

Multivariate Chain-Ladder

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Multivariate Chain–Ladder

1. The Univariate Chain–Ladder Method

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2. The Univariate Model of Schnaus

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2. The Univariate Model of Schnaus
3. The Multivariate Model

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4. The Multivariate Chain–Ladder Method

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4. The Multivariate Chain–Ladder Method
5. Conclusion

1. The Univariate Chain–Ladder Method

An Example

Accident Year i	Development Year k					
	0	1	2	3	4	5
0	1001	1855	2423	2988	3335	3483
1	1113	2103	2774	3422	3844	
2	1265	2433	3233	3977		
3	1490	2873	3880			
4	1725	3261				
5	1889					

An Example

Accident Year i	Development Year k					
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4	1725	3261				
5	1889					
					3335	3483
						1.044

An Example

Accident Year i	Development Year k					
	0	1	2	3	4	5
0	1001	1855	2423	2988	3335	3483
1	1113	2103	2774	3422	3844	4013
2	1265	2433	3233	3977		
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2	1265	2433	3233	3977		
3	1490	2873	3880			
4	1725	3261				
5	1889					
				6410	7179	
						1.044

An Example

Accident Year i	Development Year k					
	0	1	2	3	4	5
0	1001	1855	2423	2988	3335	3483
1	1113	2103	2774	3422	3844	4013
2	1265	2433	3233	3977		
3	1490	2873	3880			
4	1725	3261				
5	1889					
				6410	7179	
					1,120	1.044

An Example

Accident Year i	Development Year k					
	0	1	2	3	4	5
0	1001	1855	2423	2988	3335	3483
1	1113	2103	2774	3422	3844	4013
2	1265	2433	3233	3977	4454	
3	1490	2873	3880			
4	1725	3261				
5	1889					
				6410	7179	
					1,120	1.044

An Example

Accident Year i	Development Year k					
	0	1	2	3	4	5
0	1001	1855	2423	2988	3335	3483
1	1113	2103	2774	3422	3844	4013
2	1265	2433	3233	3977	4454	4650
3	1490	2873	3880			
4	1725	3261				
5	1889					
				6410	7179	
					1,120	1.044

An Example

Accident Year i	Development Year k					
	0	1	2	3	4	5
0	1001	1855	2423	2988	3335	3483
1	1113	2103	2774	3422	3844	4013
2	1265	2433	3233	3977	4454	4650
3	1490	2873	3880	4780	5354	5590
4	1725	3261				
5	1889					
			8430	10387		
				1,232	1,120	1.044

An Example

Accident Year i	Development Year k					
	0	1	2	3	4	5
0	1001	1855	2423	2988	3335	3483
1	1113	2103	2774	3422	3844	4013
2	1265	2433	3233	3977	4454	4650
3	1490	2873	3880	4780	5354	5590
4	1725	3261	4334	5339	5980	6243
5	1889					
		9264	12310			
			1.329	1,232	1,120	1.044

An Example

Accident Year i	Development Year k					
	0	1	2	3	4	5
0	1001	1855	2423	2988	3335	3483
1	1113	2103	2774	3422	3844	4013
2	1265	2433	3233	3977	4454	4650
3	1490	2873	3880	4780	5354	5590
4	1725	3261	4334	5339	5980	6243
5	1889	3587	4767	5873	6578	6867
	6594	12525				
		1.899	1.329	1,232	1,120	1.044

An Example

Accident Year i	Development Year k					
	0	1	2	3	4	5
0	1001	1855	2423	2988	3335	3483
1	1113	2103	2774	3422	3844	4013
2	1265	2433	3233	3977	4454	4650
3	1490	2873	3880	4780	5354	5590
4	1725	3261	4334	5339	5980	6243
5	1889	3587	4767	5873	6578	6867

Abstract Run-Off Triangle

Run-off triangle of observable aggregate claims:

Accident Year	Development Year								
	0	1	...	k	...	$n - i$...	$n - 1$	n
0	$S_{0,0}$	$S_{0,1}$...	$S_{0,k}$...	$S_{0,n-i}$...	$S_{0,n-1}$	$S_{0,n}$
1	$S_{1,0}$	$S_{1,1}$...	$S_{1,k}$...	$S_{1,n-i}$...	$S_{1,n-1}$	
⋮	⋮	⋮		⋮		⋮			
i	$S_{i,0}$	$S_{i,1}$...	$S_{i,k}$...	$S_{i,n-i}$			
⋮	⋮	⋮		⋮					
$n - k$	$S_{n-k,0}$	$S_{n-k,1}$...	$S_{n-k,k}$					
⋮	⋮	⋮							
$n - 1$	$S_{n-1,0}$	$S_{n-1,1}$							
n	$S_{n,0}$								

Chain–Ladder Estimation

- Individual Development Factors:

$$F_{i,k} := \frac{S_{i,k}}{S_{i,k-1}}$$

Chain–Ladder Estimation

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$$F_{i,k} := \frac{S_{i,k}}{S_{i,k-1}}$$

- Chain–Ladder Factors:

$$\begin{aligned}\widehat{F}_k^{\text{CL}} &:= \frac{\sum_{j=0}^{n-k} S_{j,k}}{\sum_{j=0}^{n-k} S_{j,k-1}} \\ &= \sum_{j=0}^{n-k} \frac{S_{j,k-1}}{\sum_{h=0}^{n-k} S_{h,k-1}} F_{j,k}\end{aligned}$$

(for $k \geq 1$)

Chain–Ladder Estimation

- Identity for aggregate claims:

$$S_{i,k} = S_{i,n-i} \prod_{l=n-i+1}^k F_{i,l}$$

Chain–Ladder Estimation

- Identity for aggregate claims:

$$S_{i,k} = S_{i,n-i} \prod_{l=n-i+1}^k F_{i,l}$$

- Chain–Ladder Estimators:

$$\hat{S}_{i,k}^{\text{CL}} := S_{i,n-i} \prod_{k=n-i+1}^k \hat{F}_l^{\text{CL}}$$

(for $k \geq n - i$)

Chain–Ladder Estimation:

Chain–Ladder estimation is recursive since

$$\widehat{S}_{i,n-i}^{\text{CL}} = S_{i,n-i}$$

and

$$\widehat{S}_{i,k}^{\text{CL}} = \widehat{S}_{i,k-1}^{\text{CL}} \widehat{F}_k^{\text{CL}}$$

for $k = n - i + 1, \dots, n$

Modified Chain–Ladder Estimation

- Chain–Ladder Factors:

$$\hat{F}_k^{\text{CL}} = \sum_{j=0}^{n-k} \frac{S_{j,k-1}}{\sum_{h=0}^{n-k} S_{h,k-1}} F_{j,k}$$

Modified Chain–Ladder Estimation

- Chain–Ladder Factors:

$$\widehat{F}_k^{\text{CL}} = \sum_{j=0}^{n-k} \frac{S_{j,k-1}}{\sum_{h=0}^{n-k} S_{h,k-1}} F_{j,k}$$

- Modified Chain–Ladder Factors:

$$\widehat{F}_k := \sum_{j=0}^{n-k} W_{j,k} F_{j,k}$$

$$\text{with } \sum_{j=0}^{n-k} W_{j,k} = 1$$

(for $k \geq 1$)

Modified Chain–Ladder Estimation

- Chain–Ladder Estimators:

$$\widehat{S}_{i,k}^{\text{CL}} := S_{i,n-i} \prod_{l=n-i+1}^k \widehat{F}_l^{\text{CL}}$$

Modified Chain–Ladder Estimation

- Chain–Ladder Estimators:

$$\widehat{S}_{i,k}^{\text{CL}} := S_{i,n-i} \prod_{k=n-i+1}^k \widehat{F}_l^{\text{CL}}$$

- Modified Chain–Ladder Estimators:

$$\widehat{S}_{i,k} := S_{i,n-i} \prod_{k=n-i+1}^k \widehat{F}_l$$

⇒ Competition !!!

(for $k \geq n - i$)

2. The Univariate Model of Schnaus

Truncated Run-Off Square

Accident Year	Development Year					
	0	1	...	$k - 1$	k	...
0	$S_{0,0}$	$S_{0,1}$...	$S_{0,k-1}$	$S_{0,k}$...
1	$S_{1,0}$	$S_{1,1}$...	$S_{1,k-1}$	$S_{1,k}$...
⋮	⋮	⋮		⋮	⋮	
$n - k$	$S_{n-k,0}$	$S_{n-k,1}$...	$S_{n-k,k-1}$	$S_{n-k,k}$...
$n - k + 1$	$S_{n-k+1,0}$	$S_{n-k+1,1}$...	$S_{n-k+1,k-1}$	$S_{n-k+1,k}$...
⋮	⋮	⋮	...	⋮	⋮	
$n - 1$	$S_{n-1,0}$	$S_{n-1,1}$...	$S_{n-1,k-1}$	$S_{n-1,k}$...
n	$S_{n,0}$	$S_{n-1,1}$...	$S_{n,k-1}$	$S_{n,k}$...

We denote by \mathcal{G}_{k-1} the information provided by development years $0, 1, \dots, k - 1$.

The Model of Schnaus

Assumption:

For each $k = 1, \dots, n$
there exist *random variables* F_k and V_k such that

$$E(S_{i,k} | \mathcal{G}_{k-1}) = S_{i,k-1} F_k$$

holds for all $i = 0, 1, \dots, n$.

The Model of Schnaus

Assumption:

For each $k = 1, \dots, n$

there exist *random variables* F_k and V_k such that

$$\begin{aligned} E(S_{i,k} | \mathcal{G}_{k-1}) &= S_{i,k-1} F_k \\ \text{var}(S_{i,k} | \mathcal{G}_{k-1}) &= S_{i,k-1} V_k \end{aligned}$$

holds for all $i = 0, 1, \dots, n$.

The Model of Schnaus

Assumption:

For each $k = 1, \dots, n$

there exist *random variables* F_k and V_k such that

$$E(S_{i,k} | \mathcal{G}_{k-1}) = S_{i,k-1} F_k$$

$$\text{var}(S_{i,k} | \mathcal{G}_{k-1}) = S_{i,k-1} V_k$$

$$\text{cov}(S_{i,k}, S_{j,k} | \mathcal{G}_{k-1}) = 0$$

holds for all $i, j = 0, 1, \dots, n$ such that $j \neq i$.

The Model of Schnaus

Assumption:

For each $k = 1, \dots, n$

there exist *random variables* F_k and V_k such that

$$E(S_{i,k} | \mathcal{G}_{k-1}) = S_{i,k-1} F_k$$

$$\text{var}(S_{i,k} | \mathcal{G}_{k-1}) = S_{i,k-1} V_k$$

$$\text{cov}(S_{i,k}, S_{j,k} | \mathcal{G}_{k-1}) = 0$$

holds for all $i, j = 0, 1, \dots, n$ such that $j \neq i$.

A slightly less general model is the **Model of Mack**

The Model of Schnaus

Equivalent formulation of the assumption:

For each $k = 1, \dots, n$

there exist *random variables* F_k and V_k such that

$$E(F_{i,k} | \mathcal{G}_{k-1}) = F_k$$

$$\text{var}(F_{i,k} | \mathcal{G}_{k-1}) = V_k / S_{i,k-1}$$

$$\text{cov}(F_{i,k}, F_{j,k} | \mathcal{G}_{k-1}) = 0$$

holds for all $i, j = 0, 1, \dots, n$ such that $j \neq i$.

Estimation of a non-observable $F_{i,k}$ ($i+k > n$):

- Admissible estimators:

$$\hat{F}_k := \sum_{j=0}^{n-k} W_{j,k} F_{j,k}$$

with *random variables*

$W_{0,k}, W_{1,k}, \dots, W_{n-k,k} \in \mathcal{G}_{k-1}$ satisfying

$$\sum_{j=0}^{n-k} W_{j,k} = 1.$$

Estimation of a non-observable $F_{i,k}$ ($i+k > n$):

- The Chain-Ladder Factor

$$\widehat{F}_k^{\text{CL}} := \sum_{j=0}^{n-k} \frac{S_{j,k-1}}{\sum_{h=0}^{n-k} S_{h,k-1}} F_{j,k}$$

is an admissible estimator.

Estimation of a non-observable $F_{i,k}$ ($i+k > n$):

- The Chain-Ladder Factor

$$\widehat{F}_k^{\text{CL}} := \sum_{j=0}^{n-k} \frac{S_{j,k-1}}{\sum_{h=0}^{n-k} S_{h,k-1}} F_{j,k}$$

is an admissible estimator.

- Every admissible estimator is a Modified Chain-Ladder Factor.

Estimation of a non-observable $F_{i,k}$ ($i+k > n$):

- The Chain-Ladder Factor

$$\widehat{F}_k^{\text{CL}} := \sum_{j=0}^{n-k} \frac{S_{j,k-1}}{\sum_{h=0}^{n-k} S_{h,k-1}} F_{j,k}$$

is an admissible estimator.

- Every admissible estimator is a **Modified Chain-Ladder Factor**.
- *Every* admissible estimator is unbiased in the sense that

$$E\left(\widehat{F}_k \mid \mathcal{G}_{k-1}\right) = E\left(F_{i,k} \mid \mathcal{G}_{k-1}\right)$$

Estimation of a non-observable $F_{i,k}$ ($i+k > n$):

- Optimization problem:

Minimize

$$E \left(\left(\hat{F}_k - F_{i,k} \right)^2 \mid \mathcal{G}_{k-1} \right)$$

over all admissible estimators.

Estimation of a non-observable $F_{i,k}$ ($i+k > n$):

- Optimization problem:
Minimize

$$E \left(\left(\hat{F}_k - F_{i,k} \right)^2 \mid \mathcal{G}_{k-1} \right)$$

over all admissible estimators.

- Solution: The optimization problem has the unique solution

$$\hat{F}_k^{\text{CL}} := \sum_{j=0}^{n-k} \frac{S_{j,k-1}}{\sum_{h=0}^{n-k} S_{h,k-1}} F_{j,k}$$

Estimation of $S_{i,n-i+1}$:

- Admissible estimators:

$$\widehat{S}_{i,n-i+1} := S_{i,n-i} \widehat{F}_{n-i+1}$$

with an admissible estimator \widehat{F}_{n-i+1} of $F_{i,n-i+1}$.

Estimation of $S_{i,n-i+1}$:

- The Chain–Ladder Estimator

$$\hat{S}_{i,n-i+1}^{\text{CL}} := S_{i,n-i} \hat{F}_{n-i+1}^{\text{CL}}$$

is an admissible estimator.

Estimation of $S_{i,n-i+1}$:

- The Chain–Ladder Estimator

$$\widehat{S}_{i,n-i+1}^{\text{CL}} := S_{i,n-i} \widehat{F}_{n-i+1}^{\text{CL}}$$

is an admissible estimator.

- Every admissible estimator is a Modified Chain–Ladder Estimator.

Estimation of $S_{i,n-i+1}$:

- The Chain–Ladder Estimator

$$\widehat{S}_{i,n-i+1}^{\text{CL}} := S_{i,n-i} \widehat{F}_{n-i+1}^{\text{CL}}$$

is an admissible estimator.

- Every admissible estimator is a **Modified Chain–Ladder Estimator**.
- *Every* admissible estimator is unbiased in the sense that

$$E\left(\widehat{S}_{i,n-i+1} \mid \mathcal{G}_{n-i}\right) = E\left(S_{i,n-i+1} \mid \mathcal{G}_{n-i}\right)$$

Estimation of $S_{i,n-i+1}$:

- Optimization problem:
Minimize

$$E \left(\left(\hat{S}_{i,n-i+1} - S_{i,n-i+1} \right)^2 \mid \mathcal{G}_{n-i} \right)$$

over all admissible estimators.

Estimation of $S_{i,n-i+1}$:

- Optimization problem:
Minimize

$$E \left(\left(\hat{S}_{i,n-i+1} - S_{i,n-i+1} \right)^2 \mid \mathcal{G}_{n-i} \right)$$

over all admissible estimators.

- Solution: The optimization problem has the unique solution

$$\hat{S}_{i,n-i+1}^{\text{CL}} := S_{i,n-i} \hat{F}_{n-i+1}^{\text{CL}}$$

\implies Chain-Ladder is the Winner !!!

Estimation of $S_{i,n-i+1}$:

- Proof:

$$\begin{aligned} & E \left(\left(\widehat{S}_{i,n-i+1} - S_{i,n-i+1} \right)^2 \mid \mathcal{G}_{n-i} \right) \\ &= E \left(\left(S_{i,n-i} \widehat{F}_{n-i+1} - S_{i,n-i} F_{i,n-i+1} \right)^2 \mid \mathcal{G}_{n-i} \right) \\ &= S_{i,n-i}^2 \cdot E \left(\left(\widehat{F}_{n-i+1} - F_{i,n-i+1} \right)^2 \mid \mathcal{G}_{n-i} \right) \end{aligned}$$

Sequential Optimality:

As a consequence of the previous result, the chain–ladder estimators are optimal in a sequential sense.

3. The Multivariate Model

Correlated Lines Of Business

- In the present section we consider the loss–development for $m = 2$ correlated lines of business.

Correlated Lines Of Business

- In the present section we consider the loss–development for $m = 2$ correlated lines of business.
- All definitions and results can be extended to arbitrary m .

Run-Off Triangle for Portfolio (1)

Accident Year	Development Year								
	0	1	...	k	...	$n - i$...	$n - 1$	n
0	$S_{0,0}^{(1)}$	$S_{0,1}^{(1)}$...	$S_{0,k}^{(1)}$...	$S_{0,n-i}^{(1)}$...	$S_{0,n-1}^{(1)}$	$S_{0,n}^{(1)}$
1	$S_{1,0}^{(1)}$	$S_{1,1}^{(1)}$...	$S_{1,k}^{(1)}$...	$S_{1,n-i}^{(1)}$...	$S_{1,n-1}^{(1)}$	
⋮	⋮	⋮		⋮		⋮			
i	$S_{i,0}^{(1)}$	$S_{i,1}^{(1)}$...	$S_{i,k}^{(1)}$...	$S_{i,n-i}^{(1)}$			
⋮	⋮	⋮		⋮					
$n - k$	$S_{n-k,0}^{(1)}$	$S_{n-k,1}^{(1)}$...	$S_{n-k,k}^{(1)}$					
⋮	⋮	⋮							
$n - 1$	$S_{n-1,0}^{(1)}$	$S_{n-1,1}^{(1)}$							
n	$S_{n,0}^{(1)}$								

Run-Off Triangle for Portfolio (2)

Accident Year	Development Year								
	0	1	...	k	...	$n - i$...	$n - 1$	n
0	$S_{0,0}^{(2)}$	$S_{0,1}^{(2)}$...	$S_{0,k}^{(2)}$...	$S_{0,n-i}^{(2)}$...	$S_{0,n-1}^{(2)}$	$S_{0,n}^{(2)}$
1	$S_{1,0}^{(2)}$	$S_{1,1}^{(2)}$...	$S_{1,k}^{(2)}$...	$S_{1,n-i}^{(2)}$...	$S_{1,n-1}^{(2)}$	
⋮	⋮	⋮		⋮		⋮			
i	$S_{i,0}^{(2)}$	$S_{i,1}^{(2)}$...	$S_{i,k}^{(2)}$...	$S_{i,n-i}^{(2)}$			
⋮	⋮	⋮		⋮					
$n - k$	$S_{n-k,0}^{(2)}$	$S_{n-k,1}^{(2)}$...	$S_{n-k,k}^{(2)}$					
⋮	⋮	⋮							
$n - 1$	$S_{n-1,0}^{(2)}$	$S_{n-1,1}^{(2)}$							
n	$S_{n,0}^{(2)}$								

Run-Off Triangle for Both Portfolios

Accident Year	Development Year								
	0	1	...	k	...	$n - i$...	$n - 1$	n
0	$S_{0,0}$	$S_{0,1}$...	$S_{0,k}$...	$S_{0,n-i}$...	$S_{0,n-1}$	$S_{0,n}$
1	$S_{1,0}$	$S_{1,1}$...	$S_{1,k}$...	$S_{1,n-i}$...	$S_{1,n-1}$	
⋮	⋮	⋮		⋮		⋮			
i	$S_{i,0}$	$S_{i,1}$...	$S_{i,k}$...	$S_{i,n-i}$			
⋮	⋮	⋮		⋮					
$n - k$	$S_{n-k,0}$	$S_{n-k,1}$...	$S_{n-k,k}$					
⋮	⋮	⋮							
$n - 1$	$S_{n-1,0}$	$S_{n-1,1}$							
n	$S_{n,0}$								

Notation

$$\mathbf{S}_{i,k} := \begin{pmatrix} S_{i,k}^{(1)} \\ S_{i,k}^{(2)} \end{pmatrix}$$

$$\mathbf{T}_{i,k} := \begin{pmatrix} S_{i,k}^{(1)} & 0 \\ 0 & S_{i,k}^{(2)} \end{pmatrix}$$

$$\mathbf{F}_{i,k} := \begin{pmatrix} F_{i,k}^{(1)} \\ F_{i,k}^{(2)} \end{pmatrix}$$

Truncated Run-Off Square

Accident Year	Development Year					
	0	1	...	$k - 1$	k	...
0	$\mathbf{S}_{0,0}$	$\mathbf{S}_{0,1}$...	$\mathbf{S}_{0,k-1}$	$\mathbf{S}_{0,k}$...
1	$\mathbf{S}_{1,0}$	$\mathbf{S}_{1,1}$...	$\mathbf{S}_{1,k-1}$	$\mathbf{S}_{1,k}$...
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$n - k$	$\mathbf{S}_{n-k,0}$	$\mathbf{S}_{n-k,1}$...	$\mathbf{S}_{n-k,k-1}$	$\mathbf{S}_{n-k,k}$...
$n - k + 1$	$\mathbf{S}_{n-k+1,0}$	$\mathbf{S}_{n-k+1,1}$...	$\mathbf{S}_{n-k+1,k-1}$	$\mathbf{S}_{n-k+1,k}$...
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$n - 1$	$\mathbf{S}_{n-1,0}$	$\mathbf{S}_{n-1,1}$...	$\mathbf{S}_{n-1,k-1}$	$\mathbf{S}_{n-1,k}$...
n	$\mathbf{S}_{n,0}$	$\mathbf{S}_{n-1,1}$...	$\mathbf{S}_{n,k-1}$	$\mathbf{S}_{n,k}$...

We denote by \mathcal{G}_{k-1} the information provided by development years $0, 1, \dots, k - 1$.

The Multivariate Model

Assumption:

For each $k = 1, \dots, n$ there exists a *random vector* \mathbf{F}_k and a *random matrix* \mathbf{V}_k such that

$$E(\mathbf{S}_{i,k} | \mathcal{G}_{k-1}) = \mathbf{S}_{i,k-1} \mathbf{F}_k$$

holds for all $i = 0, 1, \dots, n$.

The Multivariate Model

Assumption:

For each $k = 1, \dots, n$ there exists a *random vector* \mathbf{F}_k and a *random matrix* \mathbf{V}_k such that

$$\begin{aligned} E(\mathbf{S}_{i,k} | \mathcal{G}_{k-1}) &= \mathbf{T}_{i,k-1} \mathbf{F}_k \\ \text{var}(\mathbf{S}_{i,k} | \mathcal{G}_{k-1}) &= \mathbf{T}_{i,k-1}^{1/2} \mathbf{V}_k \mathbf{T}_{i,k-1}^{1/2} \end{aligned}$$

holds for all $i = 0, 1, \dots, n$.

The Multivariate Model

Assumption:

For each $k = 1, \dots, n$ there exists a *random vector* \mathbf{F}_k and a *random matrix* \mathbf{V}_k such that

$$E(\mathbf{S}_{i,k} | \mathcal{G}_{k-1}) = \mathbf{T}_{i,k-1} F_k$$

$$\text{var}(\mathbf{S}_{i,k} | \mathcal{G}_{k-1}) = \mathbf{T}_{i,k-1}^{1/2} \mathbf{V}_k \mathbf{T}_{i,k-1}^{1/2}$$

$$\text{cov}(\mathbf{S}_{i,k}, \mathbf{S}_{j,k} | \mathcal{G}_{k-1}) = \mathbf{O}$$

holds for all $i, j = 0, 1, \dots, n$ such that $j \neq i$.

The Multivariate Model

Assumption:

For each $k = 1, \dots, n$ there exists a random vector \mathbf{F}_k and a random matrix \mathbf{V}_k such that

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$$\text{var}(\mathbf{S}_{i,k} | \mathcal{G}_{k-1}) = \mathbf{T}_{i,k-1}^{1/2} \mathbf{V}_k \mathbf{T}_{i,k-1}^{1/2}$$

$$\text{cov}(\mathbf{S}_{i,k}, \mathbf{S}_{j,k} | \mathcal{G}_{k-1}) = \mathbf{O}$$

holds for all $i, j = 0, 1, \dots, n$ such that $j \neq i$.

A slightly less general model is the [Model of Braun \[2004\]](#).

The Multivariate Model

Equivalent formulation of the assumption:

For each $k = 1, \dots, n$ there exists a *random vector* \mathbf{F}_k and a *random matrix* \mathbf{V}_k such that

$$E(\mathbf{F}_{i,k} | \mathcal{G}_{k-1}) = F_k$$

$$\text{var}(\mathbf{F}_{i,k} | \mathcal{G}_{k-1}) = \mathbf{T}_{i,k-1}^{-1/2} \mathbf{V}_k \mathbf{T}_{i,k-1}^{-1/2}$$

$$\text{cov}(\mathbf{F}_{i,k}, \mathbf{F}_{j,k} | \mathcal{G}_{k-1}) = \mathbf{O}$$

holds for all $i, j = 0, 1, \dots, n$ such that $j \neq i$.

4. The Multivariate Chain–Ladder Method

Estimation of a non-observable $\mathbf{F}_{i,k}$ ($i+k > n$):

- Admissible estimators:

$$\widehat{\mathbf{F}}_k := \sum_{j=0}^{n-k} \mathbf{W}_{j,k} \mathbf{F}_{j,k}$$

with *random matrices*

$\mathbf{W}_{0,k}, \mathbf{W}_{1,k}, \dots, \mathbf{W}_{n-k,k} \in \mathcal{G}_{k-1}$ satisfying

$$\sum_{j=0}^{n-k} \mathbf{W}_{j,k} = \mathbf{I}.$$

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$$\sum_{j=0}^{n-k} \mathbf{W}_{j,k} = \mathbf{I}.$$

- *Every* admissible estimator is unbiased in the sense that

$$E\left(\widehat{\mathbf{F}}_k \mid \mathcal{G}_{n-i}\right) = E\left(\mathbf{F}_{i,k} \mid \mathcal{G}_{n-i}\right)$$

Estimation of a non-observable $\mathbf{F}_{i,k}$ ($i+k > n$):

Optimization problem:

Minimize

$$E \left(\left(\hat{\mathbf{F}}_k - \mathbf{F}_{i,k} \right)' \left(\hat{\mathbf{F}}_k - \mathbf{F}_{i,k} \right) \mid \mathcal{G}_{k-1} \right)$$

over all admissible estimators.

Estimation of a non-observable $\mathbf{F}_{i,k}$ ($i+k > n$):

Solution:

The optimization problem has the unique solution

$$\hat{\mathbf{F}}_k^{\text{CL}} := \left(\sum_{j=0}^{n-k} \mathbf{T}_{j,k-1}^{1/2} \mathbf{V}_k^{-1} \mathbf{T}_{j,k-1}^{1/2} \right)^{-1} \cdot \sum_{j=0}^{n-k} \mathbf{T}_{j,k-1}^{1/2} \mathbf{V}_k^{-1} \mathbf{T}_{j,k-1}^{1/2} \mathbf{F}_{j,k}$$

The unique solution $\hat{\mathbf{F}}_k^{\text{CL}}$ is said to be the **Multivariate Chain-Ladder Factor** of development year k .

Estimation of $\mathbf{S}_{i,n-i+1}$:

- Admissible estimators:

$$\widehat{\mathbf{S}}_{i,n-i+1} := \mathbf{T}_{i,n-i} \widehat{\mathbf{F}}_{n-i+1}$$

with an admissible estimator $\widehat{\mathbf{F}}_{n-i+1}$ of $\mathbf{F}_{i,n-i+1}$.

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- *Every* admissible estimator is unbiased in the sense that

$$E\left(\widehat{\mathbf{S}}_{i,n-i+1} \mid \mathcal{G}_{n-i}\right) = E\left(\mathbf{S}_{i,n-i+1} \mid \mathcal{G}_{n-i}\right)$$

Estimation of $\mathbf{S}_{i,n-i+1}$:

- Optimization problem:
Minimize

$$E \left(\left(\widehat{\mathbf{S}}_{i,n-i+1} - \mathbf{S}_{i,n-i+1} \right)^2 \mid \mathcal{G}_{n-i} \right)$$

over all admissible estimators.

Estimation of $\mathbf{S}_{i,n-i+1}$:

- Optimization problem:
Minimize

$$E \left(\left(\hat{\mathbf{S}}_{i,n-i+1} - \mathbf{S}_{i,n-i+1} \right)^2 \mid \mathcal{G}_{n-i} \right)$$

over all admissible estimators.

- Solution:
The optimization problem has the unique solution

$$\hat{\mathbf{S}}_{i,n-i+1}^{\text{CL}} := \mathbf{T}_{i,n-i} \hat{\mathbf{F}}_{n-i+1}^{\text{CL}}$$

Estimation of $\mathbf{S}_{i,n-i+1}$:

The unique solution $\widehat{\mathbf{S}}_k^{\text{CL}}$ is said to be the **Multivariate Chain–Ladder Estimator** of $\mathbf{S}_{i,n-i+1}$.

Recursive Chain–Ladder Estimation:

Define

$$\hat{\mathbf{S}}_{i,n-i}^{\text{CL}} := \mathbf{S}_{i,n-i}$$

and for $k = n - i + 1, \dots, n$

$$\hat{\mathbf{T}}_{i,k-1}^{\text{CL}} := \text{diag}(\hat{\mathbf{S}}_{i,k-1}^{\text{CL}})$$

$$\hat{\mathbf{S}}_{i,k}^{\text{CL}} := \hat{\mathbf{T}}_{i,k-1}^{\text{CL}} \hat{\mathbf{F}}_k^{\text{CL}}$$

Then $\hat{\mathbf{S}}_{i,k}^{\text{CL}}$ is said to be the **Multivariate Chain–Ladder Estimator** of $\mathbf{S}_{i,k}$.

Sequential Optimality:

Sequential optimality obtains as in the univariate case.

5. Conclusion

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Conclusion

- In the univariate case, the method is there and a stochastic model had to be found for its justification.
- In the multivariate case, an extension of the stochastic model of Schnaus leads to an extension of the method.
- The coordinates of the Multivariate Chain–Ladder Estimators usually differ from the Univariate Chain–Ladder Estimators of the Coordinates.
- The multivariate approach resolves the problem of additivity.