Modelling the Underwriting Cycle

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Abstract

A model of the insurance market is proposed which enables a better understanding of underwriting cycles. Price and quantity of insurance are determined by the equilibrium of supply and demand. Supply is driven by the amount of equity available to insurance companies. Profits an losses feed back into surplus leading to fluctuations in returns. The influence of accounting and regulatory issues on the emergence of profit is taken into account. A generalization of the model is obtained by assuming that the profit in any given fiscal year is a random variable. A further generalization is proposed which allows for capital inflows and outflows in excess of retained earnings and losses.

In general market price and quantity are defined by an implicit equation which must be solved numerically. The capital of individual suppliers is defined by a system of difference equations. A simplified version of the model leads to an explicit solution for equilibrium price and quantity. The aggregate market capital is defined by a linear difference equation with constant coefficients. The behaviour of solutions is analyzed.

The literature on underwriting cycles is reviewed. Within the context of our model it is possible to analyze the impact of different contributing factors and their interaction in an integrated framework. This leads to a better understanding of underwriting cycles.

Keywords

Underwriting cycle, insurance market, supply, demand, market equilibrium, market price, market quantity, difference equations, stability, equilibrium points, time series, stationarity, autoregressive processes.

1 The General Model

1.1 Introduction

Practitioners all experience the cyclical nature of insurance markets. Periods of low, sometimes even negative returns alternate with periods of high profits. The characteristics of the cycle, its length, its amplitude, etc. vary between market segments - e.g. private lines and commercial lines - between geographical markets and over time. Cycles however, are a universal feature of insurance markets.

Different theories have been proposed to explain the cyclical nature of insurance profits. In section 1.2 we review some of this material. In section 1.3 and 1.4 we propose our own model which is based on the assumption that cycles stem from changes in supply of and demand for insurance. The insurance supply is defined by insurance providers who are characterized by their amount of capital, their targeted return on capital and their supply elasticity i.e. the flexibility with which they respond to market returns in excess or below their own target returns. Similarly the demand for insurance is defined by insurance buyers who are characterized by their individual base demand for insurance and targeted returns on capital. The demand elasticity is a further characteristic of each individual buyer. For each period equilibrium price and equilibrium quantity result from the intersection of supply and demand curves. The price of insurance, i.e. the rate of return on risk based capital, is equal to the rate of return which clears the market i.e. to that rate of return for which offer and demand are equal. The quantity of insurance transacted during a given period, i.e. the amount of pure risk premium underwritten during that period, is equal to the quantity corresponding to the equilibrium rate of return. The net profit after taxes and dividends is reinvested. Profits and losses feed into surplus, leading to offsetting changes in supply. Base demand increases according to growth rates which are driven by the economic cycle leading to further shifts in market equilibrium. Underwriting cycles are thus driven by features which lie at the heart of the insurance business.

The speed with which profits and losses emerge can exacerbate or dampen underwriting cycles. Accounting factors and regulatory constraints thus have an impact on underwriting cycles. In a more refined version of the model capital flows in and out of the market are taken into account. The model also allows an analysis of the consequences of supply and demand shifts stemming from shocks such as catastrophe losses, industry-wide, sharp increases in loss reserves, significant changes in the value of assets, redirection of business to insurance captives or risk retention groups, etc.

The theory of difference equations as well as the theory of time series provide valuable insights into the behaviour of insurance markets. The focus of the article is on difference equations rather than on time series, i.e. on expected values rather than on realizations of random variables. However equilibrium points and characteristic roots are of relevance to both difference equations and time series. The difference equation models proposed in the article can be generalized in a straightforward way to time series models.

1.2 Related Literature

Many studies have addressed the issue of cyclical underwriting returns and have proposed competing theories to explain the phenomenon. important contributions include the ones mentioned below.

The irrational-expectation hypothesis by E. Venezian (1985): the author proposes a model for rate-making. Future loss costs are estimated based on historical loss costs using a regression analysis. Venezian shows that the random fluctuations contained in past loss costs lead to future underwriting results which are autocorrelated. The theory is supported by empirical evidence. The author analyses the performance of thirteen lines of business in the US P&C insurance industry between 1960 and 1980. He shows that the profit margins of eleven lines are best described by and autoregressive process of order two (AR(2) process). The fitted AR(2) processes are all stationary and cyclical.

The rational-expectations / institutional intervention hypothesis by J.D. Cummins and J. F. Outreville (1987): the authors postulate that cycles are created in an otherwise rational market through institutional, regulatory and accounting factors. The authors assume that the subjective expected values of future losses are equal to the objective expectations based on information available at the time rates are established. Data collection lags, regulatory lags, policy renewal lags and lags due to accounting rules lead to cyclicality in underwriting results. The data from thirteen countries for the period 1957 to 1979 is analyzed. In most countries there is evidence of cyclicality and the best fit is provided by an AR(2) process. The findings are confirmed by J. Lamm-Tennant and M.A. Weiss (1997).

The feedback of profit and losses into surplus postulated by L. A. Berger (1988) is a further attempt at explaining the cyclical nature of underwriting returns. Berger provides a fully fledged model of the underwriting cycle. The market for insurance is modelled by standard supply and demand theory. Equilibrium price and quantity result from the equilibrium of supply and demand. Profits and losses feed into surplus leading to shifts in supply which offset the profit or loss trend. The behaviour of the model is consistent with usual cycle theory. The capacity-constraint hypothesis by R. Winter (1988) emphasizes a special feature of the above theory. Winter suggests that insurers will not immediately adjust surplus when it is reduced by external shocks because in adverse circumstances, external equity is expensive. This leads prices to increase when insurance capacity or equity is reduced.

The change-in-expectations hypothesis of G. Lai and R. C. Whitt (1992) and of G. Lai et al. (2000) postulates that increased uncertainty in particular about loss costs leads to higher premiums. Lai et al. propose an economic model of the insurance market. Suppliers and buyers are characterised by their utility functions. Insurance price is derived from market equilibrium. Under certain assumptions an explicit formula is derived for insurance price and quantity. It is shown that prices increase when the variance of losses increases.

External factors are invoked as an explanation for the underwriting cycle. M. F. Grace and R. E. Hoyt (1995) find a long-run relationship between general economic changes and underwriting performance. N. A. Doherty and B. Kang (1988) suggest that fluctuation in interest rates create insurance pricing cycles.

A certain number of articles rely on data analysis alone to investigate the determinants of insurance prices. For instance M. A. Weiss and J.-H. Chung (2004) perform a multiple regression analysis with the price of US non-proportional reinsurance as the dependent variable. They find that capital strength and overall capacity are significant factors determining reinsurance prices.

1.3 Market Equilibrium

1.3.1 Supply

Let us first assume that there is only one company selling insurance. Let C denote the amount of capital of the company. Let P denote the pure risk premium corresponding to the amount of insurance sold by the company (i.e. the expected loss costs on an undiscounted basis). We refer to P as to the quantity of insurance. It is assumed that the amount of risk based capital necessary to underwrite P is equal to $\frac{P}{\kappa^s}$ where κ^s is the leverage of the company. The company is further characterized by the following quantities: r^s is the target rate of return on risk based capital and ϵ^s is the supply elasticity. The exact meaning of the last two parameters becomes clear in the light of the supply function defined below. Finally let r be the rate of return on risk based capital achieved in the market. This quantity is referred to as the price of insurance. It is assumed that the following relation holds true between the quantity of insurance (P) and the price of insurance (r)

$$P = \kappa^s C + \epsilon^s (r - r^s) \frac{P}{\kappa^s}.$$

$\kappa^s C$ is the base supply, i.e. that quantity of insurance which is provided when the market rate of return on risk based capital (r) is equal to the target rate of return of the supplier (r^s). $r^s \frac{P}{\kappa^s}$ is the loading required by the supplier for assuming the quantity of insurance risk P. $r \frac{P}{\kappa^s}$ is the loading effectively achieved in the market. $(r - r^s)\frac{P}{\kappa^s}$ is thus the economic value added. It is reasonable to assume that the amount of insurance provided in excess of the base supply $(P - \kappa^s C)$ is proportional to the economic value added, where the elasticity ϵ^s is the proportionality factor. Hence the above relation between insurance quantity and insurance price. Expressing quantity as a function of price, we obtain

$$P = \frac{\kappa^s C}{1 - \frac{\epsilon^s}{\kappa^s}(r - r^s)}.$$

The supply function is usually defined as the inverse function

$$r = r^s + \frac{\kappa^s}{\epsilon^s}(1 - \frac{\kappa^s C}{P})$$

where price is expressed as a function of quantity.

Note that both rates of return r and r^s apply to the risk based capital $\frac{P}{\kappa^s}$ and not to the effective amount of capital C. It is assumed that the owners of the company define the rate of return in terms of the effective risk they assume. This is realistic as long as the risk based capital is not significantly larger than the effective capital.

1.3.2 Demand

Let us first consider commercial insurance buyers. For the time being we assume that there is only one company buying insurance. As in the previous section, let P, the pure risk premium, denote the quantity of insurance bought. It is assumed that the amount of capital allocated by the buyer to the insurance risk is equal to $\frac{P}{\kappa^d}$ where κ^d is the leverage of the buyer. The buyer is further characterized by the following quantities: P^0 the base demand, r^d the cost of capital and ϵ^d the demand elasticity. The fair loading which the buyer is prepared to pay in excess of the expected loss costs P is thus equal to $r^d \frac{P}{\kappa^d}$. The loading required by the market is $r\frac{P}{\kappa^s}$.It is thus reasonable to assume that the following relation holds true between the quantity of insurance bought (P) and the price of insurance (r)

$$P = P^0 - \epsilon^d\left(\frac{r}{\kappa^s} - \frac{r^d}{\kappa^d}\right)P$$

Introducing the modified target rate of return of the buyer

$$r^{d'} = r^d \frac{\kappa^s}{\kappa^d}$$

we obtain a similar equation as for the supply side

$$P = P^0 - \epsilon^d(r - r^{d'})\frac{P}{\kappa^s}.$$

The quantity of insurance bought is given by the following function of price

$$P = \frac{P^0}{1 + \frac{\varepsilon^d}{\kappa^s}(r - r^{d'})} = \frac{P^0}{1 + \epsilon^d(\frac{r}{\kappa^s} - \frac{r^d}{\kappa^d})}$$

and the demand function is

$$r = r^{d'} + \frac{\kappa^s}{\epsilon^d}\left(\frac{P^0}{P} - 1\right) = r^d\frac{\kappa^s}{\kappa^d} + \frac{\kappa^s}{\epsilon^d}\left(\frac{P^0}{P} - 1\right).$$

In the case of private insurance buyers it is assumed that buyers price risks according to the expected value principle and that they all use the same loading. Let P^0 denote the aggregate base demand. Let P denote the aggregate demand and let αP denote the fair loading from the buyers' viewpoint. Assume that all buyers have the same demand elasticity ϵ^d. We have

$$P = P^0 - \epsilon^d\left(r\frac{P}{\kappa^s} - \alpha P\right) = P^0 - \epsilon^d(r - r^{d^*})\frac{P}{\kappa^s}$$

with $r^{d^*} = \alpha\kappa^s$, which is of the same form as the equation we have obtained for commercial buyers.

1.3.3 Equilibrium Price and Quantity

The equilibrium price and the equilibrium quantity result from the intersection of the supply and of the demand curve. By equating the insurance quantity bought, respectively sold, expressed as a function of price, we obtain the equilibrium price r^m

$$r^m = \frac{(\epsilon^d C) r^{d'} + (\epsilon^s \frac{P^0}{\kappa^s}) r^s}{\epsilon^d C + \epsilon^s \frac{P^0}{\kappa^s}} + \frac{P^0 - \kappa^s C}{\epsilon^d C + \epsilon^s \frac{P^0}{\kappa^s}} \tag{1.3.3.1}$$

The market rate of return is a weighted average of the supply target rate of return and of the modified demand target rate of return plus a correction term depending on the excess of base demand over base supply.

By equating the price as expressed in the supply respectively in the demand function one derives the equilibrium insurance quantity P^m

$$P^m = \frac{\epsilon^s P^0 + \epsilon^d (\kappa^s C)}{\epsilon^s + \epsilon^d + \epsilon^s \epsilon^d (\frac{r^s}{\kappa^s} - \frac{r^d}{\kappa^d})} \tag{1.3.3.2}$$

The equilibrium quantity is a weighted average of the base demand and of the base supply with an additional term in the denominator which depends on the difference in targeted return on sales of buyers and sellers.

The equilibrium loading is

$$\lambda^m = r^m \frac{P^m}{\kappa^s} = \frac{(\epsilon^d r^{d'} - \kappa^s) C + (1 + \epsilon^s \frac{r^s}{\kappa^s}) P^0}{\epsilon^s + \epsilon^d + \epsilon^s \epsilon^d (\frac{r^s}{\kappa^s} - \frac{r^d}{\kappa^d})} \tag{1.3.3.3}$$

The fact that the expression for the loading simplifies will prove to be useful later.

1.3.4 Retained Earnings and Capital Growth

It is assumed that the amount of capital of the company in the next period (C_{t+1}) is equal to the amount of capital in the current period (C_t) plus a share of the equilibrium loading generated during the current period (λ_t^m) plus a share of the return on capital generated by the financial risk of the company ($r^f C_t$).Let ρ_t denote the share of the earnings which is contributed to the capital of the next period. ($1 - \rho_t$ is the share of the earnings which is paid out as taxes and dividends.) The capital of the company in the next period is given by the following formula

$$C_{t+1} = C_t + \rho_t r^f C_t + \rho_t \lambda_t^m \tag{1.3.3.4}$$

We assume that the accounting is based on economic value, i.e. that assets and liabilities are marked to market. λ_t^m is the insurance operating return. It is the expected net present value of premiums less claims payments discounted to the middle of the period. Costs are not taken into account. $r^f C_t$ is the expected return on capital generated by the financial risk of the company. It stems from the investment income generated by the assets matching the capital and from any excess return generated by a possible mismatch between assets and liabilities. In section 3.1, we broaden the definition of this risk. For the time being we ignore random fluctuations around expected values. For more details on the model see R. Schnieper [2000].

1.4 The General Model

From now on we consider a multi-period model. (Typically a period is a year but other units of time can be considered: months, quarters, etc.) We assume that both the supply side and the demand side of the market consist of many participants.

An insurance market consists of a certain number of suppliers and of a certain number of insurance buyers. It is assumed that the suppliers specialize in providing insurance to the said group of buyers and that the buyers essentially only buy from the said group of suppliers. Examples of insurance markets are: the Swiss, the French or the UK personal lines market, the international market for large corporate risks, the worldwide reinsurance market. Whilst certain insurance groups are active in many such markets, we focus on those legal entities or departments within the group which are dedicated to a specific market.

Let $C_{k,t}$ denote the amount of capital of company k at the beginning of period t. It is assumed that base demand of all buyers increases by a factor γ_t during period t, therefore base demand of buyer l at the beginning of period t is

$$P_{l,t}^0 = \gamma_0 \cdot \gamma_1 \cdot \ldots \cdot \gamma_{t-1} \cdot P_l^0 = \Gamma_t \cdot P_l^0$$

where

$$\Gamma_t = \gamma_0 \cdot \gamma_1 \cdot \ldots \cdot \gamma_{t-1}.$$

The amount of insurance sold by company k during period t is defined by

$$P_{k,t} = \kappa^s \cdot C_{k,t} + \varepsilon_k^s (r - r_k^s) \cdot \frac{P_{k,t}}{\kappa^s} \qquad (1.4.1)$$

and the amount of insurance bought by buyer number l during period t is defined by

$$P_{l,t} = \Gamma_t \cdot P_l^0 - \varepsilon_l^d \left(\frac{r}{\kappa^s} - \frac{r_l^d}{\kappa_l^d}\right) \cdot P_{l,t} \qquad (1.4.2)$$

Note that the leverage factors, the target rates of return as well as the elasticities are time independent. Note that the leverage factor of the supply side of the market is the same for all suppliers i.e. there is a market consensus about the amount of risk based capital necessary to write a given quantity of pure risk premium. The market equilibrium for period t is reached for that value of r for which aggregate supply and aggregate demand are equal

$$\sum_{k=1}^{n_s} \frac{\kappa^s C_{k,t}}{1 - \frac{\varepsilon_k^s}{\kappa^s}(r - r_k^s)} = \sum_{l=1}^{n_d} \frac{\Gamma_t P_l^0}{1 + \varepsilon_l^d \left(\frac{r}{\kappa^s} - \frac{r_l^d}{\kappa_l^d}\right)} \qquad (1.4.3)$$

Let r_t^m denote the solution of this equation. (It is easily seen that the above equation admits exactly one solution.) r_t^m is the market rate of return achieved during period t. Let $P_{k,t}^s$ denote the amount of insurance sold by supplier k during period t. It is derived by solving (1.4.1) for $P_{k,t}$ and by inserting r_t^m for r. The amount of profit made by company k during period t

$$\lambda_{k,t}^s = r_t^m \frac{P_{k,t}^s}{\kappa^s} = \frac{r_t^m C_{k,t}}{1 - \frac{\varepsilon_k^s}{\kappa^s}(r_t^m - r_k^s)}$$

Let $\rho_{k,t}$ denote the profit retention rate of company k at the end of period t. Let r_k^f denote the rate of return generated by the financial risk of company k including the return on assets matching the equity. Let us also assume that company k does not return any capital in addition to the dividend or obtain any capital increase in addition to the retained earnings. Company k's capital at the beginning of period $t+1$ is thus

$$C_{k,t+1} = C_{k,t} + \rho_{k,t} r_k^f C_{k,t} + \rho_{k,t} \lambda_{k,t}^s$$

$$C_{k,t+1} = C_{k,t}\left(1 + \rho_{k,t} r_k^f + \frac{\rho_{k,t} r_t^m}{1 - \frac{\varepsilon_k^s}{\kappa^s}(r_t^m - r_k^s)}\right) \quad k = 1, ..., n_s \qquad (1.4.4)$$

On the other hand, the base demand from buyer l at the beginning of period $t+1$ is

$$P_{l,t+1}^0 = \Gamma_{t+1} \cdot P_l^0, \qquad l = 1, ..., n_\alpha.$$

and it is seen that if we specify the starting capital of each company

$$C_{k,0} \qquad \text{for } \; k = 1, ..., n_s$$

the system of difference equations (1.4.4) together with the implicit equation (1.4.3) for the market rate of return enables us to compute the capital of each company at the beginning of each period together with the achieved market rate of return in each period. Examples are given below. No general statement can be made about the behaviour of the solution of this system of difference equations since it depends on unspecified time-varying profit retention rates and demand growth rates.

An interesting special case arises when these quantities are time independent. Let γ_t=γ for all t, hence

$$\Gamma_t = \gamma^t$$

We also assume

$$\rho_{k,t} = \rho_k \quad \text{for all } t$$

We make the following variable changes

$$\overline{C}_{k,t} = \frac{C_{k,t}}{\gamma^t} \qquad \textit{and} \qquad \overline{P}_{l,t}^0 = \frac{P_{l,t}^0}{\gamma^t} = P_l^0$$

Definitions

$\overline{C}_{k,t}$ is the **standardized capital** of company k at the beginning of period t and $\overline{P}^0_{l,t} = P^0_l$ is the **standardized base demand** of buyer l.

The market rate of return during period t, r^m_t, is the solution of the following equation

$$\sum_{k=1}^{n_s} \frac{\kappa^s \overline{C}_{k,t}}{1 - \frac{\varepsilon^s_k}{\kappa^s}(r - r^s_k)} = \sum_{l=1}^{n_d} \frac{P^0_l}{1 + \varepsilon^d_l(\frac{r}{\kappa^s} - \frac{r^d_l}{\kappa^d_l})} \tag{1.4.3a}$$

and company $k's$ standardized capital at the beginning of period $t+1$ is given by the following difference equation

$$\overline{C}_{k,t+1} = \overline{C}_{k,t}\frac{1}{\gamma}(1 + \rho_k r^f_k + \frac{\rho_k r^m_t}{1 - \frac{\varepsilon^s_k}{\kappa^s}(r^m_t - r^s_k)}) \qquad k = 1, ..., n_s \tag{1.4.4a}$$

The following definition is borrowed from the theory of difference equations (See e.g. S.M. Elaydi [1999])

Definition

$(\overline{C}_1, ..., \overline{C}_{n_s}, r^m)$ is an equilibrium point of the system of difference equations (1.4.3a) and (1.4.4a) if and only if

$$\left.\begin{matrix} \overline{C}_{k,t_0} = \overline{C}_k \ k = 1, ..., n_s \\ r^m_{t_0} = r^m \end{matrix}\right\} \Rightarrow \left.\begin{matrix} \overline{C}_{k,t} = \overline{C}_k \ k = 1, ..., n_s \\ r^m_t = r^m \end{matrix}\right\} \text{ for all } t \geq t_0$$

Theorem

For time-homogeneous profit retention rates and demand growth rates

(i) There exists an equilibrium point $(\overline{C_1}, ..., \overline{C_{n_s}}, r^m)$ of the system of equations (1.4.3a) and (1.4.4a). It is characterized by the fact that $\overline{C_k} \neq 0$ for exactly one value of k.

(ii) Let k_0 denote the value of k for which $\overline{C_k} \neq 0$, the equilibrium rate of return is given by the following equation

$$r^m = \frac{\frac{1}{r^s_{k_0}} + \frac{\varepsilon^s_{k_0}}{\kappa^s}}{\frac{\rho_{k_0}}{\gamma - 1 - \rho_k r^f_k} + \frac{\varepsilon^s_{k_0}}{\kappa^s}} r^s_{k_0}$$

Proof

From the definition of the equilibrium point and from (1.4.4.a) we obtain

$$\frac{\rho_k r^m}{1 + \frac{\varepsilon^s_k}{\kappa^s}(r^m - r^s_k)} = \gamma - 1 - \rho_k r^f_k$$

for all k for which $\overline{C_k} \neq 0$. Hence if k is such that $\overline{C_k} \neq 0$, we obtain

$$r^m = \frac{1 + \frac{\varepsilon^s_k}{\kappa^s} r^s_k}{\frac{\rho_k}{\gamma - 1 - \rho_k r^f_k} + \frac{\varepsilon^s_k}{\kappa^s}}$$

Without loss of generality it can be assumed that the right hand side of the above equation is different for different values of k.(If this is not the case, one aggregates the suppliers with identical values.) Hence $\overline{C_k} \neq 0$ for at most one value of k. $\overline{C_k} = 0$ for all values of k is the degenerate case which is only achieved if the capital of all suppliers is zero Hence $\overline{C_k} \neq 0$. for exactly one value of k. Which proves (i) and (ii)

Q.E.D.

Remarks

1. The statement of the above theorem is an argument in favour of a homogeneous supply side of the market, i.e. of a model where all suppliers have the same target rate of return and elasticity. In such a case, the supply side behaves as one single seller with a capital equal to the aggregate capital of the market. In most insurance markets, the demand side consists of a large number of more or less homogeneous buyers. Assuming the same target rate of return, leverage and elasticity for all buyers is therefore realistic. In such a case, the demand side of the market also behaves as one single buyer with a base demand equal to the aggregate base demand of the market.

2. In those cases where market participants are heterogeneous, it can be shown that a suitably defined single buyer and single seller model provides a good approximation to the more complex market in the sense that both equilibrium price and quantity of the approximating model are not too different from equilibrium price and quantity of the original model. The approximating single buyer and single seller model is obtained by defining the leverage, target rate of return and elasticity as weighted averages. The weights on the supply side are equal to the capital of individual sellers. On the demand side the weights are equal to the base demand of each buyer.

1.5 Examples

1.1 The supply side of the market consists of two groups of sellers with $\kappa^s = 2.5$ and $\epsilon^s = 5$ for both groups and $r_1^s = 10\%$ and $r_2^s = 6\%$ respectively. It is further assumed that $C_{1,0} = 50$ [Monetary Unit] and $C_{2,0} = 50$ [MU].

The demand side of the market consists of one group of homogeneous buyers with $\kappa^d = 2.5, \epsilon^d = 5$ and $r^d = 8\%$.It is further assumed that $P^0 = 200$.

Since $\kappa^s = 2.5$, the choice of $C_0 = C_{1,0} + C_{2,0} = 100$ and of $P^0 = 200$ means that there is a supply overcapacity. In addition we assume $r^f = 2\%$ and $\rho = 0.5$ for all suppliers and $\gamma = 1.05$.

Applying (1.4.3a) and (1.4.4a) we have computed the values of the standardized variables. For comparison purposes we have also tabulated the values of the standardized variables of the approximating homogeneous model with $r_1^s =_2^s= 0.08\%$. The results are summarized in the following table

t	Inhomogeneous $\overline{C}_t$ [MU]	Model r_t^m [%]	Homogeneous $\overline{C}_t$ [MU]	Model r_t^m [%]
0	100	2.41	100	2.44
1	97.2	3.11	97.2	3.14
2	94.8	3.72	94.9	3.75
5	89.5	5.17	89.5	5.19
10	84.4	6.60	84.5	6.62
20	80.9	7.66	81.0	7.68
30	80.1	7.90	80.2	7.93

Over the above range, the approximation of the inhomogeneous model by the corresponding homogeneous model is excellent. The values are nearly identical. The results seem to converge. However, if we compute the solution of the system of difference equations for a much longer period, we obtain

t	Inhomogeneous $\overline{C}_t$ [MU]	Model r_t^m [%]	Homogeneous $\overline{C}_t$ [MU]	Model r_t^m [%]
100	79.6	7.94	80	8.00
200	79.3	7.90	80	8.00
500	78.5	7.82	80	8.00
1000	77.9	7.75	80	8.00

For practical purposes a period of such length is meaningless. It is however interesting to notice that the solutions of the system of difference equations corresponding to the inhomogeneous model no longer seem to converge.

It is also interesting to notice that the relative weight - measured by relative equity - of the first group of suppliers decreases from 50% (t=0) to 48% (t=30) to 43% (t=100) and to 5% (t=1000).

The process corresponding to the inhomogeneous model consists of an overlap of two trends: (i) Standardized equity decreases in order to adjust to lower base demand. As a consequence the rate of return increases. (ii) Over a very long period of time, the process seems to converge towards the equilibrium point described in the theorem of section 1.4. For practical purposes however, the latter trend is irrelevant.

1.2 Model parameters are as in model 1.1 except for the following two differences $P^0 = 250$ and $\gamma = 1.04$.i.e. the starting capital and base demand are in balance but targeted net capital growth rate is higher than base demand growth rate. We obtain the following values

t	Inhomogeneous $\overline{C}_t$ [MU]	Model r_t^m [%]	Homogeneous $\overline{C}_t$ [MU]	Model r_t^m [%]
0	100	7.96	100	8.00
1	100.9	7.72	101.0	7.76
2	101.8	7.52	101.8	7.55
5	103.6	7.06	103.7	7.09
10	105.5	6.62	105.6	6.65
20	106.8	6.30	106.9	6.32
30	107.1	6.22	107.3	6.24

As in the preceding example, the approximation of the inhomogeneous model by the corresponding homogeneous model is excellent. As t grows larger, we obtain for the homogeneous model $\overline{C}_t \to 107.4$ and $r_t^m \to 6.21\%$.The unbalance between the two above mentioned growth rates leads to a slight capacity oversupply and to a corresponding decrease in the rate of return.

1.3 Model parameters are as in model 1.1 except for the following differences $P^0 = 250$ and demand growth is assumed to be cyclical. The average growth factor is assumed to be $\gamma = 1.05$ as in 1.1 above, however it is assumed that during the first two periods there is an excess growth of base demand of 5%, followed by four periods with a growth deficit of 5% and finally two periods with an excess growth of 5% thus leading to a cycle of length eight. This is modelled by multiplying base demand in the standardized model by the following factors

$$(\delta_1, ...\delta_8) = (1.05, 1.05, 0.95, 0.95, 0.95, 0.95, 1.05, 1.0606)$$

The factors are applied in a cumulative way. The last factor has been chosen in a way to ensure that the product of all factors is equal to one. After a few cycles the process converges. Below we have tabulated the results of the third cycle. They do not change much after that.

t	Base Demand [MU]	Inhomogenous $\overline{C}_t$ [MU]	Model r_t^m [%]	Homogenous $\overline{C}_t$ [MU]	Model r_t^m [%]
24	250.0	98.0	8.43	98.2	8.46
25	262.5	98.3	9.58	98.5	9.61
26	275.6	99.2	10.56	99.4	10.60
27	261.8	100.7	8.91	100.9	8.94
28	248.8	101.2	7.50	101.4	7.52
29	236.3	100.9	6.28	101.1	6.31
30	224.5	100.0	5.23	100.2	5.26
31	235.7	98.6	6.81	98.8	6.84

The approximation of the inhomogeneous model by the corresponding homogeneous model is excellent. The examples illustrates the fact that irregular growth in base demand, stemming e.g. from the economic cycle leads to cyclical returns.

2. We now consider an entity which accounts on a funded basis. Lloyd's is a case in point. Profits from a given underwriting year are recognized at the end of the third development year. In the case of a Lloyd's syndicate, capital providers pledge assets or produce guarantees to cover potential losses in excess of premiums and investment income. Neither the capital nor the matching assets or corresponding investment income appear in the accounts of the syndicate. As a consequence, capacity growth is defined by the following formula

$$C_{t+1} = C_t + \rho\lambda_{t-2}$$

or using standardized quantities

$$\overline{C}_{t+1} = \frac{1}{\gamma}\overline{C}_t + \frac{\rho}{\gamma^3} r_{t-2}^m \frac{\overline{P}_{t-2}^m}{\kappa^s}$$

In a Lloyd's context, capital is committed for one year only. Emerging results act as a signal for capital providers. The amount of capital they are prepared to commit for the next underwriting year depends on the latest results. Therefore the parameter ρ in the above formula may be well in excess of one. (Admittedly since the inflow of corporate capital and the introduction of the Franchise Board, Lloyd's capital providers behave in a more sophisticated way. It is nevertheless felt that the above simplistic assumptions are an appropriate way to model the Lloyd's of old.)

We assume the same model parameters as in example 1.1 except for $P^0 = 250$, $r^f = 0$ and $\rho = 3$ respectively $\rho = 4$. We assume the same starting values for $t = 0, 1, 2$.Applying (1.4.3a) and (1.4.4a) we have computed the values of the standardized variables. We have also tabulated the values of the standardized variables of the approximating homogeneous model with $r_1^s = r_2^s = 0.08\%$. For $\rho = 3$ We have obtained the following results

	Inhomogeneous	Model	Homogeneous	Model
t	$\overline{C}_t$ [MU]	r_t^m [%]	$\overline{C}_t$ [MU]	r_t^m [%]
2	100	7.96	100	8.00
6	164.6	-4.25	164.7	-4.23
11	100.9	7.73	101.2	7.71
16	159.1	-3.46	159.1	-3.41
21	103.9	6.98	104.4	6.93
26	154.1	-2.72	154.0	-2.63

For $\rho = 4$ we have obtained the following results

	Inhomogeneous	Model	Homogeneous	Model
t	$\overline{C}_t$ [MU]	r_t^m [%]	$\overline{C}_t$ [MU]	r_t^m [%]
2	100	7.96	100	8.00
6	193.1	-7.93	193.4	-7.91
11	78.2	14.06	78.4	14.05
15	274.7	-15.36	273.7	-15.24
20	51.1	24.08	51.7	23.93
25	504.1	-25.49	498.2	-25.28
30	30.2	34.63	30.8	34.47

Note that in both tables we have recorded the local maxima and minima of C_t.In both cases, they are identical for the inhomogeneous and the approximating homogeneous process. The quality of the approximation in the range of interest is excellent in both cases. Both processes display very strong fluctuations in returns. Based on the above results it is not clear what the asymptotic behaviour of the two processes is.

2 The Simplified Model

2.1 Model Definition

We make the following

Assumptions

1. Suppliers and buyers are homogeneous. All suppliers have the same target rate of return (r^s), elasticity (ϵ^s), profit retention rate (ρ_t) and rate of return on financial risk (r^f). (The general model of section 1 already assumes that suppliers have the same leverage, κ^s.) All buyers have the same target rate of return (r^d), elasticity (ϵ^d) and leverage (κ^d).

2. The profit retention rate (ρ) and the demand growth rate (γ) are time-independent.

The first assumption can be justified on theoretical grounds or the homogeneous model can be viewed as an approximation of the true, inhomogeneous model. (See remarks of section 1.4.) Under the above assumption, the market rate of return defining equation (1.4.3) becomes

$$\frac{\kappa^s C_t}{1-\frac{\varepsilon^s}{\kappa^s}(r-r^s)} = \frac{\gamma^t P^0}{1-\varepsilon^d(\frac{r}{\kappa^s}-\frac{r^d}{\kappa^d})} \qquad (2.1.1)$$

where C_t and $\gamma^t P^0$ are the aggregate capital and aggregate base demand respectively. The above equation admits an explicit solution. The equilibrium price for period t is

$$r_t^m = \frac{(\epsilon^d C_t)r^d \frac{\kappa^s}{\kappa^d} + (\epsilon^s \frac{\gamma^t P^0}{\kappa^s})r^s + \gamma^t P^0 - \kappa^s C_t}{\epsilon^d C_t + \epsilon^s \frac{\gamma^t P^0}{\kappa^s}} \qquad (2.1.2)$$

Using the same techniques as in section 1.3, we obtain the equilibrium quantity for period t

$$P_t^m = \frac{\epsilon^s(\gamma^t P^0) + \epsilon^d(\kappa^s C_t)}{\epsilon^s + \epsilon^d + \epsilon^s\epsilon^d(\frac{r^s}{\kappa^s} - \frac{r^d}{\kappa^d})} \qquad (2.1.3)$$

inserting the above value of the equilibrium price and equilibrium quantity, we obtain the profit for period t

$$\lambda_t^m = \frac{(\epsilon^d \frac{r^d}{\kappa^d} - 1)\kappa^s}{\epsilon^s + \epsilon^d + \epsilon^s\epsilon^d(\frac{r^s}{\kappa^s} - \frac{r^d}{\kappa^d})} C_t + \frac{1 + \epsilon^s \frac{r^s}{\kappa^s}}{\epsilon^s + \epsilon^d + \epsilon^s\epsilon^d(\frac{r^s}{\kappa^s} - \frac{r^d}{\kappa^d})}\gamma^t P^0 \qquad (2.1.4)$$

which is of the form

$$\lambda_t^m = aC_t + b\gamma^t P^0 \qquad (2.1.4a)$$

with

$$a = \frac{(\epsilon^d \frac{r^d}{\kappa^d} - 1)\kappa^s}{\epsilon^s + \epsilon^d + \epsilon^s\epsilon^d(\frac{r^s}{\kappa^s} - \frac{r^d}{\kappa^d})} \ and\ b = \frac{1 + \epsilon^s \frac{r^s}{\kappa^s}}{\epsilon^s + \epsilon^d + \epsilon^s\epsilon^d(\frac{r^s}{\kappa^s} - \frac{r^d}{\kappa^d})} \qquad (2.1.4a)$$

So far we have assumed that the change in surplus from one period to the next is defined by the following equation

$$C_{t+1} = C_t + \rho r^f C_t + \rho\lambda_t^m$$

where λ_t^m is the profit generated by the business underwritten during period t. In order to reflect situations encountered in practice, the equation needs to be refined. If for instance the policies written during period t have inception dates uniformly distributed during the period and if all policies have a term of one period, the insurers will earn 50% of the premium during period t and 50% during period $t+1$. Accordingly they will recognize the profit over two periods, which leads to the following equation

$$C_{t+1} = C_t + \rho r^f C_t + 0.5\rho\lambda_t^m + 0.5\rho\lambda_{t-1}^m$$

Lloyd's syndicates for instance still account on a funded basis meaning that the profit stemming from a given underwriting year is only recognized at the end of the third development year. In addition syndicates earn no investment income on the assets matching the capital. This leads to the following difference equation

$$C_{t+1} = C_t + \rho\lambda_{t-2}^m$$

In general the aggregate capital of the insurers is defined by the following difference equation

$$C_{t+1} = C_t + \rho r^f C_t + \rho\alpha_1\lambda_t^m + ... + \rho\alpha_k\lambda_{t-k+1}^m \qquad (2.1.5)$$

where $\alpha_k = 1-\alpha_1-...-\alpha_{k-1}$ for $k \succ 1$ and $\alpha_1 = 1$ for $k = 1$. Inserting (2.1.4a) into (2.1.5), dividing both sides by $\gamma^{t+1}$and re-arranging terms we obtain the following difference equation defining the change in standardized capital from one period to the next.

$$\overline{C}_{t+1} - \tfrac{1}{\gamma}(1+\rho r^f + \rho a\alpha_1)\overline{C}_t - \tfrac{1}{\gamma^2}\rho a\alpha_2\overline{C}_{t-1} - ... - \tfrac{1}{\gamma^k}\rho a\alpha_k\overline{C}_{t-k+1}$$
$$= (\tfrac{\alpha_1}{\gamma} + \tfrac{\alpha_2}{\gamma^2} + ... + \tfrac{\alpha_k}{\gamma^k})\rho b P^o \qquad (2.1.6)$$

Thus assumptions 1 and 2 enable us to replace a system of difference equations by a single linear difference equation with constant coefficients and a constant input. Because of (2.1.3) and (2.1.4) respectively, P_t^m and λ_t^m satisfy a similar difference equation each.

2.2 Limiting Behaviour of Solutions

We recapitulate some of the well known results of the theory of difference equations. The proofs can be found e.g. in S. N. Elaydi [1999].

Consider the k-th order homogeneous difference equation

$$x(n+k) + p_1 x(n+k-1) + ... + p_k x(n) = 0 \qquad (2.2.1)$$

where the $p_i s$ are constants and $p_k \neq 0$. The following equation

$$\lambda^k + p_1\lambda^{k-1} + ... + p_k = 0 \qquad (2.2.2)$$

is called the characteristic equation of (2.2.1). Let λ_i $(i = 1...r)$ be the different solutions of (2.2.2). the $\lambda_i s$ are called the characteristic roots of the difference equation. Let m_i be the multiplicity of λ_i $(i = 1...r)$. The general solution of (2.2.1) is given by

$$x(n) = \sum_{i=1}^{r}(a_{i0} + a_{i1}n + ...a_{i,m_i-1}n^{m_i-1})\lambda_i^n \qquad (2.2.3)$$

The coefficients a_{ij} are defined by the starting conditions $x(n+k-i) = x^0_{n+k-i}$ $(i = 1, ...k)$.

We now consider inhomogeneous linear difference equation with constant coefficients and constant inputs

$$x(n+k) + p_1 x(n+k-1) + ... + p_k x(n) = c \quad (2.2.4)$$

The general solution of (2.2.4) is the sum of a particular solution of (2.2.4) and of the general solution of the corresponding homogeneous equation (2.2.1). From the definition of an equilibrium point it is seen that if (2.2.4) has an equilibrium point, it is of the form

$$x^* = \frac{c}{1 + p_1 + ... + p_k} \qquad (2.2.5)$$

x^* is a particular solution of (2.2.4). Consequently the general solution of (2.2.4) is of the form

$$y(n) = x^* + x(n) \qquad (2.2.6)$$

with $x(n)$ defined by (2.2.3). It is immediately seen that the solutions of (2.2.4) converge to x^* as $n \to \infty$ if and only if maximum $\{|\lambda_i|\,,\ i = 1...r\} \prec 1$.

Definition

The equilibrium point of a linear difference equation with constant coefficients and a constant input is **stable** if and only if the above condition is satisfied.

We now specifically consider the difference equation (2.1.6). The equilibrium point is

$$\overline{C}^* = \frac{\rho(\frac{\alpha_1}{\gamma} + ... + \frac{\alpha_k}{\gamma^k})b}{1 - \frac{1}{\gamma}(1 + \rho r^f) - \rho(\frac{\alpha_1}{\gamma} + ... + \frac{\alpha_k}{\gamma^k})a} P^0 \quad (2.2.7)$$

inserting (2.1.4a)

$$\overline{C}^* = \frac{\rho(\frac{\alpha_1}{\gamma} + ... + \frac{\alpha_k}{\gamma^k})(\kappa^s + \epsilon^s r^s)}{(1 - \frac{1}{\gamma} - \frac{1}{\gamma}\rho r^f)(\epsilon^s + \epsilon^d + \epsilon^s \epsilon^d(\frac{r^s}{\kappa^s} - \frac{r^d}{\kappa^d})) - \rho(\frac{\alpha_1}{\gamma} + ... + \frac{\alpha_k}{\gamma^k})(\epsilon^d r^d \frac{\kappa^s}{\kappa^d} - \kappa^s)} \frac{P^0}{\kappa^s} (2.2.8)$$

We now derive the value of the market price at equilibrium point. Starting with the value of r^m_t given by (2.1.2), dividing numerator and denominator by γ^t and replacing $\overline{C}_t$ by $\overline{C}^*$ we obtain

$$r^* = \frac{(\epsilon^d r^d \frac{\kappa^s}{\kappa^d} - \kappa^s)\overline{C}^* + (\epsilon^s r^s + \kappa^s)\frac{P^0}{\kappa^s}}{\epsilon^d \overline{C}^* + \epsilon^s \frac{P^0}{\kappa^s}}$$

inserting (2.2.8) gives after some simplifications

$$r^* = \frac{(1 - \frac{1}{\gamma} - \frac{1}{\gamma}\rho r^f)(\frac{\epsilon^s}{\kappa^s} r^s + 1)}{(1 - \frac{1}{\gamma} - \frac{1}{\gamma}\rho r^f)\frac{\epsilon^s}{\kappa^s} + \rho(\frac{\alpha_1}{\gamma} + ... + \frac{\alpha_k}{\gamma^k})} \quad (2.2.9)$$

Similarly we obtain the standardized market quantity at equilibrium point from (2.1.3)

$$\overline{P}^* = \frac{\epsilon^s P^0 + \epsilon^d \kappa^s \overline{C}^*}{\epsilon^s + \epsilon^d + \epsilon^s \epsilon^d (\frac{r^s}{\kappa^s} - \frac{r^d}{\kappa^d})}$$

inserting (2.2.8) gives after some simplifications

$$\overline{P}^* = \frac{(1 - \frac{1}{\gamma} - \frac{1}{\gamma}\rho r^f)\epsilon^s + \rho(\frac{\alpha_1}{\gamma} + ... + \frac{\alpha_k}{\gamma^k})\kappa^s}{(1 - \frac{1}{\gamma} - \frac{1}{\gamma}\rho r^f)(\epsilon^s + \epsilon^d + \epsilon^s \epsilon^d (\frac{r^s}{\kappa^s} - \frac{r^d}{\kappa^d})) - \rho(\frac{\alpha_1}{\gamma} + ... + \frac{\alpha_k}{\gamma^k})(\epsilon^d r^d \frac{\kappa^s}{\kappa^d} - \kappa^s)} P^0 \quad (2.2.10)$$

The market leverage at equilibrium point is also of interest

$$\kappa^* = \frac{\overline{P}^*}{\overline{C}^*} = \frac{\rho(\frac{\alpha_1}{\gamma} + ... + \frac{\alpha_k}{\gamma^k}) + (1 - \frac{1}{\gamma} - \frac{1}{\gamma}\rho r^f)\frac{\epsilon^s}{\kappa^s}}{\rho(\frac{\alpha_1}{\gamma} + ... + \frac{\alpha_k}{\gamma^k})(1 + r^s \frac{\epsilon^s}{\kappa^s})} \kappa^s$$

Remarks
1. Neither r^* nor κ^*depend on r^d, ϵ^d or κ^d.
2. Re-arranging (2.2.9) we obtain

$$r^* = \frac{\frac{1}{r^s} + \frac{\epsilon^s}{\kappa^s}}{\rho \frac{(\frac{\alpha_1}{\gamma} + ... + \frac{\alpha_k}{\gamma^k})}{(1 - \frac{1}{\gamma} - \frac{1}{\gamma}\rho r^f)} + \frac{\epsilon^s}{\kappa^s}} r^s$$

from which it is seen that $r^* \succeq r^s$ if and only if $\gamma - 1 \succeq \rho(r^f + r^s(\alpha_1 + \frac{\alpha_2}{\gamma} + ... + \frac{\alpha_k}{\gamma^{k-1}}))$. The equivalence also holds true if $\succeq$ is replaced by $\preceq$. The equilibrium point market rate of return is determined by the target rate of return of suppliers and by the balance of demand growth and targeted capital growth.

Example 1

We assume that k=1 and therefore $\alpha = 1$. (2.1.6) becomes

$$\overline{C}_{t+1} - \frac{1}{\gamma}(1 + \rho r^f + \rho a)\overline{C}_t = \frac{1}{\gamma}\rho b P^o$$

The characteristic root is

$$\lambda = \frac{1}{\gamma}(1 + \rho r^f + \rho a)$$

Case1. Let us first assume that $\lambda \neq 1$. The equilibrium point is

$$\overline{C^*} = \frac{\frac{1}{\gamma}\rho b P^o}{1 - \frac{1}{\gamma}(1 + \rho r^f + \rho a)}$$

and the general solution is

$$\overline{C_t} = \overline{C_0}(\frac{1}{\gamma}(1 + \rho r^f + \rho a))^t + \frac{1}{\gamma}\rho b P^o \frac{1 - (\frac{1}{\gamma}(1 + \rho r^f + \rho a))^t}{1 - \frac{1}{\gamma}(1 + \rho r^f + \rho a)}$$

where $\overline{C_0}$ is the initial value of $\overline{C_t}$. Thus if $|\lambda| \prec 1$, $\overline{C_t}$ converges to the equilibrium point. If $|\lambda| \succ 1$, $\overline{C_t}$ diverges for any starting point other than the equilibrium point. If $\lambda = -1$, $\overline{C_t}$ alternates between two points.

Case2. For $\lambda = 1$, the general solution is of the form

$$\overline{C_t} = \overline{C_0} + (t-1)\tfrac{1}{\gamma}\rho b P^o$$

there is no equilibrium point and $\overline{C_t}$ diverges.

We now specify the model parameters and derive capital, price and quantity at equilibrium point.

i) We assume that the parameters are the same as in the homogeneous model of example 1.1 in section 1.5. We obtain the following results: $\lambda = 0.862$ hence $\overline{C}_t \longrightarrow \overline{C}^*$, $\overline{C}^* = 80$, $r^* = 8.00\%$, and $\overline{P}^* = 200$,which confirms the results obtained in example 1.1 of section 1.5.

ii) With the parameters of the homogeneous model of example 1.2 in section 1.5 we obtain: $\lambda = 0.870$ hence $\overline{C}_t \longrightarrow \overline{C}^*$, $\overline{C}^* = 107.4$, $r^* = 6.21\%$, and $\overline{P}^* = 259.3$,which confirms the results obtained in example 1.2 of section 1.5. Comparing i) and ii) confirms remark 2 above.

iii) We use the same parameters as in ii) above except for $\gamma = 1.06$.We obtain the following results:$\lambda = 0.854$ hence $\overline{C}_t \longrightarrow \overline{C}^*$, $\overline{C}^* = 93.5$, $r^* = 9.67\%$, and $\overline{P}^* = 241.9$.The situation is opposite to ii) above which again confirms remark 2.

Example 2

We assume that $k = 2$ and that $\alpha_1 = \alpha \epsilon [0,1]$.Of our three examples, this is probably the one which is the most relevant to praxis. (2.1.6) becomes

$$\overline{C}_{t+1} - \frac{1}{\gamma}(1 + \rho r^f + \rho a \alpha)\overline{C}_t - \frac{1}{\gamma^2}\rho a(1-\alpha)\overline{C}_{t-1} = (\frac{\alpha}{\gamma} + \frac{(1-\alpha)}{\gamma^2})\rho b P^o$$

and the characteristic roots are

$$\lambda_{1,2} = \frac{1}{2\gamma}(1 + \rho r^f + \rho \alpha a \pm \sqrt{(1 + \rho r^f + \rho a \alpha)^2 + 4\rho a(1-\alpha)}).$$

The equilibrium point is

$$\overline{C^*} = \frac{(\frac{\alpha}{\gamma} + \frac{(1-\alpha)}{\gamma^2})\rho b P^o}{1 - \frac{1}{\gamma}(1 + \rho r^f + \rho a \alpha) - \frac{1}{\gamma^2}\rho a(1-\alpha)}$$

The asymptotic behaviour of $\overline{C_t}$ depends on the characteristic roots. Without loss of generality we can assume $|\lambda_1| \succeq |\lambda_2|$ .If $|\lambda_1| \prec 1$, the equilibrium point is stable and $\overline{C_t}$ converges to the equilibrium point. If the roots are complex or if λ_1 is negative, $\overline{C_t}$ oscillates, else convergence is monotonous. If $|\lambda_1| \succ 1, \overline{C_t} \rightarrow \infty$ and the equilibrium point is unstable. For a more detailed discussion, see e.g. S. N. Elaydi [1999].

We assume that the parameters are the same as in the homogeneous model of example 1.1 in section 1.5 except that $P^0 = 1$, which amounts to a choice of the monetary unit. We also assume that γ takes the values 1.04, 1.05, 1.06 and

κ^d independently takes the values 1.25, 2.5 and 5.0. For each of the 9 cases we have tabulated (λ_1, λ_2) and $\overline{C}^*$

	$\gamma = 1.04$	$\gamma = 1.05$	$\gamma = 1.06$
$\kappa^d = 1.25$	$(0.049, 0.878)$	$(0.048, 0.870)$	$(0.048, 0.862)$
	0.51	0.47	0.44
$\kappa^d = 2.5$	$(0.056, 0.865)$	$(0.056, 0.856)$	$(0.055, 0.848)$
	0.43	0.40	0.37
$\kappa^d = 5$	$(0.060, 0.858)$	$(0.059, 0.850)$	$(0.058, 0.842)$
	0.40	0.37	0.34

Since r^* and κ^* do not depend on κ^d we have tabulated these quantities in a one dimensional table

	$\gamma = 1.04$	$\gamma = 1.05$	$\gamma = 1.06$
all $\kappa^d s$	6.32%	8.17%	9.90%
	0.41	0.40	0.38

For each cell $\overline{P}^*$ can be derived from κ^* and $\overline{C}^*$. In all cases the equilibrium point is stable and we have monotonous convergence of $\overline{C}_t$ towards $\overline{C}^*$. In the next section we shall see that in the presence of random shocks $\overline{C}_t$ can nevertheless display a cyclical behaviour. As in example 1, the absolute level of profitability at equilibrium point is driven by the relation between capital growth rate and demand growth rate. Differences in leverage of buyers do not affect the market rate of return at equilibrium point, however they affect the quantity of insurance bought, the amount of capital deployed and the amount of profit generated in the market.

Example 3

We assume that $k = 2$ and and that $\alpha_1 = \alpha_2 = 0$.This corresponds to the funded accounting system. We also assume that $r^f = 0$.(2.1.6) becomes

$$\overline{C}_{t+1} - \frac{1}{\gamma}\overline{C}_t - \frac{1}{\gamma^3}\rho a\overline{C}_{t-2} = \frac{1}{\gamma^3}\rho b P^o$$

The characteristic equation is

$$\lambda^3 - \frac{1}{\gamma}\lambda^2 - \frac{1}{\gamma^3}\rho a = 0$$

The equilibrium point is

$$\overline{C^*} = \frac{\frac{1}{\gamma^3}\rho b P^o}{1 - \frac{1}{\gamma} - \frac{1}{\gamma^3}\rho a}$$

A general characterization of the limiting behaviour of $\overline{C_t}$ in the general case is cumbersome and is not attempted here.

We assume the same parameters as in example 2 of section 1.5. except that ρ can take different values. The results are tabulated below. $|\lambda|_{\max}$ denotes the maximum of the norms of the characteristic roots. If $|\lambda|_{\max} \prec 1$, the equilibrium point is stable and $\overline{C}_t$ converges to the equilibrium point. If $|\lambda|_{\max} \succ 1$, the equilibrium point is unstable and $\overline{C}_t$ diverges. If the characteristic equation has dominating complex roots, the system oscillates. In addition to these characteristics we have also recorded r^*, $\overline{C}^*$ and $\overline{P}^*$.

ρ	0.5	1	2	3	4
$\lvert\lambda\rvert_{\max}$	0.82	0.70	0.85	0.96	1.04
oscillates	*no*	*yes*	*yes*	*yes*	*yes*
r^*	10.48%	5.76%	3.03%	2.06%	1.56%
$\overline{C}^*$	90.6	109.4	122.1	127.0	129.6
$\overline{P}^*$	238.2	261.7	277.6	283.7	287.0

From the above table we now know that the system of example 2 in section 1.5 converges for $\rho = 3$ and diverges for $\rho = 4$.From the above mentioned example, however we also know that even the converging system displays strong oscillations around its equilibrium point during the first 30 periods. This is also reflected in the very high value of $\lvert\lambda\rvert_{\max}$. Based on this criteria, we see that even the process with $\rho = 2$ shows significant oscillations and only converges slowly. In addition, the above systems are characterised by supply overcapacity and generate poor returns for values of ρ in excess of 1.

Later we shall look at more sophisticated and more realistic ways to model capital inflows and outflows. We shall see that capital flows are often a contributing factor to underwriting cycles.

3 Selected Topics

In this section we consider different generalizations of the simplified model which are of practical relevance.

3.1 Stochastic Model

So far we have assumed that the profit stemming from business underwritten in any given period is derived from the equilibrium between supply and demand and is deterministic. We now generalize the model to allow for random fluctuations around the expected value.

There are different ways to take uncertainty into account within the framework of our model. One can model underwriting year risk, fiscal year risk or a combination thereof. We start with equation (2.1.5) which defines changes in surplus from one period to the next

$$C_{t+1} = C_t + \rho r^f C_t + \rho\alpha_1\lambda_t^m + ... + \rho\alpha_k\lambda_{t-k+1}^m.$$

We make the same model assumptions as in chapter 2. In addition we assume that the profit contribution from each underwriting year in each fiscal year is subject to random fluctuations. In the above formula, $\alpha_l\lambda_{t-l+1}^m$ is replaced by $\alpha_l\lambda_{t-l+1}^m + \widetilde{\epsilon}_{t-l+1,t}$ ($l = 1, ...k$), where $\widetilde{\epsilon}_{t-l+1,t}$ is the random variable which pertains to that part of the profit from underwriting year $t-l+1$ which emerges in fiscal year t. The above formula becomes

$$C_{t+1} = C_t + \rho r^f C_t + \rho\alpha_1\lambda_t^m + ... + \rho\alpha_k\lambda_{t-k+1}^m + \rho(\widetilde{\epsilon}_{t,t} + ... + \widetilde{\epsilon}_{t-k+1,t}).$$

The formula adequately reflects insurance risk pertaining to new underwriting years, i.e. underwriting years which contribute to the premium earned during the current fiscal year. It does not take into account insurance risk stemming from old underwriting years (loss development risk) nor does it take into account financial risk. A way to address the issue consists in introducing a further random variable $\widetilde{\theta}_{t+1}$ which pertains to the fiscal year and which reflects the two above mentioned risk components. The formula becomes

$$C_{t+1} = C_t+\rho r^f C_t+\rho\alpha_1\lambda_t^m+...+\rho\alpha_k\lambda_{t-k+1}^m+\rho(\widetilde{\epsilon}_{t,t}+...+\widetilde{\epsilon}_{t-k+1,t})+\widetilde{\theta}_{t+1}. \quad (3.1.1)$$

Fiscal year risk is a generalization of the concept introduced in section 1.3.4. In addition to financial risk it also entails loss development risk.

We make the following assumptions on the first and second order moments of the above random variables.

Assumptions

1. $E[\widetilde{\epsilon}_{s,t}] = 0 \;\; all\ s,t; \quad Var[\widetilde{\epsilon}_{s,t}] = (\gamma^s)^2(\alpha_{t-s+1})^2\sigma_\epsilon^2 \quad for\ 1 \le t-s+1 \le k;$ $Cov[\widetilde{\epsilon}_{s,t},\widetilde{\epsilon}_{u,v}] = 0\ for\ s \neq u\ or\ t \neq v$

2. $E[\widetilde{\theta}_t] = 0 \;\; all\ t; \quad Var[\widetilde{\theta}_t] = (\gamma^t)^2\sigma_\theta^2; \quad Cov[\widetilde{\theta}_s,\widetilde{\theta}_t] = 0 \quad for\ s \neq t$

3. $Cov[\widetilde{\epsilon}_{s,t},\widetilde{\theta}_u] = 0 \quad for \quad all\ s,\ t,\ u.$

The first assumption ensures that random fluctuations stemming from underwriting year risk are bias free. The standard deviation of random fluctuations is proportional to a scale factor depending on the underwriting year (γ^s) and to a factor depending on the earning period (α_{t-s+1}).Random fluctuations pertaining to different underwriting periods are uncorrelated. Random fluctuations pertaining to profits emerging in different fiscal years are also uncorrelated. The second assumption makes similar statements on random fluctuations stemming from fiscal year risk. Finally we assume that underwriting year risk and fiscal year risk are uncorrelated.

Remarks

1. If $\sigma_\epsilon = \sigma_\theta = 0$, (3.1.1) becomes a simple difference equation. This is the difference equation corresponding to our stochastic process.

2. Let $\widetilde{\phi}_t = \widetilde{\epsilon}_{t,t}+\widetilde{\epsilon}_{t,t+1}+...+\widetilde{\epsilon}_{t,t+k-1}$ denote the risk pertaining to underwriting year t. Because of assumption 1 we have

$Var[\widetilde{\phi}_t] = (\gamma^t)^2\sigma_\epsilon^2(\alpha_1^2+\alpha_2^2+...+\alpha_k^2).$

Theorem

Within the framework of the so defined model, the standardized centered process $\left\{\overline{C}_t-\mu\right\}$ is an autoregressive process of order k. The mean μ of the autoregressive process is equal to the equilibrium point of the corresponding difference equation.

Proof

Using (2.1.4a), re-arranging terms and dividing each term by γ^{t+1} we obtain

$$\overline{C}_{t+1}-\tfrac{1}{\gamma}(1+\rho r^f+\rho a\alpha_1)\overline{C}_t-\tfrac{1}{\gamma^2}\rho a\alpha_2\overline{C}_{t-1}-...-\tfrac{1}{\gamma^k}\rho a\alpha_k\overline{C}_{t-k+1}-$$
$$-(\tfrac{\alpha_1}{\gamma}+\tfrac{\alpha_2}{\gamma^2}+...+\tfrac{\alpha_k}{\gamma^k})\rho bP^o = \tfrac{\rho}{\gamma^{t+1}}(\widetilde{\epsilon}_{t,t}+...+\widetilde{\epsilon}_{t-k+1,t})+\tfrac{\widetilde{\theta}_{t+1}}{\gamma^{t+1}}$$

We define $\left\{\overline{C}_t^*\right\} = \left\{\overline{C}_t-\mu\right\}$ as the deviation process of $\left\{\overline{C}_t\right\}$ from its mean

μ. We introduce the following notation

$$\widetilde{\Theta}_{t+1} = \frac{\rho}{\gamma^{t+1}}(\widetilde{\epsilon}_{t,t} + ... + \widetilde{\epsilon}_{t-k+1,t}) + \frac{\widetilde{\theta}_{t+1}}{\gamma^{t+1}}$$

Inserting $\overline{C}_t = \overline{C}_t^* + \mu$ in the above equation we obtain

$$\overline{C}_{t+1}^* - \tfrac{1}{\gamma}(1 + \rho r^f + \rho a \alpha_1)\overline{C}_t^* - \tfrac{1}{\gamma^2}\rho a \alpha_2 \overline{C}_{t-1}^* - ... - \tfrac{1}{\gamma^k}\rho a \alpha_k C_{t-k+1} +$$
$$+\mu - \tfrac{1}{\gamma}(1 + \rho r^f + \rho a \alpha_1)\mu - \tfrac{1}{\gamma^2}\rho a \alpha_2 \mu - ... - \tfrac{1}{\gamma^k}\rho a \alpha_k \mu$$
$$-(\tfrac{\alpha_1}{\gamma} + \tfrac{\alpha_2}{\gamma^2} + ... + \tfrac{\alpha_k}{\gamma^k})\rho b P^o = \overline{\Theta}_{t+1}$$

The mean μ is defined by the fact that all terms on the left hand side of the equation which do not contain $\overline{C}_t^*$ must add up to 0. Hence

$$\mu = \frac{(\frac{\alpha_1}{\gamma} + ... + \frac{\alpha_k}{\gamma^k})\rho b}{(1 - \frac{1}{\gamma} - \frac{1}{\gamma}\rho r^f) - (\frac{\alpha_1}{\gamma} + ... + \frac{\alpha_k}{\gamma^k})\rho a} P^0$$

and it is seen that the mean of the stochastic process is equal to the equilibrium point of the corresponding difference equation (see (2.2.7)).

The standardized centered process is thus defined by the following autoregressive equation

$$\overline{C}_{t+1}^* - \frac{1}{\gamma}(1 + \rho r^f + \rho a \alpha_1)\overline{C}_t^* - ... - \frac{1}{\gamma^k}\rho a \alpha_k \overline{C}_{t-k+1}^* = \widetilde{\Theta}_{t+1} \quad (3.1.2).$$

We must now prove that $\left\{\widetilde{\Theta}_t\right\}$ is a white noise process. Because of assumptions 1 to 3 above, $\left\{\widetilde{\Theta}_t\right\}$ is clearly a sequence of uncorrelated random variables with zero mean. In addition

$$Var[\widetilde{\Theta}_t] = \rho^2 \sigma_\epsilon^2 \left((\frac{\alpha_1}{\gamma})^2 + ... + (\frac{\alpha_k}{\gamma^k})^2 \right) + \sigma_\theta^2 = \sigma_\Theta^2$$

is independent of t. $\left\{\widetilde{\Theta}_t\right\}$ is a white noise process and $\left\{\overline{C}_t^*\right\}$ is an AR(k) process.

Q.E.D.

We briefly recapitulate certain well known results from the theory of time series (see e.g. G.E.P. Box and G.M. Jenkins [1976]). We have seen that stability is a highly desirable property of the solutions of difference equations. The corresponding property of time series is stationarity. The AR(k) process

$$C_t - \phi_1 C_{t-1} - ... - \phi_k C_{t-k} = \widetilde{\Theta}_t$$

can be represented as as a series of random shocks

$$C_t = \sum_{j=0}^{\infty} \psi_j \widetilde{\Theta}_{t-j}.$$

Definition

The process is **stationary** if $\sum_{j=0}^{\infty} \psi_j z^j$ converges on or within the unit circle.

Stationarity ensures in particular that the process has a finite variance. It is well known that the process is stationary if and only if the zeros of

$$\phi(z) = 1 - \phi_1 z - ... - \phi_k z^k$$

all lie outside the unit circle. Thus the AR(k) process is stationary if and only if the corresponding difference equation has a stable equilibrium point.

Let ρ_l denote the autocorrelation of order l of the AR(k) process. It is easily seen that the autocorrelation function ρ_l satisfies the difference equation

$$\rho_l - \phi_1 \rho_{l-1} - ... - \phi_k \rho_{l-k} = 0 \qquad l \succ 1.$$

Therefore, if all characteristic roots λ_i $(i = 1, ...k)$ of the difference equation are different, we have

$$\rho_l = \sum_{i=1}^{k} a_i \lambda_i^l$$

else ρ_l is given by formula (2.2.3). In any case the characteristic roots of the difference equation define the behaviour of the autocorrelation function of the corresponding AR(k) process.

Example 1

We assume that $k = 1$ and therefore $\alpha = 1$.The standardized, centered process is an AR(1) process defined by

$$\overline{C}^*_{t+1} - \frac{1}{\gamma}(1 + \rho r^f + \rho a \alpha_1)\overline{C}^*_t = \widetilde{\Theta}_{t+1}.$$

For $\lambda = \frac{1}{\gamma}(1 + \rho r^f + \rho a \alpha_1)\epsilon(-1, 1)$, the process is stationary. For realistic values of the model parameters, we have λ smaller than one but close to one. The process is thus stationary and the autocorrelation function $\rho_k = \lambda^k$ slowly decays exponentially to zero. Neighboring values of the time series tend to be similar and crossings of the mean tend to be infrequent. Nevertheless the process displays a cyclical behaviour.

Example 2

We assume that k=2 and that $\alpha_1 = \alpha\epsilon[0, 1]$.The standardized, centered process is an AR(2) process defined by

$$\overline{C}^*_{t+1} - \frac{1}{\gamma}(1 + \rho r^f + \rho a \alpha)\overline{C}^*_t - \frac{1}{\gamma^2}\rho a(1 - \alpha)\overline{C}^*_{t-1} = \widetilde{\Theta}_{t+1}$$

The autocorrelation function is of the form $\rho_k = a_1 \lambda_1^k + a_2 \lambda_2^k$, where λ_1 and λ_2 are the characteristic roots of the corresponding difference equation. For a realistic choice of the model parameters both characteristic roots are real, nonnegative and smaller than one with one large root close to one. Therefore the process is stationary, the autocorrelation function remains positive and slowly dampens out. The time series displays a similar behaviour as in example 1.

For an in depth discussion of the characteristics of AR(2) processes as well as for specific examples see e.g. G.E.P. Box and G.M. Jenkins [1976].

3.2 Open Markets

So far we have assumed that the change in capital on the supply side of the market stems exclusively from retained earnings or losses. In practice many insurance markets experience significant capital inflows and outflows. These are driven by the perceived profitability of the respective markets. Investors usually base their assessment of the future profitability of a market on its historical profitability. One or many very profitable years lead to an inflow of capital. A series of highly unprofitable years drives capital out of the market.

The barriers to entry vary from market to market. Markets where the distribution is dominated by tied agents or by employees of insurance companies have high barriers to entry. It is the case of many continental European personal lines markets. Markets where the distribution is dominated by brokers have low barriers to entry. This applies e.g. to the UK market. Wholesale broker markets have the lowest barriers to entry. The international reinsurance is a case in point.

Barriers to exit are generally higher than barriers to entry. The main reasons are regulatory constraints. One should however not forget that many insurance groups are active in different market segments and can shift capital from one market segment to another without taking capital out of any legal entity.

The lag between the time when profitable business is written and the entry of new capital can be significant. Depending on the accounting system there may be a lag between the time when business is written and the time when the corresponding results emerge. Profits earned during any given fiscal year only become apparent to new investors during the following calendar year as the profit and loss accounts are made public. Setting up new insurance entities takes time.

Only extraordinary profits or losses trigger a capital flow. As a consequence the capital change at the end of a fiscal year is usually not a linear function of the profit or loss in that fiscal year or in previous fiscal years.

We propose a simple model which takes into account the features mentioned above. We assume that premium underwritten in year t is fully earned during the same year. We assume that capital inflows and outflows take place with a lag of one year. The difference equation defining the change in equity from one year to the next is

$$C_{t+1} = C_t + \rho r^f C_t + \rho \lambda_t^m + \delta_+ (\lambda_{t-1}^m - \frac{r_+^s}{\kappa^s} P_{t-1}^m)^+ + \delta_- (\lambda_{t-1}^m - \frac{r_-^s}{\kappa^s} P_{t-1}^m)^- \quad (3.2.1)$$

Where we have used the following notation

$x^+ = x$ if $x \succeq 0$ else $x^+ = 0$

$x^- = x$ if $x \preceq 0$ else $x^- = 0$

δ_+ and δ_- are two non-negative parameters to be specified later.

r^s_+ and r^s_- are the threshold returns above (respectively below) which capital flows into (respectively out of) the market. It is reasonable to assume $r^s_+ \succeq r^s \succeq r^s_-$ which we shall do from now on.

The model stipulates that any excess profit triggers a capital inflow which is equal to a multiple δ_+ of the actual excess profit. Any inadequate profit triggers a capital outflow which is equal to a multiple δ_- of the amount by which the required minimum profit has been missed. The model is defined by a piecewise linear difference equation with constant coefficients.

In the special case where $r^s_+ = r^s_- = r^s$ and $\delta_+ = \delta_- = \delta$ we have

$$C_{t+1} = C_t + \rho r^f C_t + \rho\lambda^m_t + \delta(\lambda^m_{t-1} - \frac{r^s}{\kappa^s}P^m_{t-1}) \quad (3.2.2)$$

The model assumptions of chapter 2 remain valid. Using the representation of λ^m_t given by (2.1.4a) as well as a similar representation for P^m_t which we obtain from (2.1.3)

$$P^m_t = eC_t + f\gamma^t P^0 \quad with \;\; e = \frac{\epsilon^d\kappa^s}{\epsilon^s + \epsilon^d + \epsilon^s\epsilon^d(\frac{r^s}{\kappa^s} - \frac{r^d}{\kappa^d})} \;\; and \;\; f = \frac{\epsilon^s}{\epsilon^s + \epsilon^d + \epsilon^s\epsilon^d(\frac{r^s}{\kappa^s} - \frac{r^d}{\kappa^d})}$$

we obtain the following difference equation for the standardized capital

$$\overline{C}_{t+1} = \tfrac{1}{\gamma}(\overline{C}_t + \rho r^f\overline{C}_t + \rho a\overline{C}_t + \rho bP^0) + \tfrac{1}{\gamma^2}\delta_+((a - \tfrac{r^s_+}{\kappa^s}e)\overline{C}_{t-1} + (b - \tfrac{r^s_+}{\kappa^s}f)P^0)^+ + $$
$$+ \tfrac{1}{\gamma^2}\delta_-((a - \tfrac{r^s_-}{\kappa^s}e)\overline{C}_{t-1} + (b - \tfrac{r^s_-}{\kappa^s}f)P^0)^- \quad (3.2.3)$$

We make the following case distinction

Case 1

For $\overline{C}_{t-1} \prec \frac{b - \frac{r^s_+}{\kappa^s}f}{-a + \frac{r^s_+}{\kappa^s}e}P^0 = \frac{\kappa^s + \epsilon^s(r^s - r^s_+)}{\kappa^s + \epsilon^d(r^s_+ - r^d\frac{\kappa^s}{\kappa^d})}\frac{P^0}{\kappa^s}$

(3.2.3) becomes

$$\overline{C}_{t+1} - \frac{1}{\gamma}(1 + \rho r^f + \rho a)\overline{C}_t - \frac{\delta_+}{\gamma^2}(a - \frac{r^s_+}{\kappa^s}e)\overline{C}_{t-1} = \frac{1}{\gamma}\rho bP^0 + \frac{\delta_+}{\gamma^2}(b - \frac{r^s_+}{\kappa^s}f)P^0$$

The equilibrium point $\overline{C}^*_1$ and the characteristic roots λ_1,λ_2 of the difference equation are derived in the usual way. Hence

$$\lambda_{1,2} = \frac{1}{2\gamma}((1 + \rho r^f + \rho a) \pm \sqrt{(1 + \rho r^f + \rho a)^2 - 4\delta_+(-a + \frac{e}{\kappa^s}r^s_+)}$$

which leads to the following refinement of the case distinction

i) For $\delta_+ \prec \frac{1}{4}\frac{(1+\rho r^f+\rho a)^2}{-a+\frac{e}{\kappa^s}r^s_+}$ both characteristic roots are real, positive and smaller than 1: $\overline{C}_t$ converges monotonously towards $\overline{C}^*_1$.If $\overline{C}^*_1$ is in domain 1 (i.e. the domain defined by case 1) or on the boundary of domain 1, $\overline{C}^*_1$ it is an equilibrium point. Else $\overline{C}_t$ jumps into domain 2 or 3 after a certain number of steps.

ii) For $\frac{1}{4}\frac{(1+\rho r^f+\rho a)^2}{-a+\frac{e}{\kappa^s}r^s_+} \le \delta_+ \prec \frac{\gamma^2}{-a+\frac{e}{\kappa^s}r^s_+}$ the characteristic roots are conjugate complex numbers with norm smaller than 1: $\overline{C}_t$ converges towards

$\overline{C}_1^*$. Convergence is cyclical, therefore $\overline{C}_t$ may jump into domain 2 or 3 after a certain number of steps.

iii) For $\delta_+ \succ \frac{\gamma^2}{-a+\frac{e}{\kappa^s}r_+^s}$ the characteristic roots are conjugate complex numbers with norm larger than 1: $\overline{C}_t$ is cyclical and diverges. After a certain number of steps, $\overline{C}_t$ jumps into domain 2 or 3.

Case 2

For $\frac{\kappa^s+\epsilon^s(r^s-r_+^s)}{\kappa^s+\epsilon^d(r_+^s-r^d\frac{\kappa^s}{\kappa^d})}\frac{P^0}{\kappa^s} \leq \overline{C}_t \leq \frac{\kappa^s+\epsilon^s(r^s-r_-^s)}{\kappa^s+\epsilon^d(r_-^s-r^d\frac{\kappa^s}{\kappa^d})}\frac{P^0}{\kappa^s}$

(3.2.3) becomes

$$\overline{C}_{t+1} - \frac{1}{\gamma}(1+\rho r^f+\rho a)\overline{C}_t = \frac{1}{\gamma}\rho b P^0$$

The characteristic root is

$$\lambda = \frac{1}{\gamma}(1+\rho r^f+\rho a)$$

For a realistic choice of model parameters we have $\lambda\epsilon(0,1)$: $\overline{C}_t$ converges towards $\overline{C}_2^* = \frac{\frac{1}{\gamma}\rho b}{1-\frac{1}{\gamma}-\frac{1}{\gamma}\rho r^f-\frac{\rho}{\gamma}a}$ P^0.If $\overline{C}_2^*$ is in domain 2, it is an equilibrium point and domain 2 is absorbing.

Case 3

For $\frac{\kappa^s+\epsilon^s(r^s-r_-^s)}{\kappa^s+\epsilon^d(r_-^s-r^d\frac{\kappa^s}{\kappa^d})}\frac{P^0}{\kappa^s} \prec \overline{C}_{t-1}$

(3.2.3) becomes

$$\overline{C}_{t+1} - \frac{1}{\gamma}(1+\rho r^f+\rho a)\overline{C}_t - \frac{\delta_-}{\gamma^2}(a-\frac{r_-^s}{\kappa^s}e)\overline{C}_{t-1} = \frac{1}{\gamma}\rho b P^0 + \frac{\delta_-}{\gamma^2}(b-\frac{r_-^s}{\kappa^s}f)P^0$$

The finer case distinction is similar to case 1 above and is omitted here.

Numerical Example

We assume the same parameters as for the homogeneous model of example 1.1 of section 1.5 except that we do not specify P^0 but use the parameter to denote the monetary unit. In addition we assume that $r_+^s = r^s = 0.08$ and $r_-^s = 0$.We shall consider the implications of different choices for δ_+ and δ_-. The different domains are defined as follows

Domain 1: $\overline{C}_{t-1} \prec 0.4P^0$

Domain 2: $0.4P^0 \preceq \overline{C}_{t-1} \preceq 0.552P^0$

Domain 3: $0.552P^0 \prec \overline{C}_{t-1}$.

$\overline{C}_1^* = 0.4P^0$ irrespective of the value of δ_+.$\overline{C}_2^* = 0.4P^0$.Thus $\overline{C}_1^* = \overline{C}_2^* = \overline{C}^*$ is an equilibrium point both for domain 1 and 2. $\overline{C}_3^*\epsilon[0.4P^0\ ,0.552P^0)$ for $\delta_-\epsilon[0,\infty)$. Thus $\overline{C}_3^*$ is not an equilibrium point for domain 3.

Domain 1: for $\delta_+ \prec 0.819$, $\overline{C}_t$ converges monotonously towards $\overline{C}^*$.For $0.819 \prec \delta_+ \prec 4.41, \overline{C}_t$ converges towards $\overline{C}^*$. Convergence is cyclical, therefore $\overline{C}_t$ may jump into domain 2 or 3 after a certain number of steps. For $4.41 \prec \delta_+$, $\overline{C}_t$ is cyclical and diverges. After a certain number of steps, $\overline{C}_t$ jumps into domain 2 or 3.

Domain 3: the corresponding thresholds for δ_- are 0.975 and 5.25. However irrespectively of the choice of δ_-, $\overline{C}_t$ will move out of domain 3, either through monotonous or cyclical convergence towards $\overline{C}_3^*$, which is inside domain 2, or because $\overline{C}_t$ diverges.

From the above analysis we see that for $\delta_+ \prec 4.41$ and $\delta_- \prec 5.25$, $\overline{C}_t$ eventually converges to $\overline{C}^*$ irrespectively of the starting values $\overline{C}_0$ and $\overline{C}_1$. However $\overline{C}_t$ may display strong oscillations. We have computed an example with $\delta_+ = \delta_- = 4$ and $\overline{C}_0 = \overline{C}_1 = 0.3P^0$. We have obtained $\overline{C}_2 = 0.40P^0$ and $\overline{C}_3 = 0.49P^0$. After this significant initial overshooting, the process converges monotonously to $\overline{C}^* = 0.40P^0$.

For $\delta_+ \succ 4.41$ and $\delta_- \succ 5.25$ no general statement can be made. The process can converge to the equilibrium point after significant fluctuations or it can diverge by oscillating between domain 2 and 3.

The technique we have used to analyze the model can in principle be applied to any piecewise linear model with constant coefficients.

Remark

The special case (3.2.2) above, leads to a more straightforward solution. The difference equation is

$$\overline{C}_{t+1} - \frac{1}{\gamma}(1 + \rho r^f + \rho a)\overline{C}_t - \frac{\delta}{\gamma^2}(a - \frac{r^s}{\kappa^s}e)\overline{C}_{t-1} = \frac{1}{\gamma}\rho b P^0 + \frac{\delta}{\gamma^2}(b - \frac{r^s}{\kappa^s}f)P^0$$

and the characteristic roots are

$$\lambda_{1,2} = \frac{1}{2\gamma}((1 + \rho r^f + \rho a) \pm \sqrt{(1 + \rho r^f + \rho a)^2 - 4\delta(-a + \frac{e}{\kappa^s}r^s)}.$$

Because typically $\delta \succ 1$, $\lambda_1$and λ_2 are likely to be complex or even to be larger than one in norm. Hence the solutions are cyclical or even unstable. Open market models are much more prone to cycles than markets which do not experience capital in- and outflows. This is confirmed by the fact that markets with low barriers to entry tend to experience strong underwriting cycles.

With $r^s_+ = r^s_- = 0.08$ and with the other parameters as above, The solution of the difference equation is a dampened sine wave for $4.41 \succeq \delta_+ = \delta_- \succ 0.819$ and diverges for $\delta_+ = \delta_- \succ 4.41$.

3.3 Random Shocks

Let us assume that we want to model the impact of a stress scenario on an insurance market. As an example we assume that the assets overperform during a period of five years leading to a 4% net (after tax and dividends) extraordinary increase in capital p.a. At the end of the period, the overperformance is corrected at once and the capital suffers a drop of 17.8%. In addition a further drop of 7.5% of the remaining capital occurs as a consequence of market wide significant adverse loss developments. This scenario is close to what happened in the late 1990s and early 2000s. After a protracted period of very favorable stock market performance, a dramatic correction occurred in 2001 and 2002.

It was accompanied by significant adverse loss developments in respect of US business underwritten in the late 1990s. It is estimated that on a worldwide basis, the amount of equity deployed to underwrite insurance risks dropped by approximately 25%.

One possibility to model such risks is to use the stochastic model of section 3.1 and to make the appropriate assumptions on the distribution of $\widetilde{\theta}_t$. A better insight can be gained by generalizing the simplified model of chapter 2 through the introduction of a deterministic correction factor. We therefore assume that the difference equation defining the change in surplus from one period to the next is

$$\overline{C}_{t+1} - \frac{1}{\gamma}(1 + \rho r^f + \rho a + \phi_t)\overline{C}_t = \frac{1}{\gamma}\rho b P^o$$

ϕ_t is the correction factor. Note that the difference equation has now time dependent parameters. The market rate of return during a given period is defined by the standardized version of (2.1.1).

We have used the parameters of the homogeneous version of model 1.1 in section 1.5 except that we have set $P^0 = 1$ and $\overline{C}_0 = 0.4$. To reflect the above described scenario, we have assumed $(\phi_1, ...\phi_5, \phi_6, \phi_7, ...) = (0.04, ...0.04, -0.24, 0, ...)$.We have obtained the following results

t	$\overline{C}_t$ [MU]	r^m_t [%]	P^m_t [MU]
0	0.400	8.00	1.00
5	0.462	4.38	1.08
6	0.348	11.47	0.94
10	0.371	9.86	0.96
15	0.386	8.87	0.98
20	0.394	8.41	0.99

It is seen that during the first phase, capital grows too quickly leading to an oversupply of capacity and depressing the market rate of return. After year five a correction takes place, capital is destroyed, the market rate of return soars and the amount of insurance bought declines. Over time the system slowly comes back to normal.

It is realistic to assume that the extraordinary returns achieved during year six and the immediately following years trigger an inflow of capital into the market. We have therefore considered a combination of this example and the example of section 3.2. The process also converges back to normal but it does so after overshooting with a relative maximum of capital reached in year nine. The larger δ_+, the more severe the overshooting. This is an example of an underwriting cycle reinforced by the interaction of different factors.

3.4 Summary and Conclusions

From the models proposed and from the different examples, we can draw the following conclusions:

The equilibrium market rate of return is driven by the target rate of return of suppliers and by the balance of demand growth and targeted capital growth.

At equilibrium point, the parameters characterizing the buyers $(r^d, \kappa^d, \epsilon^d)$ have an influence on the quantity of insurance being transacted but not on its price.

Cycles are induced by the **supply side**. Profits and losses feed back into surplus leading to shifts in supply which offset the profit or loss trend. Capital inflows and outflows triggered by extraordinary profits and losses induce underwriting cycles. These supply induced cycles are exacerbated by reporting lags stemming from accounting or regulatory factors. Random shocks (e.g. adverse loss developments or catastrophes) create fluctuations in results and can induce cycles through a depletion of the capital of insurance companies. The changes-in-expectations hypothesis mentioned in section 1.2 can be viewed as a shift in suppliers' leverage.

Cycles are induced by the **demand side**. A temporary acceleration or deceleration of demand growth leads to fluctuations in the rate of return. Demand shifts, such as sudden decreases in base demand can also trigger fluctuations in returns. They usually occur as a reaction to a dramatic contraction in supply.

Cycles are induced by **external factors**. Significant changes in the value of assets can lead to underwriting cycles because they have an impact on the amount of capital available to insurance companies. Changes in interest rates have no direct impact on insurance returns. They lead to changes in the face value of insurance premiums but have no direct influence on insurance price or quantity.

The above factors seldom act in isolation but **interact** with each other, thus possibly exacerbating fluctuations in results and creating underwriting cycles. The simplified model and its generalizations enable a simultaneous evaluation of the impact of different factors. The influence of each factor can be quantified by analyzing the solutions of difference equations.

The stochastic version of example 2 leads to an AR(2) processes for the capital, the profit and the pure risk premium. This is in line with the findings of the different articles which have fitted time series to real life data.

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