DEPENDENCE MATTERS!

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"The new system should provide supervisors with the appropriate tools to assess the "overall solvency" of an insurance undertaking. This means that the system should not only consist of a number of quantitative ratios and indicators, but also cover qualitative aspects that influence the risk-standing of an undertaking (management, internal risk control, competitive situation etc.). … The solvency system should encourage and give an incentive to insurance undertakings to measure and manage their risks. In this regard, there is a clear need for developing common EU principles on risk management and supervisory review. Furthermore the quantitative solvency requirements should cover the most significant risks to which an insurance undertaking is exposed. This risk-oriented approach would lead to the recognition of internal models (either partial or full) provided these improve the undertaking's risk management and better reflect its true risk profile than a standard formula."

European Commission, Internal Market DG, March 2003

important topic here:

recognizing and modeling of dependencies between risks

Definition. A *copula* is a function C of d variables on the unit d-cube $[0,1]^d$ with the following properties:

1. the range of C is the unit interval $[0,1]$;
2. $C(\mathbf{u})$ is zero for all $\mathbf{u}$ in $[0,1]^d$ for which at least one coordinate equals zero;
3. $C(\mathbf{u}) = u_k$ if all coordinates of $\mathbf{u}$ are 1 except the k-th one;
4. C is d-increasing in the sense that for every $\mathbf{a} \le \mathbf{b}$ in $[0,1]^d$ the measure $\Delta C_{\mathbf{a}}^{\mathbf{b}}$ assigned by C to the d-box $[\mathbf{a},\mathbf{b}] = [a_1, b_1] \times \cdots \times [a_d, b_d]$ is nonnegative, i.e.

$$\Delta C_{\mathbf{a}}^{\mathbf{b}} := \sum_{(\varepsilon_1,\cdots,\varepsilon_n)\in\{0,1\}^d} (-1)^{\sum_{i=1}^{d}\varepsilon_i} C\left(\varepsilon_1 a_1 + (1-\varepsilon_1) b_1, \cdots, \varepsilon_d a_d + (1-\varepsilon_d) b_d\right) \ge 0.$$

In other words:

A copula C is a multivariate distribution function of a random vector that has uniform margins.

Sklar's Theorem. Let H denote a n-dimensional distribution function with margins $F_1, \cdots, F_n$. Then there exists a n-copula C such that for all real $(x_1, \cdots, x_n)$,

$$H(x_1, \cdots, x_n) = C\left(F_1(x_1), \cdots, F_n(x_n)\right).$$

If all the margins are continuous, then the copula is unique, and is determined uniquely on the ranges of the marginal distribution functions otherwise. Moreover, the converse of the above statement is also true. If we denote by $F_1^{-1}, \cdots, F_n^{-1}$ the generalized inverses of the marginal distribution functions, then for every $(u_1, \cdots, u_n)$ in the unit n-cube,

$$C(u_1, \cdots, u_n) = H\left(F_1^{-1}(u_1), \cdots F_n^{-1}(u_n)\right).$$

Fréchet-Hoeffding bounds:

$$\underbrace{\max(u_1 + \cdots + u_d - d + 1, 0)}_{=\mathcal{W}(u_1,\cdots,u_d)} \leq C(u_1, \cdots, u_d) \leq \underbrace{\min(u_1, \cdots, u_d)}_{=\mathcal{M}(u_1,\cdots,u_d)}$$

$\mathcal{W}(u_1, \cdots, u_d)$ is generally not a copula for $d \geq 3$, but inequality is sharp.

Two random variables with copula $\mathcal{M}(u_1, u_2)$ are called *co-monotonic* (highest degree of **positive** dependence), two random variables with copula $\mathcal{W}(u_1, u_2)$ are called *counter-monotonic* (highest degree of **negative** dependence).

Wide-spread fallacy: Co-monotonic risks increase total risk, counter-monotonic risks decrease total risk ($\rightarrow$ diversification effect)

This is wrong in general!

Definition. Let $d, n \in \mathbb{N}$ and define intervals $I_{i_1,\cdots,i_d}(n) := \bigtimes_{j=1}^{d} \left(\frac{i_j - 1}{n}, \frac{i_j}{n} \right]$ for all possible choices $i_1, \cdots, i_d \in N_n := \{1, \cdots, n\}$. If $a_{i_1,\cdots,i_d}(n)$ are non-negative real numbers with the property

$$\sum_{(i_1,\cdots,i_d) \in J(i_k)} a_{i_1,\cdots,i_d}(n) = \frac{1}{n}$$

for all $k \in \{1, \cdots, d\}$ and $i_k \in \{1, \cdots, n\}$, with $J(i_k) := \{(j_1, \cdots, j_n) \in N_n{}^d \mid j_k = i_k\}$, then the function $c_n := \sum_{(i_1,\cdots,i_d) \in N_n{}^d} a_{i_1,\cdots,i_d}(n) \mathbb{1}_{I_{i_1,\cdots,i_d}(n)}$ is the density of a d-dimensional copula, called *grid-type copula* with parameters $\{a_{i_1,\cdots,i_d}(n) | (i_1, \cdots, i_d) \in N_n{}^d\}$. Here $\mathbb{1}_A$ denotes the indicator random variable of the event A, as usual.

It is easy to see that in case of an absolutely continuous d-dimensional copula C, with continuous density

$$c(u_1,\cdots,u_d)=\frac{\partial^d}{\partial u_1\cdots\partial u_d}C(u_1,\cdots,u_d),\ (u_1,\cdots,u_d)\in(0,1)^d,$$

c can be approximated arbitrarily close by a density of a grid-type copula. The classical *multivariate mean-value-theorem* of calculus tells us here that we only have to choose

$$a_{i_1,\cdots,i_d}(n):=\int_{\frac{i_d-1}{n}}^{\frac{i_d}{n}}\cdots\int_{\frac{i_1-1}{n}}^{\frac{i_1}{n}}c(u_1,\cdots,u_d)\,du_1\cdots du_d=\Delta C_{\boldsymbol{\alpha}_n}^{\boldsymbol{\beta}_n},\quad i_1,\cdots,i_d\in N_n$$

with $\alpha_{nk}=\dfrac{i_k-1}{n},\ \beta_{nk}=\dfrac{i_k}{n},\ k=1,\cdots,d.$

Lemma. Let $U_1,\cdots,U_d$ be independent standard uniformly distributed random variables and let f_d and F_d denote the density and cumulative distribution function of $S_d := \sum_{i=1}^{d} U_i$, resp., for $d \in \mathbb{N}$. Then

$$f_d(x) = \frac{1}{2(d-1)!}\sum_{k=0}^{d}(-1)^k \binom{d}{k}(x-k)^{d-1}\operatorname{sgn}(x-k)$$

$$F_d(x) = \frac{1}{2d!}\sum_{k=0}^{d}(-1)^k \binom{d}{k}\left((-k)^d + (x-k)^d \operatorname{sgn}(x-k)\right)$$

for $0 \le x \le d$.

This follows e.g. from USPENSKY (1937), Example 3, p.277, who attributes this result already to Laplace.

Theorem. Let $(X_1,\cdots,X_d)$ be a random vector whose joint cumulative distribution function is given by a grid-type copula with density $c_n := \sum_{(i_1,\cdots,i_d)\in N_n^d} a_{i_1,\cdots,i_d}(n)\mathbb{1}_{I_{i_1,\cdots,i_d}(n)}$. Then the density and cumulative distribution function $\tilde{f}_d(n;\bullet)$ and $\tilde{F}_d(n;\bullet)$, resp., for the sum $S_d := \sum_{i=1}^{d} X_i$ is given by

$$\tilde{f}_d(n;x) = n \sum_{(i_1,\cdots,i_d)\in N_n^d} a_{i_1,\cdots,i_d}(n)\cdot f_d\left(nx+d-\sum_{i=1}^{d} j_i\right)$$

$$\tilde{F}_d(n;x) = \sum_{(i_1,\cdots,i_d)\in N_n^d} a_{i_1,\cdots,i_d}(n)\cdot F_d\left(nx+d-\sum_{i=1}^{d} j_i\right)$$

for $0\le x\le d$, with

$$f_d(x) = \frac{1}{2(d-1)!}\sum_{k=0}^{d}(-1)^k\binom{d}{k}(x-k)^{d-1}\operatorname{sgn}(x-k)$$

$$F_d(x) = \frac{1}{2d!}\sum_{k=0}^{d}(-1)^k\binom{d}{k}\left((-k)^d+(x-k)^d\operatorname{sgn}(x-k)\right)$$

for $0\le x\le d$.

Example.

$$A(n) = \left[a_{ij}(n)\right] = \begin{bmatrix} a & b & 1/3 - a - b \\ c & 1 - 4a - 2b - 2c & -2/3 + 4a + 2b + c \\ 1/3 - a - c & -2/3 + 4a + b + 2c & 2/3 - 3a - b - c \end{bmatrix}$$

with suitable real numbers $a, b, c \in \left[0, 1/3\right]$. It follows that the covariance of the corresponding random variables X_1, X_2 is given by

$$E\left(X_1 X_2\right) - E\left(X_1\right) E\left(X_2\right) = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij}(n) \frac{\left(2i - 1\right)(2j - 1)}{36} = 0$$

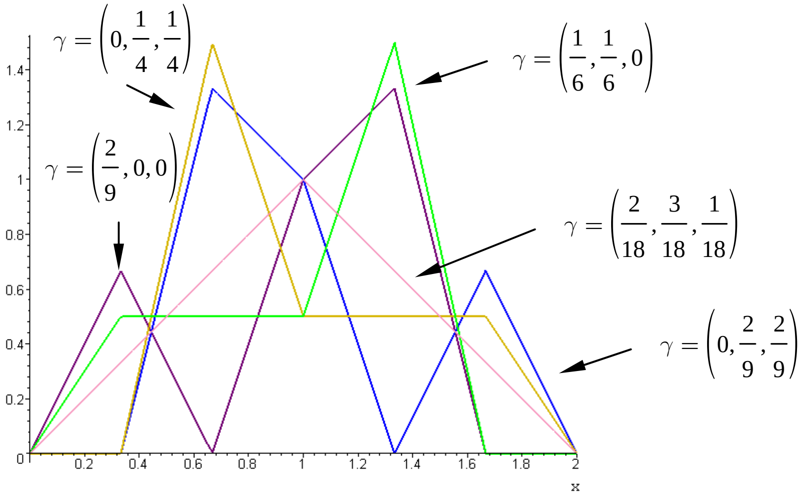
i.e. the risks X_1, X_2 are uncorrelated but dependent.

The density and cumulative distribution function of the aggregated risk $S_2 := X_1 + X_2$ are thus, by the above Theorem, given by

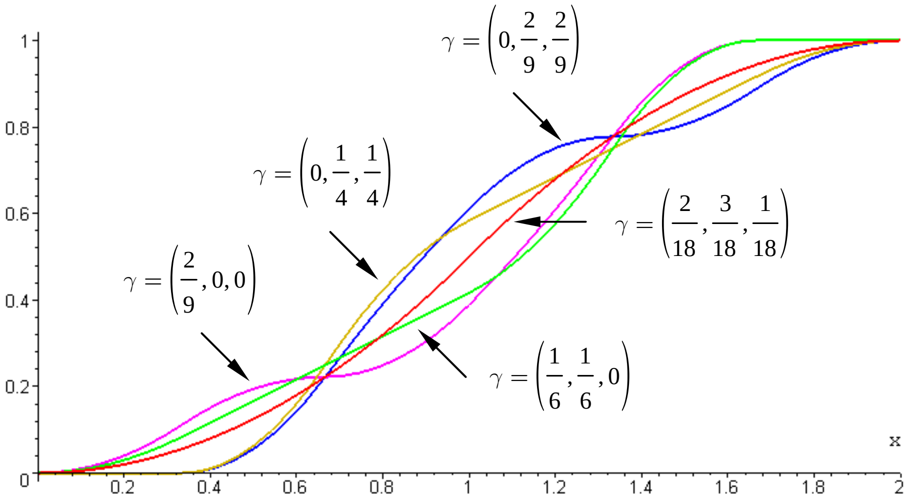
$$\tilde{f}_2(3;\gamma;x) = \begin{cases} 9ax, & 0 \le x \le \frac{1}{3} \\ 3(2a - \{b+c\}) + 9(-a + \{b+c\})x, & \frac{1}{3} \le x \le \frac{2}{3} \\ 9(4a + 3\{b+c\}) - 10 + 3(5 - 18a - 12\{b+c\})x, & \frac{2}{3} \le x \le 1 \\ 32 - 9(16a + 7\{b+c\}) + 9(14a + 6\{b+c\} - 3)x, & 1 \le x \le \frac{4}{3} \\ -28 + 3(52a + 19\{b+c\}) + 3(6 - 33a - 12\{b+c\})x, & \frac{4}{3} \le x \le \frac{5}{3} \\ 6(2 - 9a - 3\{b+c\}) + 3(-2 + 9a + 3\{b+c\})x, & \frac{5}{3} \le x \le 2 \\ 0, & \text{otherwise}; \end{cases}$$

$$
\tilde{F}_2(3;\gamma;x)=\begin{cases}
0, & x\le 0\\
\dfrac{9a}{2}x^2, & 0\le x\le \dfrac{1}{3}\\
\dfrac{9}{2}(-a+\{b+c\})x^2+3(2a-\{b+c\})x+\dfrac{1}{2}(-2a+\{b+c\}), & \dfrac{1}{3}\le x\le \dfrac{2}{3}\\
\dfrac{5}{2}(3-18a-12\{b+c\})x^2+(-10+36a+27\{b+c\})x+ \\
\qquad +\dfrac{1}{6}(20-66a-57\{b+c\}), & \dfrac{2}{3}\le x\le 1\\
\dfrac{9}{2}(-3+14a+6\{b+c\})x^2+(32-144a-63\{b+c\})x+ \\
\qquad +\dfrac{1}{6}(-106+237a+213\{b+c\}), & 1\le x\le \dfrac{4}{3}\\
\dfrac{9}{2}(2-22a-4\{b+c\})x^2+(-28+156a-57\{b+c\})x+ \\
\qquad +\dfrac{1}{6}(134-726a-267\{b+c\}), & \dfrac{4}{3}\le x\le \dfrac{5}{3}\\
\dfrac{3}{2}(-6+9a+3\{b+c\})x^2+3(4-6a-18\{b+c\})x+ \\
\qquad +(-11+54a+18\{b+c\}), & \dfrac{5}{3}\le x\le 2\\
1, & x\ge 2.
\end{cases}
$$

The following graphs show five different densities $\tilde{f}_2(3;\gamma;\bullet)$ and cumulative distribution functions $\tilde{F}_2(3;\gamma;\bullet)$ for the sum $S_2 = X_1 + X_2$, for various choices of $\gamma = (a,b,c)$.



Plots of $\tilde{f}_2(3;\gamma;\bullet)$ for various choices of γ



Plots of $\tilde{F}_2(3;\gamma;\bullet)$ for various choices of γ

For finding the “worst” VaR scenario in this setup we have to minimize the cumulative distribution function

$$\tilde{F}_2(3;\gamma;x) = \frac{3}{2}(-6+9a+3\{b+c\})x^2 + 3(4-6a-18\{b+c\})x + (-11+54a+18\{b+c\})$$

at the point $x = \frac{5}{3}$, which is the solution of the following linear programming problem:

$$\min!\ 6a+2b+2c \text{ under the conditions}$$

$$\begin{aligned}
a + b &\le \frac{1}{3} & 4a + b + 2c &\ge \frac{2}{3}\\
a + c &\le \frac{1}{3} & 4a + 2b + c &\ge \frac{2}{3}\\
3a + b + c &\le \frac{2}{3} & a,b,c &\ge 0.
\end{aligned}$$

The solution of this problem is given by all $\gamma = \left(0, b, c\right)$ fulfilling the condition $b + c = \frac{4}{9}$. In particular, $b = c = \frac{2}{9}$ is a feasible solution, which is among the cases shown above (blue line).

In a similar way, we can determine the "best" VaR scenario here, which is uniquely determined by the parameter $\gamma = \left(\frac{2}{9}, 0, 0\right)$. This case is also shown above (pink line).

Naturally, it is also possible to express the quantile function $Q(3; \gamma; \bullet)$ for all choices of γ in an explicit way, by solving the appropriate quadratic equations in the representation of $\tilde{F}_2(3; \gamma; \bullet)$ above.

For the three cases "worst" VaR scenario, independence and "best" VaR scenario, we obtain

$$Q(3;\gamma;u)=\begin{cases}\begin{cases}\frac{4}{3}+\frac{1}{3}\sqrt{9u-7}, & \frac{7}{9}\le u\le\frac{8}{9}\\ 2-\sqrt{1-u}, & \frac{8}{9}\le u\le 1,\end{cases} & \gamma=\left(0,\frac{2}{9},\frac{2}{9}\right)\\ 2-\sqrt{2(1-u)},\quad \frac{7}{9}\le u\le 1, & \gamma=\left(\frac{1}{9},\frac{1}{9},\frac{1}{9}\right)\\ \frac{5}{3}-\frac{1}{2}\sqrt{2(1-u)},\quad \frac{7}{9}\le u\le 1, & \gamma=\left(\frac{2}{9},0,0\right).\end{cases}$$

Likewise, for the Expected Shortfall[1], we obtain

$$\mathrm{ES}(3;\gamma;u)=\frac{1}{1-u}\int_u^1 Q(3;\gamma;v)\,dv=\begin{cases}\begin{cases}\dfrac{\dfrac{38-36u}{27}-\dfrac{2}{81}\sqrt{9u-7}^3}{1-u}, & \dfrac{7}{9}\le u\le\dfrac{8}{9} \\ 2\left(1-\dfrac{1}{3}\sqrt{1-u}\right), & \dfrac{8}{9}\le u\le 1,\end{cases} & \gamma=\left(0,\dfrac{2}{9},\dfrac{2}{9}\right) \\ 2-\dfrac{1}{3}\sqrt{2(1-u)}, \quad \dfrac{7}{9}\le u\le 1, & \gamma=\left(\dfrac{1}{9},\dfrac{1}{9},\dfrac{1}{9}\right) \\ \dfrac{5}{3}-\dfrac{1}{3}\sqrt{2(1-u)}, \quad \dfrac{7}{9}\le u\le 1, & \gamma=\left(\dfrac{2}{9},0,0\right).\end{cases}$$

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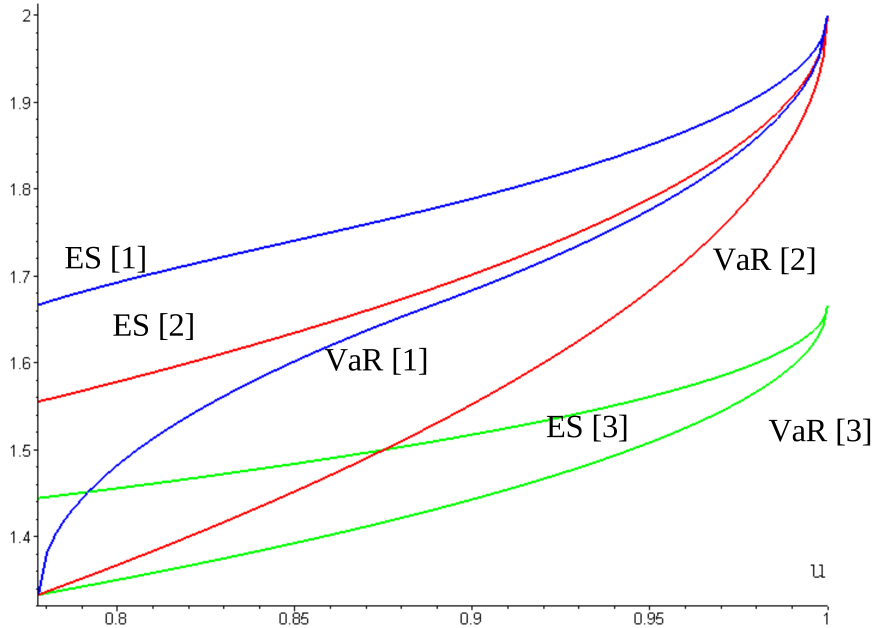
[1] Here we denote $\mathrm{ES}_u = ES(n;\gamma;u)=E\left(S_d \mid S_d \ge VaR_u\right)$ with $VaR_u = Q(n;\gamma;u)$ for $0<u<1$. In the literature, one often finds the complementary notation with $\varepsilon = 1-u$.



VaR and Expected Shortfall for "worst" case [1], independence case [2], and "best" case [3]

For instance, we obtain

	worst case	independence	best case
$VaR_{0,9}$	1,6838	1,5528	1,4430
$ES_{0,9}$	1,7892	1,7019	1,5269
$VaR_{0,99}$	1,9000	1,8586	1,5960
$ES_{0,99}$	1,9333	1,9057	1,6225

Example. Suppose that the risks X_1 and X_2 are each uniformly distributed and their joint cumulative distribution function is a copula of Clayton or Gumbel type, resp., i.e.

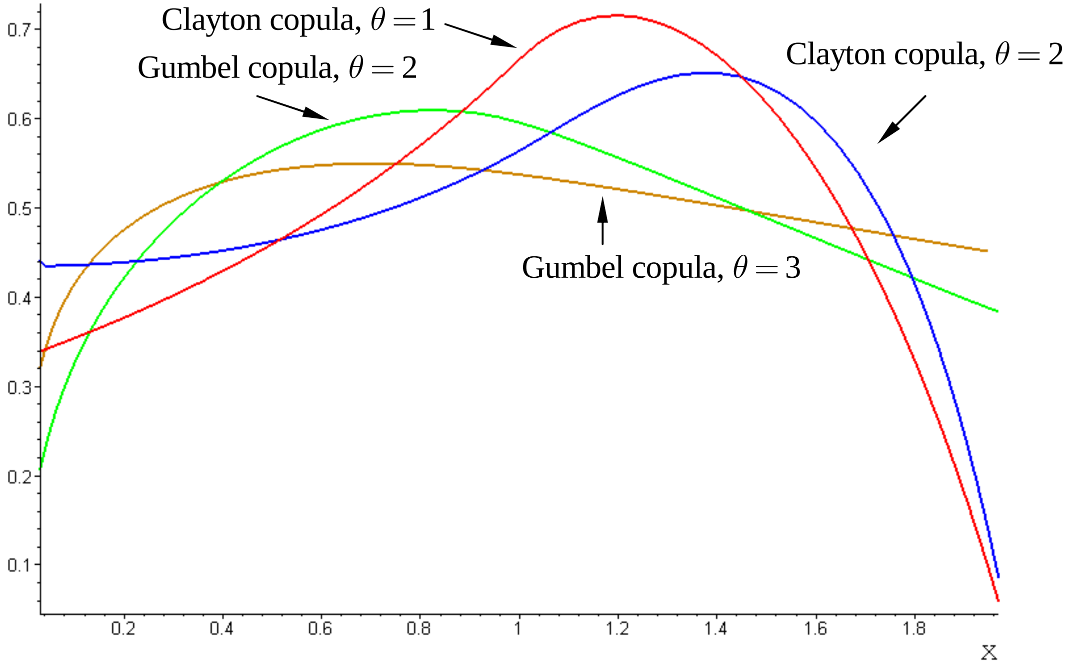
$$C_1(\theta;u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta} \quad \text{(Clayton)} \quad \text{or}$$

$$C_2(\theta;u,v) = \exp\left(-\left\{(-\ln u)^{\theta} + (-\ln v)^{\theta}\right\}^{1/\theta}\right) \quad \text{(Gumbel)}$$

for $(u,v) \in (0,1)^2$,

with $\theta \geq 1$.

We consider the distribution of the aggregated risk $S_2 := X_1 + X_2$. The following graph shows the calculated densities for S_2 under these copulas, for a grid-type copula approximation, with 10000 subsquares of the unit interval of equal area each.



approximation of densities for two aggregate dependent risks

Example (extension to joint distribution with compact support)**.** We consider five dependent risks $X_1, \cdots, X_5$ with different marginal distributions and joint density given by

$$f(x_1, \cdots, x_5) = \sum_{i=1}^{20} \sum_{j=1}^{20} \sum_{k=1}^{20} \sum_{l=1}^{20} \sum_{m=1}^{20} \alpha(i, j, k, l, m) \cdot \mathbb{1}_{I(i,j,k,l,m)}(x_1, \cdots, x_5)$$

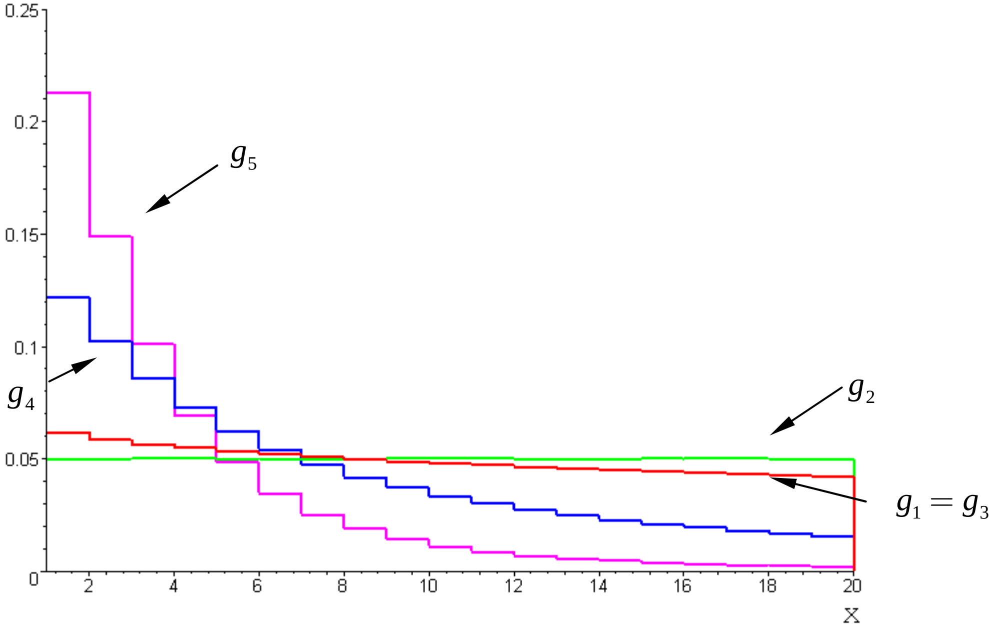
with $I(i, j, k, l, m) := (i-1, i] \times (j-1, j] \times (k-1, k] \times (l-1, l] \times (m-1, m]$ and

$$\alpha(i, j, k, l, m) = \frac{1}{K} \cdot \frac{\frac{4}{3} + \sin(i+j+k)}{i + \sqrt{j} + k + l^2 + m^3 - 4} \quad \text{for } i, j, k, l, m \in \{1, \cdots, 20\},$$

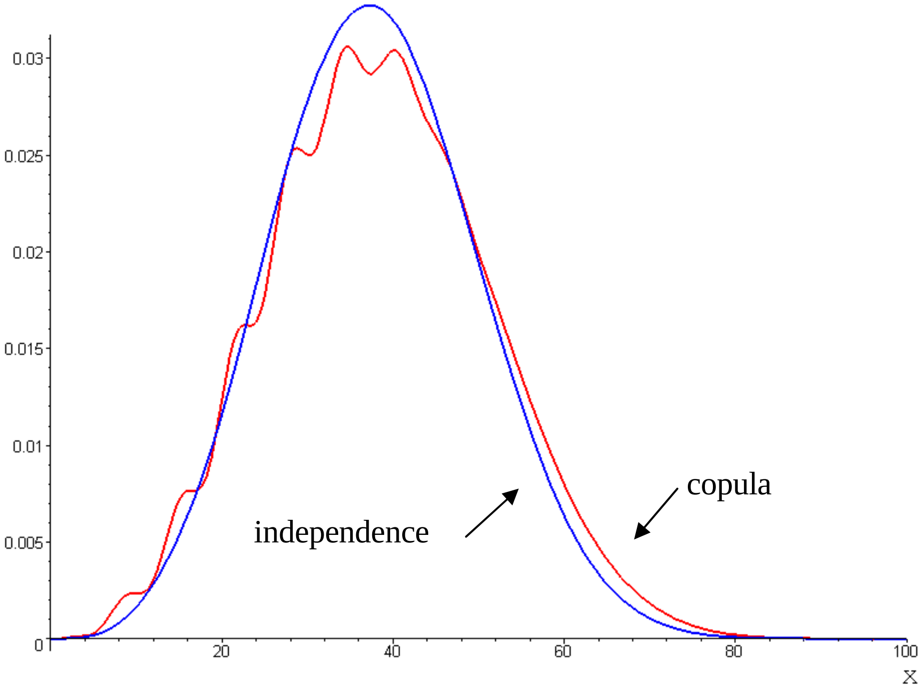
with the normalizing constant

$$K = \sum_{i=1}^{20} \sum_{j=1}^{20} \sum_{k=1}^{20} \sum_{l=1}^{20} \sum_{m=1}^{20} \frac{\frac{4}{3} + \sin(i+j+k)}{i + \sqrt{j} + k + l^2 + m^3 - 4} \approx 12198.$$

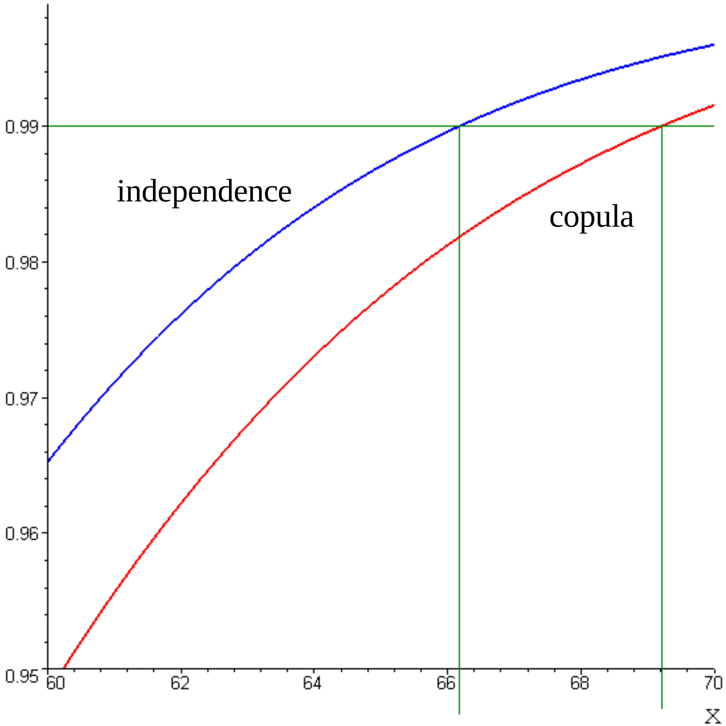
(support of the joint distribution: $20^5 = 3\,200\,000$ disjoint hypercubes in $\mathbb{R}^5$)



plot of marginal densities



density of five aggregated risks, dependent vs. independent case



cumulative distribution function, dependent vs. independent case

Lemma. Suppose that the risks X_1 and X_2 follow a Pareto distribution with density

$$f(x) = \beta(1+x)^{-(1+\beta)}, \ \ x \geq 0 \quad (\beta > 0)$$

each. Then the density g and cumulative distribution function G of the aggregated risk $S_2 := X_1 + X_2$ can be explicitly calculated in the following cases:

<u>$\beta = 1/2:$</u>

Case 1: X_1 and X_2 are independent:

$$g(z) = \frac{z}{(2+z)^2\sqrt{1+z}} \approx \frac{1}{\sqrt{1+z}^3}, \ \ G(z) = 1 - 2\frac{\sqrt{1+z}}{2+z}, \ \ z \geq 0$$

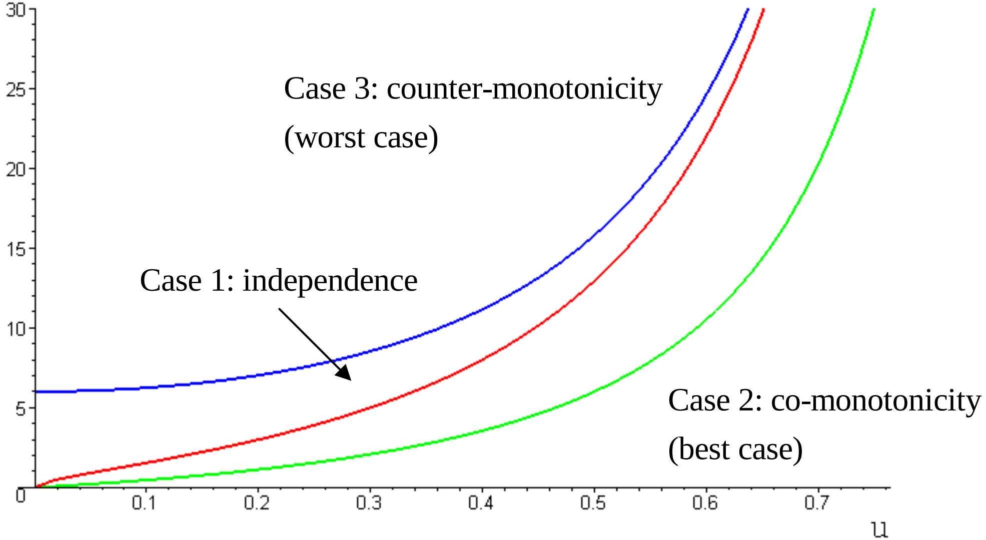
Case 2: X_1 and X_2 are co-monotonic:

$$g(z) = \frac{1}{4\sqrt{1+z/2}^{\,3}} \approx \frac{1}{\sqrt{2}\sqrt{1+z}^{\,3}}, \quad G(z) = 1 - \sqrt{\frac{2}{2+z}}, \; z \geq 0$$

Case 3: X_1 and X_2 are counter-monotonic:

$$g(z) = \frac{4+z-2\sqrt{3+z}}{\sqrt{z+6-4\sqrt{3+z}}\;\sqrt{3+z}\sqrt{2+z}^{\,3}} \approx \frac{1}{\sqrt{1+z}^{\,3}},$$

$$G(z) = \frac{\sqrt{z^2+8z+12-(8+4z)\sqrt{3+z}}}{2+z}, \quad z \geq 6.$$



VaR’s for the three cases

$\beta = 1$:

Case 1: X_1 and X_2 are independent:

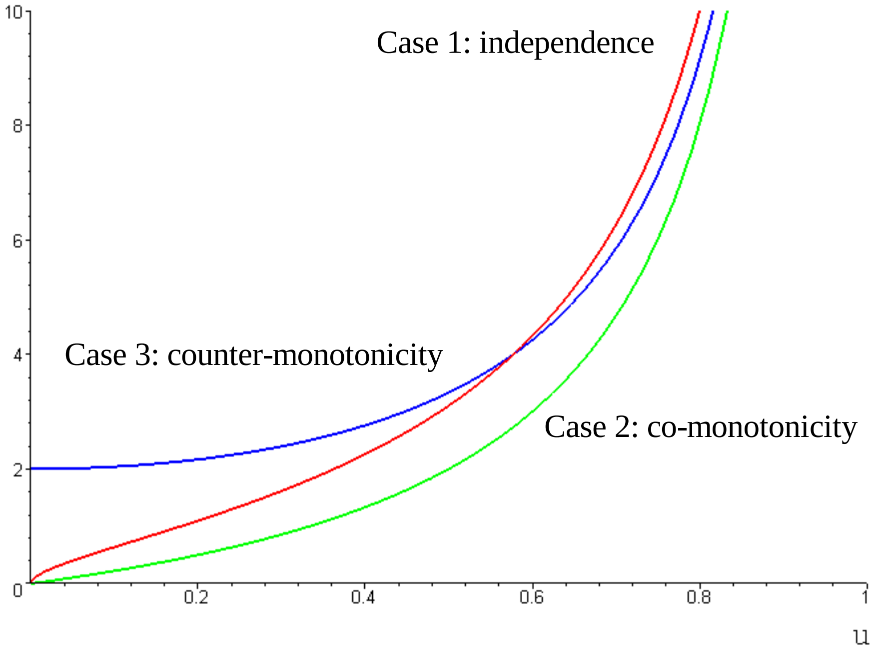
$$g(z) = 4\frac{\ln(1+z)}{(2+z)^3} + \frac{2z}{(1+z)(2+z)^2} \approx \frac{2}{(1+z)^2}, \quad G(z) = \frac{z^2 + 2z - 2\ln(1+z)}{(2+z)^2}, \quad z \ge 0$$

Case 2: X_1 and X_2 are co-monotonic:

$$g(z) = \frac{1}{2(1+z/2)^2} = \frac{2}{(2+z)^2} \approx \frac{2}{(1+z)^2}, \quad G(z) = 1 - \frac{2}{2+z}, \quad z \ge 0$$

Case 3: X_1 and X_2 are counter-monotonic:

$$g(z) = \frac{2}{\sqrt{z-2}(2+z)^{3/2}} \approx \frac{2}{(1+z)^2}, \quad G(z) = \sqrt{\frac{z-2}{z+2}}, \quad z \ge 2.$$



VaR's for the three cases

$\underline{\beta = 2:}$

Case 1: X_1 and X_2 are independent:

$$g(z) = \frac{48\ln(1+z)}{(2+z)^5} + \frac{4z\left(10+10z+z^2\right)}{(2+z)^4(1+z)^2} \approx \frac{4}{(1+z)^3},$$

$$G(z) = z\frac{z^3+7z^2+16z+6}{(2+z)^3(1+z)} - \frac{12}{(2+z)^5}\ln(1+z), \quad z \geq 0$$

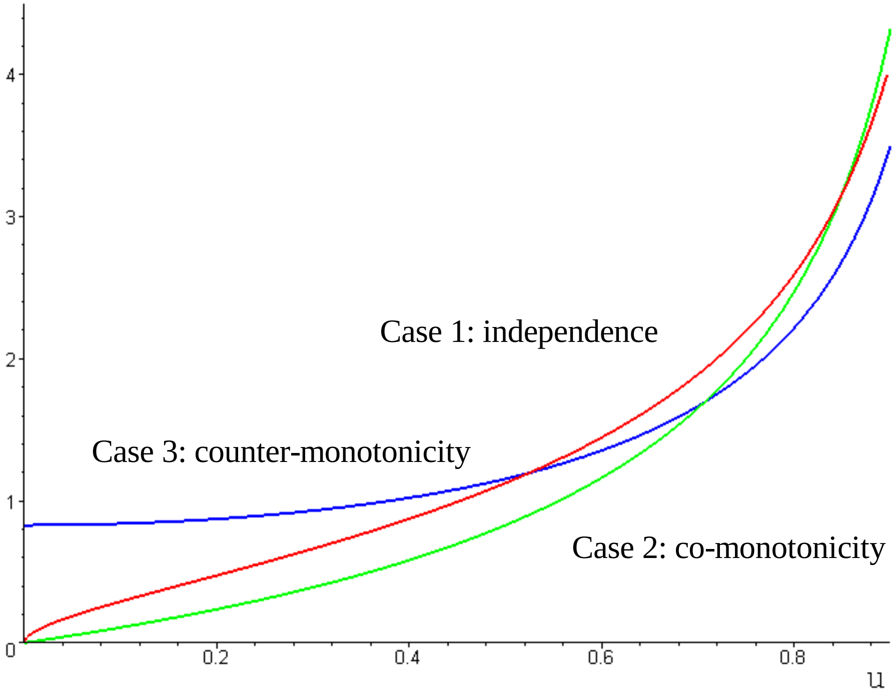
Case 2: X_1 and X_2 are co-monotonic:

$$g(z) = \frac{8}{(2+z)^3} \approx \frac{8}{(1+z)^3}, \quad G(z) = 1 - \frac{4}{(2+z)^2}, \quad z \geq 0$$

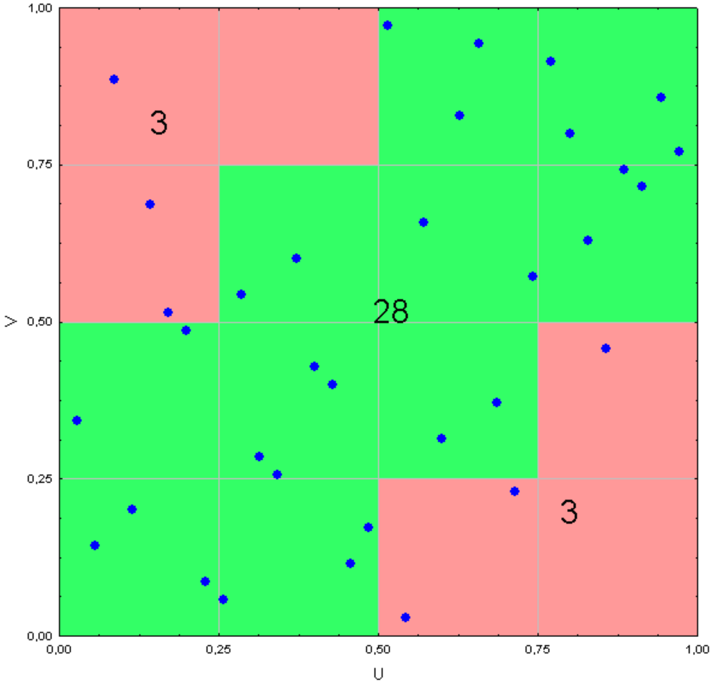
Case 3: X_1 and X_2 are counter-monotonic:

$$g(z) = \frac{4(2+z)}{\sqrt{z^2+4z+2-2\sqrt{z^2+4z+5}}\left(\sqrt{z^2+4z+5}-1\right)^2\sqrt{z^2+4z+5}} \approx \frac{4}{(1+z)^3},$$

$$G(z) = \frac{1}{(2+z)^2}\sqrt{(2+z)^4-4(2+z)^2-8-8\sqrt{(2+z)^2+1}}, \quad z \geq 2.$$



VaR's for the three cases



empirical copula for windstorm vs. flooding, from real data

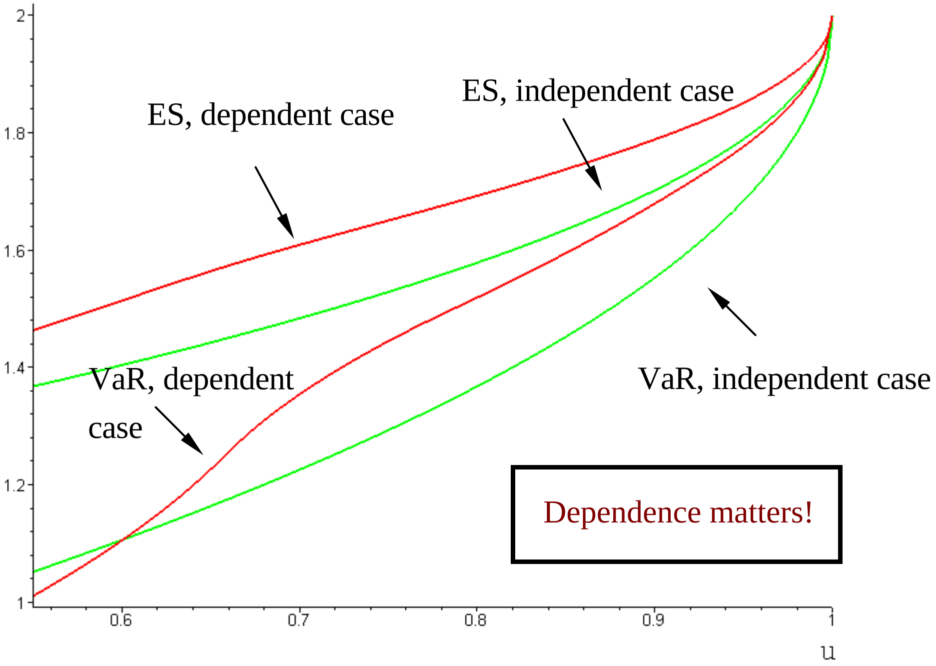
The data can be well fitted to a 4×4 grid-type copula represented by the following weight matrix:

$$A = \begin{bmatrix} \frac{13}{136} & \frac{12}{136} & \frac{8}{136} & \frac{1}{136} \\ \frac{8}{136} & \frac{15}{136} & \frac{7}{136} & \frac{4}{136} \\ \frac{8}{136} & \frac{7}{136} & \frac{7}{136} & \frac{12}{136} \\ \frac{5}{136} & 0 & \frac{12}{136} & \frac{17}{136} \end{bmatrix}$$

From this , we obtain the following density f and cumulative distribution function F for the aggregated risk $S := U + V$:

$$f(x)=\begin{cases} \frac{26}{17}x, & 0\le x\le \frac{1}{4} \\ \frac{14}{17}x+\frac{3}{17}, & \frac{1}{4}\le x\le \frac{1}{2} \\ \frac{22}{17}x-\frac{1}{17}, & \frac{1}{2}\le x\le \frac{3}{4} \\ \frac{32}{17}-\frac{22}{17}x, & \frac{3}{4}\le x\le 1 \\ \frac{28}{17}-\frac{18}{17}x, & 1\le x\le \frac{5}{4} \\ \frac{26}{17}x-\frac{27}{17}, & \frac{5}{4}\le x\le \frac{3}{2} \\ \frac{33}{17}-\frac{14}{17}x, & \frac{3}{2}\le x\le \frac{7}{4} \\ 4-2x, & \frac{7}{4}\le x\le 2 \\ 0, & \text{otherwise} \end{cases}$$

$$F(x)=\begin{cases} 0, & x<0 \\ \frac{13}{17}x^2, & 0\le x\le \frac{1}{4} \\ \frac{7}{17}x^2+\frac{3}{17}x-\frac{3}{136}, & \frac{1}{4}\le x\le \frac{1}{2} \\ \frac{11}{17}x^2-\frac{1}{17}x+\frac{5}{136}, & \frac{1}{2}\le x\le \frac{3}{4} \\ -\frac{11}{17}x^2+\frac{32}{17}x-\frac{47}{68}, & \frac{3}{4}\le x\le 1 \\ -\frac{9}{17}x^2+\frac{28}{17}x-\frac{39}{68}, & 1\le x\le \frac{5}{4} \\ \frac{13}{17}x^2-\frac{27}{17}x+\frac{197}{136}, & \frac{5}{4}\le x\le \frac{3}{2} \\ -\frac{7}{17}x^2+\frac{33}{17}x-\frac{163}{136}, & \frac{3}{2}\le x\le \frac{7}{4} \\ -x^2+4x-3, & \frac{7}{4}\le x\le 2 \\ 1, & x>2. \end{cases}$$



VaR and ES, dependence vs. independence