A Bonus-Malus System as a Markov Set-Chain

Małgorzata Niemiec
Warsaw School of Economics
Institute of Econometrics

Contents

A Bonus-Malus System as a Markov Set-Chain

1. Markov set-chain

Definition 1*

Let N^1 be a compact set of $r \times r$ stochastic matrices. Let consider Markov chains with the state space $S = \{1, 2, \ldots, r\}$, having all their transition matrices in N^1. A *Markov set-chain* is the sequence

$$N^1, N^2, N^3, \ldots,$$

where $N^k = \{\mathbf{P}: \mathbf{P} = \mathbf{P}_1 \mathbf{P}_2 \cdots \mathbf{P}_k$, where $\mathbf{P}_i \in N^1$ for all $i = 1, 2, \ldots, k\}$ for each k.

*Hartfiel D. J. [1998], *Markov Set–Chains*, Springer Verlag, New York

A Bonus-Malus System as a Markov Set-Chain

1. Markov set-chain

Definition 2*

A *matrix interval* is the interval

$$[\mathbf{K}, \mathbf{Q}] = \{\mathbf{P}: \mathbf{K} \leq \mathbf{P} \leq \mathbf{Q}\},$$

where $\mathbf{P}=[p_{ij}]$ denotes a $r\times r$ stochastic matrix and $\mathbf{K}=[k_{ij}]$ and $\mathbf{Q}=[q_{ij}]$ are nonnegative $r\times r$ matrices such that $\mathbf{K} \leq \mathbf{Q}$.

Definition 3*

A matrix interval $[\mathbf{K}, \mathbf{Q}]$ is *tight* if $k_{ij} = \min\limits_{\mathbf{P}\in[\mathbf{K},\mathbf{Q}]} p_{ij}$ and $q_{ij} = \max\limits_{\mathbf{P}\in[\mathbf{K},\mathbf{Q}]} p_{ij}$ for all i and j.

*Hartfiel D. J. [1998], *Markov Set–Chains*, Springer Verlag, New York

2. Model of bonus-malus system

A system employed in automobile insurance is called a *bonus-malus system* when*:

- in each period (usually a year) all policyholders of a given tariff group are divided into a finite number of classes C_i (i=1, 2,..., r) and their premium depends only on the class they belong to;
- the class of a policyholder for a given period is determined uniquely by the class in the preceding period and the number of claims reported in that period.

*Lemaire J. [1995], *Bonus–malus systems in automobile insurance*, Kluwer Academic Publishers, Boston

A Bonus-Malus System as a Markov Set-Chain

2. Model of bonus-malus system

The classical model of a bonus-malus system – a finite homogeneous Markov chain with the state space $S = \{1, 2,\ldots, r\}$ and transition matrix*

$$\mathbf{P}(\lambda) = \left[p_{ij}(\lambda)\right] = \sum_{k=0}^{\infty} p_k(\lambda)\mathbf{T}_k ,$$

where $p_k(\lambda)$ is the probability that the policyholder with claim frequency λ reports k claims in one period and $\mathbf{T}_k$ - the matrix of transition rules.

*Lemaire J. [1995], *Bonus–malus systems in automobile insurance*, Kluwer Academic Publishers, Boston

A Bonus-Malus System as a Markov Set-Chain

2. Model of bonus-malus system

Assumptions:

1. the number of claims of a policyholder characterised by λ conforms to a Poisson distribution;
2. $\lambda^{(1)}$ and $\lambda^{(2)}$ such that

   $0<\lambda^{(1)}< \lambda^{(2)}<1$ (1)

   determine the interval $[\lambda^{(1)}, \lambda^{(2)}]$ of the claim frequency variability.

Under the above assumptions:

$$p_k(\lambda)\in[min\{p_k(\lambda^{(1)}); p_k(\lambda^{(2)})\}, max\{p_k(\lambda^{(1)}); p_k(\lambda^{(2)})\}], \quad (2)$$

for $k = 0, 1, 2,\ldots$ and $\lambda\in[\lambda^{(1)}, \lambda^{(2)}]$.

2. Model of bonus-malus system

Since $\lambda \in [\lambda^{(1)}, \lambda^{(2)}]$, the lower and upper bounds on the transition matrix interval are:

$$\mathbf{K} = \sum_{k=0}^{\infty} min\{p_k^{(1)}, p_k^{(2)}\}\mathbf{T}_k = [min\{p_{ij}^{(1)}, p_{ij}^{(2)}\}], \qquad (3)$$

$$\mathbf{Q} = \sum_{k=0}^{\infty} max\{p_k^{(1)}, p_k^{(2)}\}\mathbf{T}_k = [max\{p_{ij}^{(1)}, p_{ij}^{(2)}\}], \qquad (4)$$

where upper indices $^{(1)}$ and $^{(2)}$ indicate that a given probability is calculated for $\lambda^{(1)}$ and $\lambda^{(2)}$ respectively.

$$\mathbf{P}(\lambda) \in [\mathbf{K}, \mathbf{Q}] \quad \text{for } \lambda \in [\lambda^{(1)}, \lambda^{(2)}]$$

2. Model of bonus-malus system

Theorem 1

Let a Markov set-chain be a model of a bonus-malus system under assumptions (1)-(2). Let [**K**, **Q**] be its transition matrix interval, where **K** and **Q** are given by formulas (3) and (4). Then *the Markov set-chain* is *ergodic*.

Theorem 2

Let a Markov set-chain be a model of a bonus-malus system under assumptions (1)-(2). Let [**K**, **Q**] be its transition matrix interval, where **K** and **Q** are given by formulas (3) and (4). Then *the interval* [**K**, **Q**] is *tight*.

A Bonus-Malus System as a Markov Set-Chain

3. Example

Bonus-malus system of PZU SA (April 2003)

Class number	Premium level (in percentage)	Class number after						
		0	1	2	3	4	5	6 or more
		claims						
1	200	2	1	1	1	1	1	1
2	150	3	1	1	1	1	1	1
3	130	4	1	1	1	1	1	1
4	115	5	2	1	1	1	1	1
5	100	6	3	1	1	1	1	1
6	90	7	4	2	1	1	1	1
7	80	8	5	3	1	1	1	1
8	80	9	6	4	2	1	1	1
9	70	10	7	5	3	1	1	1
10	60	11	8	6	4	2	1	1
11	50	12	9	7	5	3	1	1
12	50	13	10	8	6	4	2	1
13	40	13	11	9	7	5	3	1

3. Example

Claim frequency interval: [0.1; 0.2]

Bounds on the interval of stationary probability distributions:

$$\mathbf{l}^{\infty} = \begin{bmatrix} 0.00002 \\ 0.00004 \\ 0.00011 \\ 0.00022 \\ 0.00056 \\ 0.00108 \\ 0.00297 \\ 0.00506 \\ 0.01621 \\ 0.02180 \\ 0.08555 \\ 0.06722 \\ 0.51409 \end{bmatrix}' \qquad \mathbf{h}^{\infty} = \begin{bmatrix} 0.00246 \\ 0.00358 \\ 0.00534 \\ 0.00771 \\ 0.01174 \\ 0.01652 \\ 0.02633 \\ 0.03497 \\ 0.06154 \\ 0.07160 \\ 0.15314 \\ 0.13363 \\ 0.77898 \end{bmatrix}'$$

A Bonus-Malus System as a Markov Set-Chain

3. Example

Lower bound on the interval of mean first passage times:

$$
\overline{\mathbf{M}}^{l} = \begin{bmatrix}
407.33 & 1.11 & 2.33 & 3.68 & 5.06 & 6.46 & 7.87 & 9.29 & 10.71 & 12.13 & 13.54 & 14.96 & 16.38 \\
496.29 & 279.46 & 1.22 & 2.57 & 3.95 & 5.36 & 6.77 & 8.18 & 9.60 & 11.02 & 12.44 & 13.86 & 15.28 \\
604.95 & 339.87 & 187.12 & 1.35 & 2.73 & 4.14 & 5.55 & 6.96 & 8.38 & 9.80 & 11.22 & 12.64 & 14.05 \\
737.67 & 413.65 & 226.82 & 129.76 & 1.38 & 2.79 & 4.20 & 5.61 & 7.03 & 8.45 & 9.87 & 11.29 & 12.70 \\
800.51 & 503.98 & 275.52 & 156.67 & 85.15 & 1.40 & 2.82 & 4.23 & 5.65 & 7.07 & 8.49 & 9.90 & 11.32 \\
855.54 & 546.35 & 335.25 & 189.79 & 102.13 & 60.53 & 1.41 & 2.83 & 4.24 & 5.66 & 7.08 & 8.50 & 9.92 \\
886.27 & 583.36 & 362.86 & 230.53 & 123.16 & 72.04 & 37.98 & 1.42 & 2.83 & 4.25 & 5.67 & 7.09 & 8.51 \\
909.07 & 603.70 & 386.87 & 248.98 & 149.14 & 86.39 & 44.48 & 28.59 & 1.42 & 2.84 & 4.25 & 5.67 & 7.09 \\
922.60 & 618.60 & 399.72 & 264.93 & 160.48 & 104.23 & 52.73 & 33.01 & 16.25 & 1.42 & 2.84 & 4.26 & 5.67 \\
931.58 & 627.13 & 408.91 & 273.12 & 170.15 & 111.65 & 63.12 & 38.72 & 17.93 & 13.97 & 1.42 & 2.84 & 4.26 \\
936.66 & 632.54 & 413.84 & 278.77 & 174.73 & 117.86 & 66.95 & 46.01 & 20.30 & 15.15 & 6.53 & 1.42 & 2.84 \\
939.47 & 635.28 & 416.68 & 281.47 & 177.64 & 120.45 & 70.00 & 48.34 & 23.51 & 16.90 & 6.06 & 7.48 & 1.42 \\
940.56 & 636.46 & 417.73 & 282.71 & 178.60 & 121.84 & 70.76 & 50.07 & 23.88 & 19.35 & 5.81 & 7.23 & 1.28
\end{bmatrix}
$$

A Bonus-Malus System as a Markov Set-Chain

3. Example

Upper bound on the interval of mean first passage times:

$$\overline{\mathbf{M}}^{h} = \begin{bmatrix}
48039.25 & 1.22 & 2.71 & 4.54 & 6.52 & 8.64 & 10.84 & 13.10 & 15.41 & 17.74 & 21.39 & 23.99 & 24.82 \\
53090.48 & 22425.35 & 1.49 & 3.31 & 5.30 & 7.42 & 9.62 & 11.88 & 14.19 & 16.52 & 20.17 & 22.77 & 23.60 \\
58672.94 & 24782.61 & 9314.63 & 1.82 & 3.80 & 5.92 & 8.13 & 10.39 & 12.69 & 15.02 & 18.68 & 21.28 & 22.11 \\
64842.53 & 27387.79 & 10292.87 & 4518.94 & 1.98 & 4.10 & 6.31 & 8.57 & 10.87 & 13.20 & 16.85 & 19.46 & 20.28 \\
66351.92 & 30267.07 & 11374.11 & 4992.74 & 1785.76 & 2.12 & 4.32 & 6.59 & 8.89 & 11.22 & 14.87 & 17.47 & 18.30 \\
67461.81 & 30970.92 & 12569.22 & 5516.52 & 1972.05 & 927.84 & 2.20 & 4.47 & 6.77 & 9.10 & 12.75 & 15.35 & 16.18 \\
67806.03 & 31488.27 & 12860.73 & 6095.58 & 2178.12 & 1023.87 & 336.15 & 2.26 & 4.57 & 6.90 & 10.55 & 13.15 & 13.98 \\
68007.59 & 31648.20 & 13074.79 & 6236.26 & 2406.07 & 1130.20 & 369.92 & 197.78 & 2.30 & 4.63 & 8.29 & 10.89 & 11.71 \\
68079.66 & 31741.53 & 13140.39 & 6339.38 & 2460.80 & 1247.94 & 407.47 & 216.98 & 61.71 & 2.33 & 5.98 & 8.58 & 9.41 \\
68116.41 & 31774.42 & 13178.33 & 6370.47 & 2500.68 & 1275.68 & 449.19 & 238.43 & 66.59 & 45.87 & 3.65 & 6.25 & 7.08 \\
68130.08 & 31790.82 & 13191.16 & 6388.15 & 2512.11 & 1295.72 & 458.33 & 262.38 & 72.22 & 49.08 & 11.69 & 3.90 & 4.73 \\
68135.98 & 31796.44 & 13197.14 & 6393.64 & 2518.24 & 1300.98 & 464.68 & 267.16 & 78.69 & 52.87 & 11.16 & 14.88 & 2.37 \\
68137.60 & 31798.44 & 13198.65 & 6395.81 & 2519.55 & 1303.48 & 465.67 & 270.31 & 79.19 & 57.30 & 10.81 & 14.65 & 1.95
\end{bmatrix}$$

3. Example

The difference between the upper and lower bounds on the interval of mean first passage times:

47631.92	**0.12**	**0.39**	**0.86**	**1.46**	**2.18**	**2.97**	**3.81**	**4.70**	**5.61**	**7.85**	**9.03**	**8.44**
52594.18	22145.89	**0.27**	**0.74**	**1.34**	**2.06**	**2.85**	**3.70**	**4.58**	**5.50**	**7.73**	**8.91**	**8.32**
58067.99	24442.74	9127.51	**0.47**	**1.07**	**1.79**	**2.58**	**3.43**	**4.31**	**5.23**	**7.46**	**8.64**	**8.05**
64104.86	26974.14	10066.05	4389.18	**0.60**	**1.32**	**2.11**	**2.96**	**3.84**	**4.75**	**6.99**	**8.17**	**7.58**
65551.41	29763.09	11098.59	4836.07	1700.61	**0.72**	**1.51**	**2.36**	**3.24**	**4.15**	**6.39**	**7.57**	**6.98**
66606.28	30424.57	12233.97	5326.73	1869.92	867.31	**0.79**	**1.64**	**2.52**	**3.44**	**5.67**	**6.85**	**6.26**
66919.75	30904.91	12497.87	5865.04	2054.96	951.83	298.17	**0.85**	**1.73**	**2.65**	**4.88**	**6.06**	**5.47**
67098.51	31044.50	12687.92	5987.28	2256.93	1043.81	325.44	169.18	**0.89**	**1.80**	**4.03**	**5.21**	**4.62**
67157.06	31122.93	12740.67	6074.45	2300.32	1143.71	354.74	183.97	**45.46**	**0.91**	**3.15**	**4.33**	**3.74**
67184.84	31147.29	12769.41	6097.35	2330.53	1164.04	386.07	199.71	**48.66**	**31.90**	**2.23**	**3.42**	**2.82**
67193.41	31158.28	12777.31	6109.37	2337.38	1177.86	391.38	216.37	**51.92**	**33.93**	**5.16**	**2.49**	**1.89**
67196.50	31161.16	12780.46	6112.17	2340.60	1180.53	394.68	218.82	**55.17**	**35.97**	**5.09**	**7.39**	**0.95**
67197.03	31161.98	12780.91	6113.10	2340.95	1181.64	394.92	220.23	**55.31**	**37.94**	**5.00**	**7.42**	**0.66**

Lower bound on the interval of mean first passage times:

$$\overline{\mathbf{M}}^{l} = \begin{bmatrix}
\mathbf{407.33} & 1.11 & 2.33 & 3.68 & 5.06 & 6.46 & 7.87 & 9.29 & 10.71 & 12.13 & 13.54 & 14.96 & 16.38 \\
\mathbf{496.29} & \mathbf{279.46} & 1.22 & 2.57 & 3.95 & 5.36 & 6.77 & 8.18 & 9.60 & 11.02 & 12.44 & 13.86 & 15.28 \\
\mathbf{604.95} & \mathbf{339.87} & \mathbf{187.12} & 1.35 & 2.73 & 4.14 & 5.55 & 6.96 & 8.38 & 9.80 & 11.22 & 12.64 & 14.05 \\
\mathbf{737.67} & \mathbf{413.65} & \mathbf{226.82} & \mathbf{129.76} & 1.38 & 2.79 & 4.20 & 5.61 & 7.03 & 8.45 & 9.87 & 11.29 & 12.70 \\
\mathbf{800.51} & \mathbf{503.98} & \mathbf{275.52} & \mathbf{156.67} & \mathbf{85.15} & 1.40 & 2.82 & 4.23 & 5.65 & 7.07 & 8.49 & 9.90 & \underline{\mathbf{11.32}} \\
\mathbf{855.54} & \mathbf{546.35} & \mathbf{335.25} & \mathbf{189.79} & \mathbf{102.13} & \mathbf{60.53} & 1.41 & 2.83 & 4.24 & 5.66 & 7.08 & 8.50 & 9.92 \\
\mathbf{886.27} & \mathbf{583.36} & \mathbf{362.86} & \mathbf{230.53} & \mathbf{123.16} & \mathbf{72.04} & 37.98 & 1.42 & 2.83 & 4.25 & 5.67 & 7.09 & 8.51 \\
\mathbf{909.07} & \mathbf{603.70} & \mathbf{386.87} & \mathbf{248.98} & \mathbf{149.14} & \mathbf{86.39} & 44.48 & 28.59 & 1.42 & 2.84 & 4.25 & 5.67 & 7.09 \\
\mathbf{922.60} & \mathbf{618.60} & \mathbf{399.72} & \mathbf{264.93} & \mathbf{160.48} & \mathbf{104.23} & \mathbf{52.73} & 33.01 & 16.25 & 1.42 & 2.84 & 4.26 & 5.67 \\
\mathbf{931.58} & \mathbf{627.13} & \mathbf{408.91} & \mathbf{273.12} & \mathbf{170.15} & \mathbf{111.65} & \mathbf{63.12} & 38.72 & 17.93 & 13.97 & 1.42 & 2.84 & 4.26 \\
\mathbf{936.66} & \mathbf{632.54} & \mathbf{413.84} & \mathbf{278.77} & \mathbf{174.73} & \mathbf{117.86} & \mathbf{66.95} & 46.01 & 20.30 & 15.15 & 6.53 & 1.42 & 2.84 \\
\mathbf{939.47} & \mathbf{635.28} & \mathbf{416.68} & \mathbf{281.47} & \mathbf{177.64} & \mathbf{120.45} & \mathbf{70.00} & 48.34 & 23.51 & 16.90 & 6.06 & 7.48 & 1.42 \\
\mathbf{940.56} & \mathbf{636.46} & \mathbf{417.73} & \mathbf{282.71} & \mathbf{178.60} & \mathbf{121.84} & \mathbf{70.76} & \mathbf{50.07} & 23.88 & 19.35 & 5.81 & 7.23 & 1.28
\end{bmatrix}$$

A Bonus-Malus System as a Markov Set-Chain

3. Example

Upper bound on the interval of mean first passage times:

$$\overline{\mathbf{M}}^h = \begin{bmatrix}
48039.25 & 1.22 & 2.71 & 4.54 & 6.52 & 8.64 & 10.84 & 13.10 & 15.41 & 17.74 & 21.39 & 23.99 & 24.82 \\
53090.48 & 22425.35 & 1.49 & 3.31 & 5.30 & 7.42 & 9.62 & 11.88 & 14.19 & 16.52 & 20.17 & 22.77 & 23.60 \\
58672.94 & 24782.61 & 9314.63 & 1.82 & 3.80 & 5.92 & 8.13 & 10.39 & 12.69 & 15.02 & 18.68 & 21.28 & 22.11 \\
64842.53 & 27387.79 & 10292.87 & 4518.94 & 1.98 & 4.10 & 6.31 & 8.57 & 10.87 & 13.20 & 16.85 & 19.46 & 20.28 \\
66351.92 & 30267.07 & 11374.11 & 4992.74 & 1785.76 & 2.12 & 4.32 & 6.59 & 8.89 & 11.22 & 14.87 & 17.47 & \underline{\mathbf{18.30}} \\
67461.81 & 30970.92 & 12569.22 & 5516.52 & 1972.05 & 927.84 & 2.20 & 4.47 & 6.77 & 9.10 & 12.75 & 15.35 & 16.18 \\
67806.03 & 31488.27 & 12860.73 & 6095.58 & 2178.12 & 1023.87 & 336.15 & 2.26 & 4.57 & 6.90 & 10.55 & 13.15 & 13.98 \\
68007.59 & 31648.20 & 13074.79 & 6236.26 & 2406.07 & 1130.20 & 369.92 & 197.78 & 2.30 & 4.63 & 8.29 & 10.89 & 11.71 \\
68079.66 & 31741.53 & 13140.39 & 6339.38 & 2460.80 & 1247.94 & 407.47 & 216.98 & 61.71 & 2.33 & 5.98 & 8.58 & 9.41 \\
68116.41 & 31774.42 & 13178.33 & 6370.47 & 2500.68 & 1275.68 & 449.19 & 238.43 & 66.59 & 45.87 & 3.65 & 6.25 & 7.08 \\
68130.08 & 31790.82 & 13191.16 & 6388.15 & 2512.11 & 1295.72 & 458.33 & 262.38 & 72.22 & 49.08 & 11.69 & 3.90 & 4.73 \\
68135.98 & 31796.44 & 13197.14 & 6393.64 & 2518.24 & 1300.98 & 464.68 & 267.16 & 78.69 & 52.87 & 11.16 & 14.88 & 2.37 \\
68137.60 & 31798.44 & 13198.65 & 6395.81 & 2519.55 & 1303.48 & 465.67 & 270.31 & 79.19 & 57.30 & 10.81 & 14.65 & 1.95
\end{bmatrix}$$

4. Conclusions

The application of a Markov set-chain to the analysis of a bonus-malus system enables us to:

- broaden the scope of research in comparison with classical Markov chains;
- relax the assumption of the constant claim frequency of a policyholder;
- examine the consequences of claim frequency changes within a given interval;
- determine the variability range of stationary distribution as well as of mean first passage times.

Thank you

Małgorzata Niemiec