XXXVI[th] ASTIN COLLOQUIUM
Zurich, 4– 6 September 2005

DELTA METHOD and RESERVING

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Introduction

Presentation of methods based on reserve's moments allowing:

- Estimating:
  - *predictive distribution,*
  - *moments, percentiles (VaR) and their function*
- With estimation risk measure , confidence interval

Within the GLM approach (including Log-Poisson)

I. Notations

- For a given line of business, claims are assumed to be closed in (*n* + 1) years.

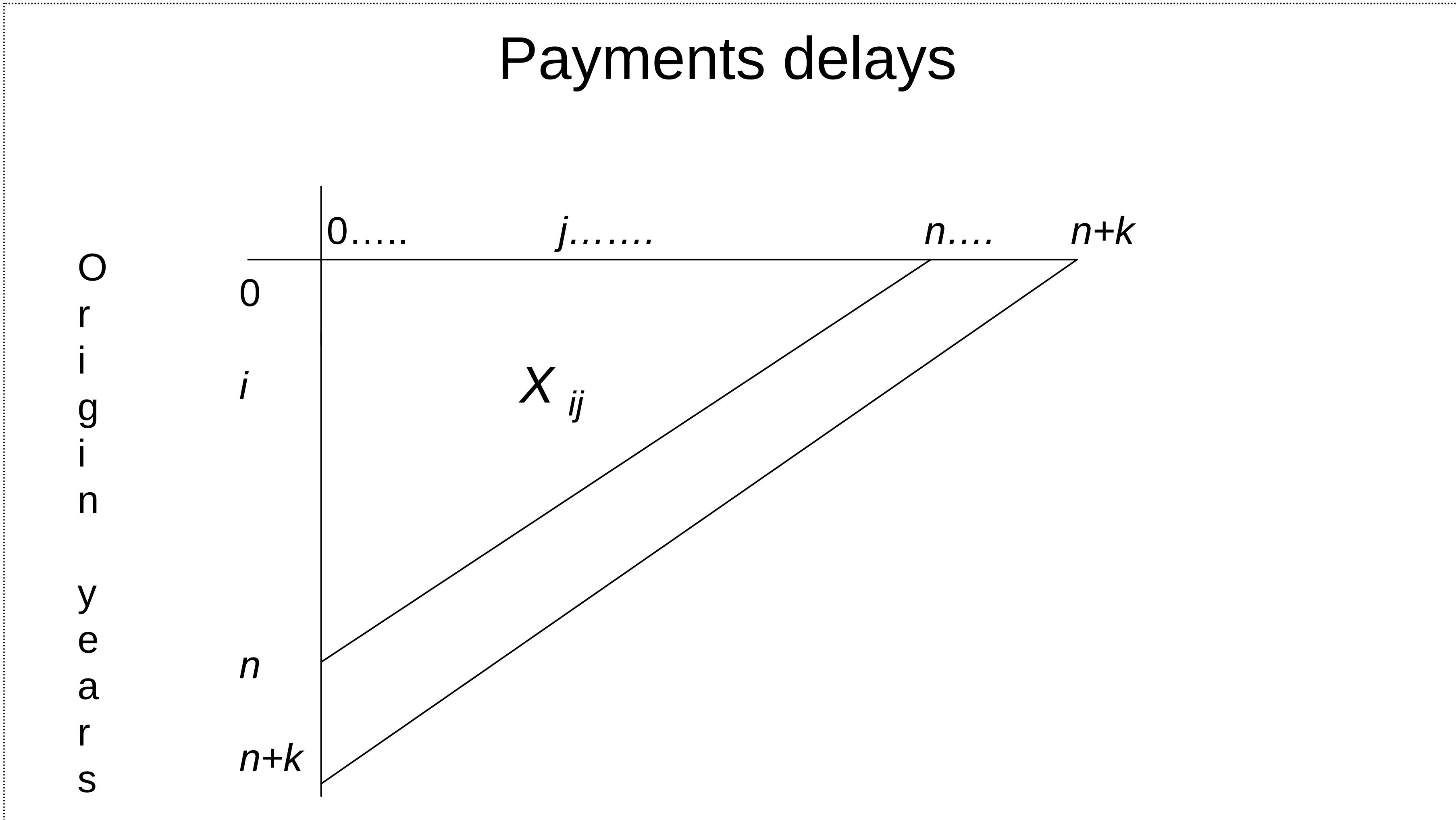
- GLM approach is based on *incremental payments r.v:*

$$X_{ij} \ , \ i, j = 0, \ldots, n$$

  assumed to be **independent**

- Among the $(n+1)^2$ X_{ij} r. v. inside the run-off triangle those located in the upper triangle have been observed

I. Notations



I. Notations

- Reserve for the *i*th origin year

$$R_i = \sum_{h=n-i+1}^{n} X_{ih}$$

- Total reserve

$$R = \sum_{i=1}^{n} R_i$$

- *Remark*: To analyze future annual cash flows by integrating new business, cash flows for the accident year $(n+k)$:

$$\sum_{i+j=n+k}^{n} X_{ij}$$

II. Parameters

2.1. Interest Parameters

Interest parameters are linked to the d.f. F_R : $\Pi\left(F_R\right)$

- Indicators depending on moments of *R* :

mean, dispersion (variance, standard deviation), margins such :

$$E(R)+\gamma\,\sigma(R)$$

- Other indicators: (tail) VaR (percentiles),TailVaR,… probability of insufficiency,…

Need : estimation of d.f. F_R directly or by inversion of m.g.f.

$$M_R\left(s\right)=E\left(s^R\right)$$

II. Parameters

2.2. Estimation

For an estimator $\hat{\Pi} = \hat{\Pi}\left[\left(X_{ij}\right)_{i+j\leq n}\right]$ of $\Pi\left(F_R\right)$ uncertainty related to this estimation will be measured by:

- asymptotic variance: $V_{as}(\hat{\Pi})$
- standard deviation : $s.e._{as}(\hat{\Pi}) = \sqrt{V_{as}(\hat{\Pi})}$ $\frac{s.e._{as}(\hat{\Pi})}{\hat{\Pi}}$

In addition, a level 95% asymptotic confidence interval for $\Pi\left(F_R\right)$ is

$$P\left\{A\left[\left(X_{ij}\right)_{i+j\leq n}\right] \leq \Pi\left(F_R\right) \leq B\left[\left(X_{ij}\right)_{i+j\leq n}\right]\right\} \to 0,95$$

III.GLM models

3.1. Random component

- independent «responses» : $X_{ij}\left(i,j=0,...,n\right)$

- with "exponential" type dist.

$$f\left(x_{ij};\theta_{ij},\phi\right)=\exp\left\{\left[\theta_{ij}x_{ij}-b\left(\theta_{ij}\right)\right]\Big/\phi+c\left(x_{ij},\phi\right)\right\}$$

- θ_{ij} *: natural parameters*
- $\phi>0$ *: dispersion parameter.*
- *b , c specific functions, b being 'regular"*

Ex. : Poisson, Normal, Gamma, IG, Tweedie dist…

- Moments: $\mu_{ij}=E\left(X_{ij}\right)=b'\left(\theta_{ij}\right)$

$V\left(X_{ij}\right)=\phi b''\left(\theta_{ij}\right)=\phi V\left(\mu_{ij}\right)$ *With V* ***variance function***

III.GLM models

3.1. Random component

- Third moment: $\mu_3(X_{ij}) = \phi^2 b'''(\theta_{ij}) = V(\mu_{ij})V'(\mu_{ij})$

- Skewness : $\gamma_1(X_{ij}) = \dfrac{\mu_3(X_{ij})}{\left[V(X_{ij})\right]^{3/2}} = \sqrt{\phi}\,\dfrac{b'''(\theta_{ij})}{\left[b''(\theta_{ij})\right]^{3/2}} = \sqrt{\phi}\,\dfrac{V'(\mu_{ij})}{\sqrt{V(\mu_{ij})}}$

- m.g.f. and cumulant g.f:

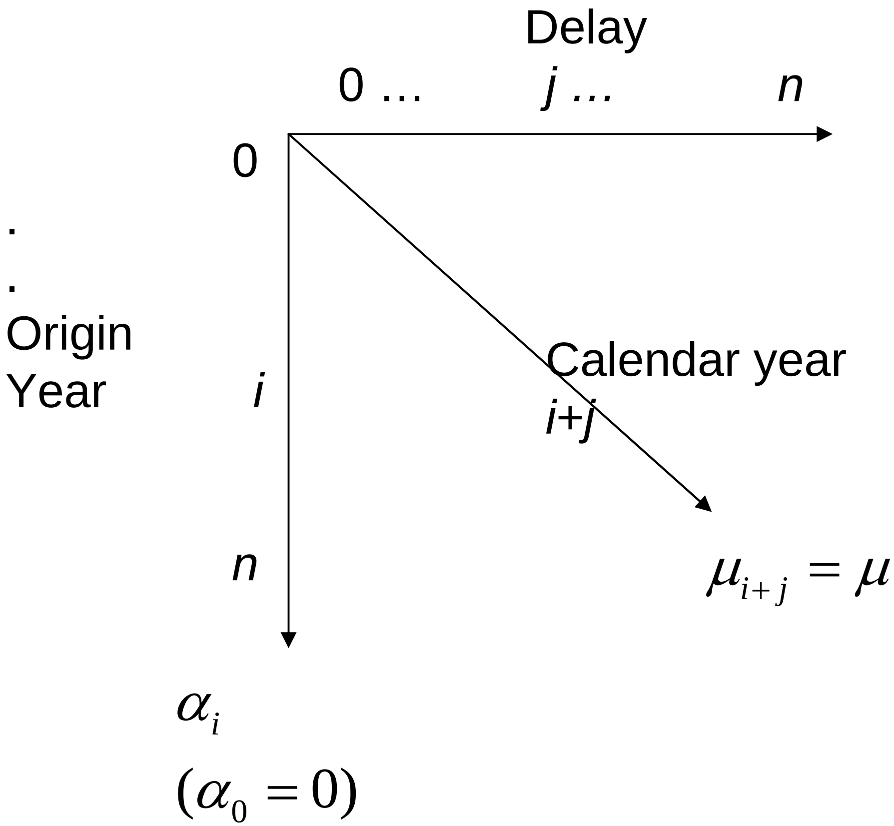
$$M_{X_{ij}}(s) = \exp\left\{\frac{1}{\phi}\left[b(\theta_{ij} + s\phi) - b(\theta_{ij})\right]\right\} \;, C_{X_{ij}}(s) = \log M_{X_{ij}}(s) = \frac{1}{\phi}\left[b(\theta_{ij} + s\phi) - b(\theta_{ij})\right]$$

- moments of X_{ij} functions of (θ_{ij}, ϕ) , then of (μ_{ij}, ϕ) :

$$\kappa_r(X_{ij}) = \phi^{r-1} b^{(r)}(\theta_{ij})$$

III.GLM models

3.2. Systematic Component, link function



$\beta_j \quad (\beta_0 = 0)$

regression parameters :

$$\xi = \left[\mu, (\alpha_i)_{i=1,\dots,n}, (\beta_j)_{j=1,\dots,n} \right]$$

III.GLM models

3.2. Systematic Component, link function

- systematic component : $\eta_{ij} = \mu + \alpha_i + \beta_j \; (i, j = 0, \ldots, n)$

- link function : *monotone and derivable real function g :*

$$\eta_{ij} = g(\mu_{ij}) \Leftrightarrow \mu_{ij} = g^{-1}(\eta_{ij})$$

- Identity link : $\eta_{ij} = \mu_{ij} = \mu + \alpha_i + \beta_j$

- Log link : $\eta_{ij} = \log \mu_{ij} \Leftrightarrow \mu_{ij} = e^{\mu + \alpha_i + \beta_j}$

IV. Estimation

- based on:

  - upper triangle likelihood $L\left[\left(x_{ij}\right)_{i+j\leq n};\mu,(\alpha_i),\left(\beta_j\right),\phi\right]$

  - and Wedderburn equations $\dfrac{\delta \log L}{\delta\xi} = 0$

m.l.e of	
$\hat{\xi} = \left[\hat{\mu},(\hat{\alpha}_i),\left(\hat{\beta}_j\right)\right]$	$\xi = \left[\mu,(\alpha_i),\left(\beta_j\right)\right]$
$\hat{\eta}_{ij} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j \;,\; \hat{\mu}_{ij} = g^{-1}\left(\hat{\eta}_{ij}\right)$	$\eta_{ij} \;,\; \mu_{ij} = E\left(X_{ij}\right)$
As: $R = \sum_{i+j\succ n} X_{ij} \Rightarrow E(R) = \sum_{i+j\succ n} \mu_{ij}$ $\hat{E}(R) = \sum_{i+j\succ n} \hat{\mu}_{ij}$	$E(R)$

IV.Estimation

4.1. Estimation risk (Delta method)

- m.l.e.: - $\hat{\xi}$ $\quad AN\left[\xi,\Sigma_{as}(\hat{\xi})\right] \quad \Sigma_{as}(\hat{\xi}) = I^{-1}(\xi)$

  - $\hat{\eta} = (\hat{\eta}_{ij}) \quad AN\left[\eta,\Sigma_{as}(\hat{\eta})\right] \quad \Sigma_{as}(\hat{\eta}) = J_\eta \Sigma_{as}(\hat{\xi}) J'_\eta$

  J_η *jacobian matrix of* $\quad \eta : \xi \rightarrow \eta(\xi) = (\eta_{ij})$

$$\frac{\partial \eta_{ij}}{\partial \mu} = 1, \ \frac{\partial \eta_{ij}}{\partial \alpha_k} = \begin{cases} 1 & if \quad k = i \\ 0 & \quad k \neq i \end{cases}, \ \frac{\partial \eta_{ij}}{\partial \beta_l} = \begin{cases} 1 & if \quad l = j \\ 0 & \quad l \neq j \end{cases}$$

  - $\hat{\mu} = (\hat{\mu}_{ij}) \quad AN\left[\mu,\Sigma_{as}(\hat{\mu})\right] \quad \Sigma_{as}(\hat{\mu}) = D\Sigma_{as}(\hat{\eta})D$

  D Jacobian matrix (diagonal) of $\quad (\eta_{ij}) \rightarrow \left[g^{-1}(\eta_{ij})\right] = \left[\mu_{ij}\right]$

IV. Estimation

4.1. Estimation risk (Delta method)

- Then $\left[\hat{E}(R_i)\right]_{i=1,\dots,n} \quad AN\left\{\left[E(R_i)\right]_{i=1,\dots,n}, \Sigma_{as}\left\{\left[\hat{E}(R_i)\right]\right\}\right\}$

  With - $\Sigma_{as}\left\{\left[\hat{E}(R_i)\right]\right\} = J_\mu \Sigma_{as}(\hat{\mu}) J_\mu{}'$

  \- J_μ *Jacobian matrix of* $(\mu_{ij}) \rightarrow \left[E(R_i)\right]$

- And $\hat{E}(R) \quad AN\left\{E(R), \sigma_{as}^2\left[\hat{E}(R)\right]\right\}$

  With: - $\sigma_{as}^2\left[\hat{E}(R)\right] = J_R \Sigma_{as}\left\{\left[\hat{E}(R_i)\right]\right\} J_R{}'$

  \- $J_R = (1,1,\dots,1)$ *Jacobian matrix of* $\left[E(R_i)\right]_{i=1,\dots,n} \rightarrow E(R)$

$\Rightarrow$ Asymptotic s.e. and confidence interval for $E(R)$

<u>using only products of matrix (spreadsheet)</u>

IV.Estimation

4.2. Extensions

- Same approach could be applied to:

  - variance: $V(R) = \sum_{i=1}^{n} \sum_{j \succ n-i} V(X_{ij}) = \phi \sum_{i=1}^{n} \sum_{j \succ n-i} V(\mu_{ij})$

  - more generally, to cumulants:

$$\kappa(R) = \sum_{i=1}^{n} \sum_{j \succ n-i} \kappa(X_{ij})$$

  - Then to any regular function of moments of *R*

- Giving only a variance function and dispersion parameter by quasi-likelihood $V(\mu) \succ 0 \ , \ \phi$

  Ex. : over-dispersed Poisson

V. Predictive distribution

5.1. Inversion of the m.g.f

- m.g.f:

$$M_R(s) = \prod_{i=1}^{n} \prod_{i+j \succ n} M_{X_{ij}}(s) = \exp\left\{ \frac{1}{\phi} \sum_{i=1}^{n} \sum_{j=n-i+1}^{n} \left\{ b\left[b'^{-1}(g^{-1}(\eta_{ij})) + s\phi \right] - b\left[b'^{-1}(g^{-1}(\eta_{ij})) \right] \right\} \right\}$$

with $(\hat{\eta}_{ij})$ *for m.l.e.*

- Inversion by F.F.T. if no standard m.g.f.

V. Predictive distribution

5.2. Approximated distributions using moments

From $R = \sum_{i=1}^{n} \sum_{j=n-i+1}^{n} X_{ij}$ and independence of X_{ij}, moments of *R* are functions of (μ_{ij}):

$$\mu = E(R) = \sum_{i=1}^{n} \sum_{j=n-i+1}^{n} \mu_{ij}$$

$$\sigma^2 = V(R) = \sum_{i=1}^{n} \sum_{j=n-i+1}^{n} V(X_{ij}) = \phi \sum_{i=1}^{n} \sum_{j=n-i+1}^{n} V(\mu_{ij})$$

$$\mu_3 = \mu_3(R) = \sum_{i=1}^{n} \sum_{j=n-i+1}^{n} \mu_3(X_{ij}) = \phi^2 \sum_{i=1}^{n} \sum_{j=n-i+1}^{n} V'(\mu_{ij}) V(\mu_{ij})$$

$$\gamma_1 = \gamma_1(R) = \frac{\mu_3}{\sigma^3}$$

(m.l.e. with $\hat{\mu}_{ij}$)

V. Predictive distribution

5.2. Approximated distributions using moments

- Using :

  - NP-approximation,
  - Gamma approximations (Translated, Bowers),…

based on *m.l.e.* of (μ, σ, γ_1)

For instance

$$F_R(x) \approx F^{(NP)}(x) = \Phi\left[\frac{-3}{\gamma_1} + \sqrt{\frac{9}{\gamma_1^2} + 1 + \frac{6}{\gamma_1}\left(\frac{x-\mu}{\sigma}\right)}\right]$$

$$q_{1-\eta}^{(NP)} = \mu + \sigma\left[\frac{\gamma_1}{6} q_{1-\eta}^2 + q_{1-\eta} - 1\right]$$

- We obtain *m.l.e.* of approximations of d.f. , VaR,…

enhanced by their asymptotic s.e. and confidence interval

VI.Example

incremental claims amounts for some line of Marine business

Years	0	1	2	3	4	5	6	7
0	1 381	4 399	4 229	435	465	205	110	67
1	859	6 940	2 619	1 531	517	572	287	
2	6 482	6 463	3 995	1 420	547	723		
3	2 899	16 428	5 521	2 424	477			
4	3 964	15 872	8 178	3 214				
5	6 809	24 484	27 928					
6	11 155	38 229						
7	10 641							

chain ladder reserve : 133750

Comparing models by extended quasi-likelihood :

1. Log / Gamma

2/.Log / overdispersed Poisson

VI.Example

	Overdispersed Poisson model				Bootstrap (1000 samples)	
	Estimation of Φ : Deviance		Estimation of Φ : Pearson residuals			
	Estimates of $E(R_i)$	$se(R_i)/R_i$	Estimates of $E(R_i)$	$se(R_i)/R_i$	Estimates of $E(R_i)$	$se(R_i)/R_i$
1	80	329%	80	348%	71	331%
2	442	134%	442	142%	427	123%
3	1 631	70%	1 631	74%	1 624	62%
4	2 811	53%	2 811	56%	2 777	46%
5	117 86	32%	11 786	34%	11 706	28%
6	41 864	20%	41 864	21%	41 799	18%
7	75 137	31%	75 137	33%	75 595	29%
Total	133 750	19%	133 750	20%	134 000	20%

VI.Example

Estimates of the percentiles of *R* directly and using Normal Power approximation

Gamma approximations not available ($\gamma_1 = 0.075$)

	0.50	0.75	0.80	0.90	0.95	0.99
$\hat{q}_{1-\eta}(R)$	133750	140733	142463	147019	150781	157836
$\hat{q}_{1-\eta}^{(NP)}(R)$	123396	130441	132205	136885	140789	148205

VI.Example

	Gamma Model					
	Estimation of Φ : maximisation of the likelihood		**Estimation of Φ : Deviance**		**Estimation of Φ : Pearson**	
	Estimates of *E*(R_i)	**se(R_i)/R_i**	**Estimates of *E*(R_i)**	**se(R_i)/R_i**	**Estimates of *E*(R_i)**	**se(R_i)/R_i**
1	101	37%	101	49%	101	50%
2	494	25%	494	33%	494	34%
3	1 286	22%	1 286	29%	1 286	30%
4	2 793	22%	2 793	28%	2 793	29%
5	11 262	23%	11 262	30%	11 262	31%
6	36 702	27%	36 702	35%	36 702	36%
7	69 563	36%	69 563	48%	69 563	49%
Total	122 200	22%	122 200	29%	122 200	30%

VI.Example

Estimates of the percentiles of *R* by Normal Power and Gamma approx.

$1-\eta$	0.50	0.75	0.80	0.90	0.95	0.99
$q_{1-\eta}^{(NP)}(R)$	112621	127662	131632	142550	152071	171164
$q_{1-\eta}^{(GT)}(R)$	132 027	146 885	150 820	161 671	171 188	190 445

Conclusion