer less than or equal to z and by $\{z\}$ the smallest integer greater than or equal to z; when x is a positive integer and $m_i = \infty$, we let $\{x/m_i\} = 1$.

We let $\sum_{k=a}^b = 0$ when $b < a$.

3 The first binomial method

3A. Sundt (1992) studied the classes $\mathcal{R}_k$ of distributions $p \in \mathcal{P}_0$ that satisfy a recursion in the form

$$p(n) = \sum_{u=1}^{\min(k,n)} \left(a(u) + \frac{b(u)}{n} \right) p(n-u); \qquad (n = 1, 2, \dots) \tag{3.1}$$

we denote this distribution by $R_k[a,b]$, and when $k = 1$, we drop the argument of a and b. In particular, he proved the following results:

Lemma 3.1. *A distribution in $\mathcal{P}_0$ can be represented as $R_k[a,b]$ if and only if its probability generating function ψ satisfies*

$$\frac{d}{ds} \ln \psi(s) = \frac{\sum_{u=1}^k (ua(u) + b(u)) s^{u-1}}{1 - \sum_{u=1}^k a(u) s^u}.$$

Lemma 3.2. *The convolution between a distribution in $\mathcal{R}_k$ and a distribution in $\mathcal{R}_l$ is a distribution in $\mathcal{R}_{k+l}$.*

Lemma 3.3. *If p is $R_k[a,b]$ and $h \in \mathcal{P}_+$, then $p \vee h$ is $R_\infty[c,d]$ with*

$$c(y) = \sum_{u=1}^{\min(k,y)} a(u) h^{u*}(y); \quad d(y) = y \sum_{u=1}^{\min(k,y)} \frac{b(u)}{u} h^{u*}(y). \quad (y = 1, 2, \dots) \tag{3.2}$$

Lemma 3.2 easily follows from Lemma 3.1.

Panjer (1981) studied the class $\mathcal{R}_1$. In particular, he proved the following lemma.

Lemma 3.4. *The binomial distribution*

$$p(n) = \binom{m}{n} \pi^n (1-\pi)^{m-n} \qquad (n = 0,1,2,\ldots,m;\ 0 < \pi < 1;\ m = 1,2,\ldots) \tag{3.3}$$

is $R_1[-\pi/(1-\pi), (m+1)\pi/(1-\pi)]$.

We denote this distribution by $\operatorname{bin}(m,\pi)$.

We now have the main tools for evaluation of f_i as defined in subsection 1A: We express the binomial claim number distribution of the policies in cell (i,j) as an $\mathcal{R}_1$ distribution. Then we express the distribution of the aggregate number of claims from these cells as an $\mathcal{R}_{J_i}$ distribution, and finally we evaluate f_i recursively by Lemma 3.3 and (3.1).

3B. Theorem 8 in Sundt (1992) gives explicit expressions for the functions a and b when expressing the convolution of k $\mathcal{R}_1$ distributions in the form $R_k[a,b]$. In practice, it seems more convenient to do this recursively. We shall deduce such a recursion and start with the following lemma.

Lemma 3.5. *The convolution of* $R_{k-1}[a,b]$ *and* $R_1[c,d]$ *is* $R_k[\alpha,\beta]$ *with*

$$\alpha(u) = a(u) - ca(u-1) \tag{3.4}$$

$$\beta(u) = b(u) - cb(u-1) - da(u-1). \tag{3.5}$$

for $u = 1,2,\ldots,k$ *with* $a(0) = -1$ *and* $a(k) = b(k) = b(0) = 0$.

Proof. Let ψ_{k-1}, ψ, and ψ_k denote the probability generating functions of $R_{k-1}[a,b]$, $R_1[c,d]$, and $R_k[\alpha,\beta]$. By using Lemma 3.1, we obtain

$$\frac{d}{ds}\ln\psi_k(s) = \frac{d}{ds}\ln\psi_{k-1}(s)\psi(s) = \frac{d}{ds}\ln\psi_{k-1}(s) + \frac{d}{ds}\ln\psi(s) =$$

$$\frac{\sum_{u=1}^{k-1}(ua(u)+b(u))s^{u-1}}{1-\sum_{u=1}^{k-1}a(u)s^u} + \frac{c+d}{1-cs} =$$

$$\frac{\left(\sum_{u=1}^{k-1}(ua(u)+b(u))s^{u-1}\right)(1-cs) + (c+d)\left(1-\sum_{u=1}^{k-1}a(u)s^u\right)}{\left(1-\sum_{u=1}^{k-1}a(u)s^u\right)(1-cs)} =$$

$$\frac{c+d+\sum_{u=1}^{k-1}\left((ua(u)+b(u))s^{u-1} - (c(ua(u)+b(u)) + (c+d)a(u))s^u\right)}{1-\left(cs+\sum_{u=1}^{k-1}(a(u)s^u - ca(u)s^{u+1})\right)} =$$

$$\frac{\sum_{u=1}^{k}(u\alpha(u)+\beta(u))s^{u-1}}{1-\sum_{u=1}^{k}\alpha(u)s^u}$$

with α given by (3.4) and

$$u\alpha(u) + \beta(u) = ua(u) + b(u) - c((u-1)a(u-1) + b(u-1)) - \\ (c+d)a(u-1). \qquad (u = 1, 2, \ldots, k)$$

Solving for $\beta(u)$ and insertion of (3.4) gives (3.5).

This completes the proof of Lemma 3.5. Q.E.D.

Theorem 3.1. *For $k = 1, 2, \ldots,$ the convolution of* $\mathrm{bin}(m_1, \pi_1), \mathrm{bin}(m_2, \pi_2), \ldots, \mathrm{bin}(m_k, \pi_k)$ *is $R_k[\alpha_k, \beta_k]$ where α_k and β_k can be evaluated recursively by*

$$\alpha_k(u) = \alpha_{k-1}(u) + \frac{\pi_k}{1-\pi_k}\alpha_{k-1}(u-1) \tag{3.6}$$

$$\beta_k(u) = \beta_{k-1}(u) + \frac{\pi_k}{1-\pi_k}(\beta_{k-1}(u-1) - (m_k+1)\alpha_{k-1}(u-1)) \tag{3.7}$$

for $k = 2, 3, \ldots$ and $u = 1, 2, \ldots, k$ with

$$\begin{gathered} \alpha_{k-1}(0) = -1; \quad \alpha_{k-1}(k) = \beta_{k-1}(k) = \beta_{k-1}(0) = 0 \\ \alpha_1(1) = -\frac{\pi_1}{1-\pi_1}; \quad \beta_1(1) = (m_1+1)\frac{\pi_1}{1-\pi_1}. \end{gathered} \tag{3.8}$$

Proof. Formula (3.8) follows from Lemma 3.4, and by letting

$$\begin{gathered} a = \alpha_{k-1}; \quad b = \beta_{k-1}; \quad \alpha = \alpha_k; \quad \beta = \beta_k \\ c = -\frac{\pi_k}{1-\pi_k}; \quad d = (m_k+1)\frac{\pi_k}{1-\pi_k} \end{gathered}$$

in Lemma 3.5, we obtain (3.6) and (3.7).

This completes the proof of Theorem 3.1. Q.E.D.

3C. As pointed out at the end of subsection 3A, f_i is a compound distribution with severity distribution h_i, and the counting distribution is a convolution of J_i binomial distributions. Thus, the counting distribution can be expressed as $R_{J_i}[a_i, b_i]$, and the functions a_i and b_i can be evaluated by Theorem 3.1. From Lemma 3.3 and (3.1), we obtain that f_i can be evaluated recursively by

$$f_i(x) = \sum_{y=1}^{\min(J_i m_i, x)} \left(c_i(y) + \frac{d_i(y)}{x}\right) f_i(x-y) \qquad (x = 1, 2, \ldots) \tag{3.9}$$

with

$$c_i\left(y\right)=\sum_{u=\{y/m_i\}}^{\min(J_i,y)}a_i\left(u\right)h_i^{u*}\left(y\right);\ d_i\left(y\right)=y\sum_{u=\{y/m_i\}}^{\min(J_i,y)}\frac{b_i\left(u\right)}{u}h_i^{u*}\left(y\right)\qquad(3.10)$$

$$(y=1,2,\ldots,J_im_i)$$

$$f_i\left(0\right)=q_i\left(0\right)=\prod_{j=1}^{J}q_{ij}\left(0\right)=\prod_{j=1}^{J}p_j\left(0\right)^{n_{ij}}=\prod_{j=1}^{J}\left(1-\pi_j\right)^{n_{ij}}.\qquad(3.11)$$

3D. We can now summarise the algorithm for evaluation of f as follows:

1. For each non-empty cell (i,j), express the claim number distribution q_{ij} as a binomial distribution.

2. For each i:

   (a) Express the claim number distribution q_i in the form $R_{J_i}\left[a_i,b_i\right]$ by using Theorem 3.1.

   (b) Express the aggregate claims distribution f_i in the form $R_\infty\left[c_i,d_i\right]$ with c_i and d_i given by (3.10).

   (c) Evaluate f_i recursively by (3.9) and (3.11).

3. Evaluate

   $$f=*_{i=1}^{I}f_i\qquad(3.12)$$

   by brute force convolution.

4 Earlier methods

4.1 Brute force convolution

The most obvious way to evaluate f is by brute force convolution, that is, if $g_i\in\mathcal{P}_0$ for $k=1,2,\ldots$, then we evaluate $*_{i=1}^{k}g_i$ recursively by

$$\left(*_{i=1}^{k}g_i\right)(x)=\left(\left(*_{i=1}^{k-1}g_i\right)*g_k\right)(x)=\sum_{y=0}^{x}g_k\left(y\right)\left(*_{i=1}^{k-1}g_i\right)(x-y).\qquad(4.1)$$

$$(x=0,1,2,\ldots;\ k=2,3,\ldots)$$

We apply this procedure to evaluate

$$f=*_{i=1}^{I}*_{j=1}^{J}f_{ij}.\qquad(4.2)$$

In particular, this method can be applied by first evaluating $f_i = *_{j=1}^J f_{ij}$ for each i and then $f = *_{i=1}^I f_i$.

For each non-empty cell (i,j), we evaluate g_{ij} by (2.1) and then $f_{ij} = g_{ij}^{n_{ij}*}$ by brute force convolution or De Pril's (1985) recursion

$$f_{ij}(x) = \frac{1}{g_{ij}(0)} \sum_{y=1}^{\min(m_i,x)} \left((n_{ij}+1) \frac{y}{x} - 1 \right) g_{ij}(y) f_{ij}(x-y), \qquad (x = 1, 2, \ldots, n_{ij} m_i) \tag{4.3}$$

which can also be expressed as a recursion for a compound binomial distribution

$$f_{ij}(x) = \frac{\pi_j}{1-\pi_j} \sum_{y=1}^{\min(m_i,x)} \left((n_{ij}+1) \frac{y}{x} - 1 \right) h_i(y) f_{ij}(x-y) \qquad (x = 1, 2, \ldots, n_{ij} m_i)$$

(cf. Lemma 3.4). Using the number of algebraic operations as measure of efficiency, Sundt & Dickson (2000) compare these methods and discuss the most efficient way of applying brute force convolution.

4.2 De Pril's methods

Inspired by De Pril (1989), Sundt (1995) pointed out that for any $g \in \mathcal{P}_0$ there exists a unique function φ_g such that g is $R_\infty[0, \varphi_g]$, and called φ_g the De Pril transform of g. From (3.1), we immediately get

$$g(x) = \frac{1}{x} \sum_{y=1}^{x} \varphi_g(y) g(x-y) \qquad (x = 1, 2, \ldots) \tag{4.4}$$

and solving for $\varphi_g(x)$ gives

$$\varphi_g(x) = \frac{1}{g(0)} \left(x g(x) - \sum_{y=1}^{x-1} \varphi_g(y) g(x-y) \right). \qquad (x = 1, 2, \ldots) \tag{4.5}$$

Furthermore, Lemma 3.3 gives that

$$\varphi_{p \vee h}(x) = x \sum_{y=1}^{x} \frac{\varphi_p(y)}{y} h^{y*}(x). \qquad (x = 1, 2, \ldots) \tag{4.6}$$

Sundt (1995) also showed that

$$\varphi_{*_{i=1}^{k} g_i} = \sum_{i=1}^{k} \varphi_{g_i} \qquad (g_1, g_2, \dots, g_k \in \mathcal{P}_0) \tag{4.7}$$

In De Pril's first exact method, for each non-empty cell (i, j), we evaluate g_{ij} by (2.1) and its De Pril transform $\varphi_{g_{ij}}$ by (4.5). From (4.7), we obtain

$$\varphi_f = \sum_{i=1}^{I} \sum_{j=1}^{J} n_{ij} \varphi_{g_{ij}}, \tag{4.8}$$

from which we evaluate f by (4.4), starting with

$$f(0) = \prod_{i=1}^{I} \prod_{j=1}^{J} (1 - \pi_j)^{n_{ij}}. \tag{4.9}$$

For De Pril's second exact method, we utilise that

$$\varphi_{p_j}(x) = -\left(\frac{\pi_j}{\pi_j - 1}\right)^x. \qquad (x = 1, 2, \dots) \tag{4.10}$$

By insertion of (4.6) in (4.8) and some reorganisation, we obtain

$$\varphi_f(x) = x \sum_{i=1}^{I} \sum_{y=\{x/m_i\}}^{x} \frac{h_i^{y*}(x)}{y} \sum_{j=1}^{J} n_{ij} \varphi_{p_j}(y), \ (x = 1, 2, \dots) \tag{4.11}$$

and insertion of (4.10) gives

$$\varphi_f(x) = -x \sum_{i=1}^{I} \sum_{y=\{x/m_i\}}^{x} \frac{h_i^{y*}(x)}{y} \sum_{j=1}^{J} n_{ij} \left(\frac{\pi_j}{\pi_j - 1}\right)^y, \ (x = 1, 2, \dots) \tag{4.12}$$

by which we evaluate φ_f. Finally, we evaluate f recursively by (4.4), starting with (4.9).

The approximations studied by De Pril (1989) are based on approximating each φ_{p_j} in (4.11) in De Pril's second exact method by a function that is equal to zero for all y greater than some finite r, and he gave error bounds for the approximations of f. Such approximations have later been discussed by Dhaene & De Pril (1994) and Dhaene & Sundt (1998).

As an example of these approximations, we shall consider the rth order De Pril approximation $f^{(r)}$, which is obtained by simply summing up to $y = r$ in the second summation in (4.12), that is,

$$\varphi_{f^{(r)}}(x) = -x \sum_{i=1}^{I} \sum_{y=\{x/m_i\}}^{\min(r,x)} \frac{h_i^{y*}(x)}{y} \sum_{j=1}^{J} n_{ij} \left(\frac{\pi_j}{\pi_j - 1}\right)^y, \qquad (x = 1, 2, \dots)$$

which is equal to zero when $x > r \max m_i$. Finally, we evaluate $f^{(r)}$ recursively by (4.4) with $f^{(r)}(0) = f(0)$.

4.3 Dhaene-Vandebroek's method

By inserting (4.8) in (4.4), we obtain

$$f(x) = \frac{1}{x} \sum_{i=1}^{I} \sum_{j=1}^{J} n_{ij} \psi_{ij}(x) \qquad (x = 1, 2, \dots) \tag{4.13}$$

with

$$\psi_{ij}(x) = \sum_{y=1}^{x} \varphi_{g_{ij}}(y) f(x-y). \qquad (x = 1, 2, \dots)$$

Dhaene & Vandebroek (1995) showed that for each cell (i, j) we have the recursion

$$\psi_{ij}(x) = \frac{1}{g_{ij}(0)} \sum_{y=1}^{\min(m_i, x)} (y f(x-y) - \psi_{ij}(x-y)) g_{ij}(y). \ (x = 1, 2, \dots) \tag{4.14}$$

with $\psi_{ij}(0) = 0$.

5 The second binomial method

For $i = 1, 2, \dots, I$ and $x = 1, 2, \dots$, let

$$\psi_i(x) = \sum_{y=1}^{x} \varphi_{f_i}(y) f(x-y) = \sum_{y=1}^{x} \varphi_{*_{j=1}^{J} g_{ij}^{n_{ij}*}}(y) f(x-y) =$$

$$\sum_{y=1}^{x} \sum_{j=1}^{J} n_{ij} \varphi_{g_{ij}}(y) f(x-y) = \sum_{j=1}^{J} n_{ij} \psi_{ij}(x).$$

Then

$$f(x) = \frac{1}{x} \sum_{i=1}^{I} \psi_i(x). \qquad (x = 1, 2, \dots) \tag{5.1}$$

From Theorem 11 in Sundt (1995), we obtain that

$$\psi_i(x) = \sum_{y=1}^{\min(J_i m_i, x)} ((y c_i(y) + d_i(y)) f(x-y) + c_i(y) \psi_i(x-y)). \tag{5.2}$$

$$(x = 1, 2, \dots; i = 1, 2, \dots, I)$$

Using this, we can evaluate f in the following way:

1. For each non-empty cell (i, j), express the claim number distribution q_{ij} as a binomial distribution.

2. For each i:

   (a) Express the claim number distribution q_i in the form $R_{J_i}[a_i, b_i]$ by using Theorem 3.1.

   (b) Express the aggregate claims distribution f_i in the form $R_\infty[c_i, d_i]$ with c_i and d_i given by (3.10)

3. For each x:

   (a) For each i, evaluate $\psi_i(x)$ recursively by (5.2).

   (b) Evaluate $f(x)$ by (5.1).

6 Comparison

6A. A characteristic feature of De Pril's model is the two-way classification. We know more than just that we have IJ cells and that the aggregate claims distribution of each policy in cell (i, j) is f_{ij}; we know that $f_{ij} = p_j \vee h_i$. Thus, all cells with the same i have the same severity distribution, and all cells with the same j have the same claim frequency, that is, cells with the same i or j have something in common, as opposed to just knowing that all the f_{ij}s are different. One would expect this additional information to be reflected in the method of evaluating f and simplify it. This is the case with the two binomial methods and De Pril's second exact method (as well as the approximations based on it), but not with De Pril's first exact method and Dhaene-Vandebroek's method. From these considerations, one should expect that one should just concentrate on De Pril's second exact method and the two binomial methods. Unfortunately, it is not that simple in general. In particular, the outcome would depend on whether the severity distributions have finite or infinite support.

One way of comparing the efficiency of methods, is counting elementary algebraic operations; summation, subtraction, multiplication, and division. Such operations are sometimes classified as bar operations (summation and subtraction) and dot operations (multiplication and division) as bar operations are normally less time-consuming on computers than dot operations. Some authors (e.g. Kuon, Reich, & Reimers (1987), Sundt & Dickson (2000), and Dickson & Sundt (2001)) count bar operations and dot operations separately whereas others (e.g. Bühlmann (1984), Dhaene & Vandebroek (1995), and Dhaene, Ribas, & Vernic (2005)) count only dot operations. Sundt & Dickson (2000) discuss reasons for not putting too much emphasis on the counting of algebraic operations. When restricting to dot operations, we also have the additional issue that dot operations can sometimes be converted to bar operations, but that should not be overdone; for a large integer n, it does not seem reasonable to calculate nx as $\sum_{i=1}^{n} x$, but we shall do it for $n = 2$.

It may seem that the effort of pedantically counting all the algebraic operations is not proportional to the value of this measure of efficiency. To only get an idea of the efficiency of the methods, we shall restrict to dot operations and not go too much into detail. Furthermore, we count as if the support of all the severity distributions is the set of all positive integers. This does not mean that the methods break down when a distribution has probability zero for some positive integer. However, in that case, we may count multiplications even when one of the factors is equal to zero. As optimality criterion, we use the number of dot operations needed for evaluating $f(x)$ for $x = 0, 1, 2, \ldots, s$.

A more extensive comparison between some of the earlier methods by counting dot operations is done by Dhaene, Ribas, & Vernic (2005). Dickson & Sundt (2001) compare some methods for evaluation of two compound $\mathcal{R}_1$ distributions.

6B. For a method A, we denote by $\tau_{\mathrm{A}}(s)$ the number of dot operations needed for evaluation of $f(x)$ for $x = 0, 1, 2, \ldots, s$. The indicator A can have the following values:

Indicator	**Method**
B1	first binomial method
B2	second binomial method
BFC	brute force convolution
DP1	De Pril's first exact method
DP2	De Pril's second exact method
DV	Dhaene-Vandebroek's method
DP(r)	rth order De Pril approximation

The last method is not fully comparable with the others as it is not an exact method, only an approximation.

As a measure of efficiency between two methods A and B, we use $\varepsilon_{\mathrm{A,B}} = \lim_{s\uparrow\infty} \tau_{\mathrm{B}}(s) / \tau_{\mathrm{A}}(s)$.

Under our present assumptions, the dot operations can typically be classified in three groups when s is sufficiently large:

1. Initial operations that we need to do only once. This is a bit imprecise; there could be recursions there too, like for the evaluation of c_i and

d_i in (3.9), but we mean that the number of dot operations does not depend on s. Let us call them α_{A}.

2. Operations that we have to do for each value of x and where the number of dot operations is the same for all values of x. Let us call them β_{A}.

3. Operations that we have to do for each value of x and where the number of dot operations is proportional with x. Let the proportionality factor be γ_{A}, so that we need $\gamma_{\mathrm{A}} x$ operations at the step for x.

If Method A satisfies this condition, then

$$\tau_{\mathrm{A}}(s) = \alpha_{\mathrm{A}} + \beta_{\mathrm{A}} s + \gamma_{\mathrm{A}} \sum_{x=1}^{s} x = \alpha_{\mathrm{A}} + \beta_{\mathrm{A}} s + \gamma_{\mathrm{A}} \frac{s(s+1)}{2}.$$

Thus, if it is satisfied by both Methods A and B with γ_{A} and γ_{B} positive, then

$$\varepsilon_{\mathrm{A,B}} = \lim_{s \uparrow \infty} \frac{\tau_{\mathrm{B}}(s)}{\tau_{\mathrm{A}}(s)} = \lim_{s \uparrow \infty} \frac{\alpha_{\mathrm{B}} + \beta_{\mathrm{B}} s + \gamma_{\mathrm{B}} \frac{s(s+1)}{2}}{\alpha_{\mathrm{A}} + \beta_{\mathrm{A}} s + \gamma_{\mathrm{A}} \frac{s(s+1)}{2}} = \frac{\gamma_{\mathrm{B}}}{\gamma_{\mathrm{A}}},$$

that is, we need to consider only γ_{A} and γ_{B}. With this procedure, we avoid counting the dot operations for initiating the methods. Hence, for methods with heavy initiation, like the binomials method, the performance would be worse for low values of s.

6C. We shall now count the γ operations for each of various methods.

We start with De Pril's first exact method. Here, the γ operations arise for each value of y in the summations in (4.4) and (4.5). At each place, we have one dot operation, and we use (4.4) on one distribution and (4.5) on $J_\bullet$ distributions. Hence, $\gamma_{\mathrm{DP1}} = J_\bullet + 1$.

For Dhaene-Vandebroek's method, we rewrite (4.14) as

$$\psi_{ij}(x) = \frac{1}{g_{ij}(0)} \sum_{y=0}^{x-1} v_{ij}(x,y)\, g_{ij}(x-y) \qquad (x = 1, 2, \dots)$$

with

$$v_{ij}(x,y) = (x-y) f(y) - \psi_{ij}(y),$$

which can be evaluated recursively by

$$\begin{aligned} v_{ij}(x,y) &= v_{ij}(x-1,y) + f(y) \qquad (x = y+1, y+2, \dots) \\ v_{ij}(y,y) &= -\psi_{ij}(y). \end{aligned}$$

Hence, whereas at first glance it seemed that we needed two dot operations for each y in (4.14), we have now reduced it to one. As we have to apply (4.14) on $J_\bullet$ distributions, we obtain $\gamma_{\mathrm{DV}} = J_\bullet$, so that Dhaene-Vandebroek's method is always better than De Pril's first exact method when all the severity distributions have infinite support.

We now turn to the first binomial method. Here, the γ operations arise in (3.9), for $h_i^{u*}(x)$ for $u = 1, 2, \ldots, J_i$ in (3.10), and when taking the convolution of the f_is in (3.12).

Let us first consider the h_i^{u*}s. The natural way to do this, is to apply (4.1) on $h_i^{u*} = h_i^{(u-1)*} * h_i$. However, Sundt & Dickson (2000) point out that when u is even, it is usually more efficient to use $h_i^{u*} = h_i^{\frac{u}{2}*} * h_i^{\frac{u}{2}*}$. As they considered distributions in $\mathcal{P}_0$ whereas we have distributions in $\mathcal{P}_+$, we cannot immediately transfer the results.

For the direct method, we utilise that $h_i(0) = 0$ and, thus, $h_i^{u*}(y) = 0$ when $y < u$, so that

$$h_i^{u*}(x) = \sum_{y=u-1}^{x-1} h_i^{(u-1)*}(y)\, h_i(x-y), \qquad (x = u, u+1, \ldots) \tag{6.1}$$

which needs $x - u + 1$ dot operations.

When u is even, for $x = u, u+1, \ldots$, we have

$$h_i^{u*}(x) = \sum_{y=\frac{u}{2}}^{x-\frac{u}{2}} h_i^{\frac{u}{2}*}(y)\, h_i^{\frac{u}{2}*}(x-y) =$$

$$\begin{cases} 2\sum_{y=\frac{u}{2}}^{\frac{x+1}{2}} h_i^{\frac{u}{2}*}(y)\, h_i^{\frac{u}{2}*}(x-y) & (x \text{ odd}) \\ 2\sum_{y=\frac{u}{2}}^{\frac{x}{2}-1} h_i^{\frac{u}{2}*}(y)\, h_i^{\frac{u}{2}*}(x-y) + h_i^{\frac{u}{2}*}\left(\frac{x}{2}\right) h_i^{\frac{u}{2}*}\left(\frac{x}{2}\right), & (x \text{ even}) \end{cases} \tag{6.2}$$

which gives $(x-u+3)/2$ dot operations when x is odd and $(x-u+2)/2$ when x is even.

When counting the γ operations, it is the operations of order x that we have to take into account, that is, one when u is odd, and $1/2$ when u is even. Counting as if u is odd half the times, we count as if on the average each of these convolutions contributes to γ_{B1} with $3/4$. For each i, we have to perform $J_i - 1$ such convolutions, so for all is together, we get a contribution to γ_{B1} of $\frac{3}{4}(J_\bullet - I)$.

At first glance, it seems that in (3.9), we need two dot operations for each y. However, by rewriting it as

$$f_i(x) = \frac{1}{x}\sum_{y=1}^{x} w_i(x,y)\, f_i(x-y) \qquad (x = 1, 2, \ldots)$$

with

$$w_i(x,y) = xc_i(y) + d_i(y),$$

which can be evaluated recursively by

$$\begin{aligned} w_i(x,y) &= w_i(x-1,y) + c_i(y) \qquad (y = 1,2,\ldots,x-1) \\ w_i(x,x) &= xc_i(x) + d_i(x), \end{aligned}$$

we have converted one of the dot operations to bar operation, so that we have only one dot operation left. We have I recursions like (3.9), so totally they contribute with I to γ_{B1}. Finally, each of the $I-1$ convolutions of f_is in (3.12) contributes to γ_{B1} with one. Thus,

$$\gamma_{\mathrm{B1}} = \frac{3}{4}(J_\bullet - I) + I + I - 1 = \frac{3J_\bullet + 5I}{4} - 1.$$

Comparing with Dhaene-Vandebroek's method, we see that $\gamma_{\mathrm{B1}} < \gamma_{\mathrm{DV}}$ when $J_\bullet > 5I - 4$. In particular, this implies that the first binomial method is always better than Dhaene-Vandebroek's method when there are at least five non-empty cells for each value of i.

For the second binomial method, the work in the h_i^{u*}s is the same as for the first binomial method, that is, the contribution to γ_{B2} is $\frac{3}{4}(J_\bullet - I)$. Then, for each value of y in (5.2), there are two dot operations (assuming that $yc_i(y)+d_i(y)$ is evaluated only once for each (y,i), and there are I such recursions. Hence,

$$\gamma_{\mathrm{B2}} = \frac{3}{4}(J_\bullet - I) + 2I = \frac{3J_\bullet + 5I}{4}.$$

As $\gamma_{\mathrm{B1}} - \gamma_{\mathrm{B2}} = 1$ this method is slightly beaten by the first binomial method, so that we have to discard it.

For De Pril's second exact method, we also have to evaluate $h_i^{u*}(x)$s. However, whereas for the binomial methods we needed these convolutions up to order J_i in (3.10), for De Pril's second exact method, we need them up to order x in (4.12) so that $\tau_{\mathrm{DPT2}}(s)$ becomes a polynomial of third order. Hence, for large s, it will perform worse than the other methods. However, in this connection it should be emphasised that the strength of De Pril's second method is as a basis for approximations that will perform better with respect to dot operations.

On the other hand, for the rth order De Pril approximation, we need the convolutions only up to order r. We have to do this for I distributions, and their total contribution to $\gamma_{\mathrm{DP}(r)}$ is $\frac{3}{4}(r-1)I$. In addition, we have to apply (4.4) once, so that

$$\gamma_{\mathrm{DP}(r)} = \frac{3}{4}(r-1)I + 1. \tag{6.3}$$

With brute force convolutions, the $J_\bullet - 1$ convolutions in (4.2) contribute with $J_\bullet - 1$ to γ_{BFC}. In addition, we have to evaluate $f_{ij} = g_{ij}^{n_{ij}*}$ when $n_{ij} > 1$. At first glance, using (4.3) for one f_{ij} seems to give a contribution of two to γ_{BFC}, but by proceeding like we did with (4.14) and (3.9) above, we can reduce it to one. However, when $n_{ij} = 2$, we can reduce it to 1/2 by proceeding similar to (6.2). Nevertheless, when comparing with Dhaene-Vandebroek's method and the first binomial method, we see that there cannot be more than two cells with more than one policy if brute force convolution should be the best method.

These considerations lead to the following conclusions about the exact methods under the present conditions:

1. De Pril's exact methods and the second binomial method can be discarded.

2. Brute force convolution can be discarded unless there are only one or two cells with more than one policy.

3. The first binomial method is better than Dhaene-Vandebroek's method when $J_\bullet > 5I - 4$.

It is logical that it is when there are many non-empty cells with the same severity distribution that the binomial methods perform best, as that is when the extra information through the two-way classification is most significant.

6D. The race between the first binomial method and Dhaene-Vandebroek's method as well as De Pril's first method and the second binomial method is remarkably close. It is amazing that it is the application of (6.2) that rescues the first binomial method. If we had used the direct method (6.1) for all u, then we would have got

$$\gamma_{\mathrm{B1}} = J_\bullet - I + I + I - 1 = J_\bullet + I - 1,$$

that is, then the first binomial method is not better than Dhaene-Vandebroek's method for any value of I, and it is better than De Pril's first exact method only when $I = 1$. Furthermore, the first binomial method is more complex to program, even when using (6.1) for all u.

It seems surprising if utilisation of the extra information from the two-way classification does not give a larger gain for exact methods. On the other hand, from (6.3) we see that the gain is very pronounced for De Pril approximations of low order.

6E. We have

$$\gamma_{\mathrm{B1}} = \sum_{i=1}^{I} \frac{3J_i + 5}{4} - 1 \tag{6.4}$$

$$\gamma_{\mathrm{DV}} = \sum_{i=1}^{I} J_i$$

and analogous if we consider only a subset of the possible is.

We could also use a combination of these two methods where we use Dhaene-Vandebroek's method when $J_i < (3J_i + 5)/4$, that is, $J_i < 5$, and the first binomial method when $J_i \geq 5$. Then we get one extra convolution between the aggregate claims distribution of all the policies treated with the first binomial method and the aggregate claims distribution of all the other policies. Let $\mathcal{A} = \{i : J_i \geq 5\}$. Then the γ value of this combined method is

$$\gamma_{\mathrm{comb}} = \sum_{i \in \mathcal{A}} \frac{3J_i + 5}{4} + \sum_{i \in \widetilde{\mathcal{A}}} J_i = \gamma_{\mathrm{DV}} - \sum_{i \in \mathcal{A}} \frac{J_i - 5}{4} = \gamma_{\mathrm{bin}} - \left(\sum_{i \in \widetilde{\mathcal{A}}} \frac{5 - J_i}{4} - 1 \right).$$

Hence, the combined method is at least as good as Dhaene-Vandebroek's method, and it is better than the first binomial method when $\sum_{i \in \widetilde{\mathcal{A}}} (5 - J_i) > 4$.

Now let us instead replace the first binomial method with the second binomial method. Then we have to drop the subtraction of one in (6.4). However, that disadvantage is canceled by the advantage that we do not need the extra convolution that we have to do when combining Dhaene-Vandebroek's method with the first binomial method. Hence, in combination with Dhaene-Vandebroek's method, the two binomial methods are equally good with respect to γ operations. However, the second binomial method is more similar to Dhaene-Vandebroek's methods and seems to easier to program for such combinations as we do not need to perform the brute force convolutions of the f_is, so we recommend that method.

7 The first binomial method in the individual life model

Let us finally consider the first binomial method in the individual life model where each severity distribution is concentrated in one point, that is, for each i, h_i is concentrated in one positive integer m_i. We now have

$$f_i(x) = \begin{cases} q_i(x/m_i) & (x = 0, m_i, 2m_i, \ldots, m_i n_i) \\ 0 & (\text{otherwise}) \end{cases}$$

and evaluate q_i by

$$q_i(x) = \sum_{y=1}^{J_i} \left(a_i(y) + \frac{b_i(y)}{x} \right) q_i(x-y). \qquad (x = 1, 2, \ldots, n_i)$$

Then

$$\left(*_{i=1}^{k} f_i\right)(x) = \sum_{y=\max\left(0, \left\{\frac{x - \sum_{i=1}^{k-1} m_i n_i}{m_k}\right\}\right)}^{[x/m_k]} q_k(y) \left(*_{i=1}^{k-1} f_i\right)(x - m_k y), \tag{7.1}$$

$$\left(x = 0, 1, 2, \ldots, \textstyle\sum_{i=1}^{k} m_i n_i;\ k = 2, 3, \ldots, I\right)$$

and finally we get $f = *_{i=1}^{I} f_i$.

We would normally have $\left(*_{i=1}^{k} f_i\right)(x) = 0$ for many values of x, and, hence, we ought to try to avoid evaluating the product sum in (7.1) for such values of x. One way to avoid some of these values, is to choose the monetary unit so large that the m_is do not have any common factor. Furthermore, even in that case, $\left(*_{i=1}^{k} f_i\right)(x) > 0$ if and only if x can be written in the form

$$x = \sum_{i=1}^{k} v_i m_i. \qquad (v_i = 0, 1, 2, \ldots, n_i;\ i = 0, 1, 2, \ldots, k) \tag{7.2}$$

Hence, we ought to evaluate the product sum only when this condition is satisfied. This can be tested quite simply. We know that

$$\left(*_{i=1}^{k} f_i\right)(0) = q_k(0) \left(*_{i=1}^{k-1} f_i\right)(0) > 0,$$

whereas, for $x = 1, 2, \ldots, \sum_{i=1}^{k} m_i n_i$, $\left(*_{i=1}^{k} f_i\right)(x) > 0$ if and only if at least one of the conditions $\left(*_{i=1}^{k} f_i\right)(x - m_k) > 0$ and $\left(*_{i=1}^{k-1} f_i\right)(x - m_k) > 0$ is satisfied.

We could also reduce the summation in (7.1) to the set

$$\{y = 0, 1, 2, \ldots, [x/m_k] : x - m_k y \text{ satisfies (7.2) with } k \text{ replaced with } k-1\}.$$

However, then it seems simpler to just test whether $\left(*_{i=1}^{k-1} f_i\right)(x - m_k y) > 0$, so this seems to be stretching too far the criterion of minimising the number of dot operations.

The multiplication in $x - m_k y$ in (7.1) could be avoided by evaluating $t_k(x, y) = x - m_k y$ recursively by

$$t_k(x, y) = t_k(x, y-1) - m_k \qquad (y = 1, 2, \ldots, [x/m_k])$$
$$t_k(x, 0) = x.$$

For a discussion of application of the earlier methods in the individual life model, we refer to Dhaene, Ribas, & Vernic (2005). The procedure we have suggested for the convolution of $f_1, f_2, \ldots, f_I$, can of course also be applied in connection with the brute force convolution method of subsection 4.1.

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