A Bonus-Malus System as a Markov Set-Chain

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Abstract

The purpose of this paper is to analyse a bonus-malus system within the framework of the theory of an ergodic Markov set-chain. It is shown that this type of a Markov chain enables to evaluate the system, even in its steady-state, under the assumption that its transition probabilities change in a definite range. We set up a model that allows for determining the consequences of changes in the claim frequency of a policyholder. As a numerical example we examine the bonus-malus system employed by one of the Polish insurance company.

Keywords:

Markov set-chain, bonus-malus system, claim frequency

1. Introduction

In the analysis of a bonus-malus system it is commonly assumed that the claim frequency of an individual policyholder remains unaltered. This assumption ensures constant transition probabilities and enables to model the system with the use of a homogeneous Markov chain (see e.g. Lemaire [1995]). However, it is known that the claim frequency of a driver may fluctuate from time to time for various reasons. Therefore, in this paper we relax the above mentioned assumption and apply an ergodic Markov set-chain defined by Hartfiel [1998].

The Markov set-chain constitutes the specific generalisation of the idea of classical Markov chains. Its fundamental assumption consists in allowing for changes of transition probabilities at each step, though these changes are restricted by some lower and upper bounds. It is assumed that the transition probabilities belong to a given compact set, usually defined as an interval, and their exact values are not known. Thus, the variability of the probabilities is possible, which may broaden the scope of the analysis of phenomena modelled to date in the framework of the theory of a homogeneous Markov chain. Although a Markov set-chain can be treated as a kind of a nonhomogeneous Markov chain, for the purpose of its application there is no need to determine rules of transition matrix changes at each step. Moreover, it is easy to examine its long run behaviour, which in case of a nonhomogeneous Markov chain is often difficult and restricts its use.

This paper is organised as follows. In Section 2 the model of a bonus-malus system is constructed within the framework of the Markov set-chain theory. Section 3 is devoted to a numerical example, which shows how the fluctuation of the claim frequency of a policyholder may influence some asymptotic measures used to evaluate bonus-malus systems. Section 4 provides final conclusions. Since the concept of Hartfiel [1998] is relatively new and so far has not been extensively applied, Appendix describes briefly the theoretical basis and properties of an ergodic Markov set-chain.

2. The model of a bonus-malus system

As stated in Lemaire [1995], a system employed in automobile insurance is called a bonus-malus system when:

- all policyholders of a given tariff group are divided into a finite number of classes, denoted by C_i ($i = 1, 2, ..., r$), and their premium depends only on the class they belong to, and
- the class of a policyholder for a given period (usually a year) is determined uniquely by the class in the preceding period and the number of claims reported in that period.

Such a system is defined by the initial class C_{i_0}, premium scale $\mathbf{b} = [b_1, b_2, ..., b_r]'$, where b_i is the premium level in class C_i, as well as transition rules i.e. rules governing the transfer of a policyholder from one class to another when the number of his or her claims is known.

The transition rules are represented by means of $r \times r$ matrices

$$\mathbf{T}_k = [t_{ij}^{(k)}],$$

where $t_{ij}^{(k)} = \begin{cases} 1 & \text{if} \quad T_k(i) = j \\ 0 & \text{if} \quad T_k(i) \neq j \end{cases}$ and $T_k(i) = j$ denotes the transfer of a policyholder reporting k claims from class C_i into class C_j in the next period. The probability of moving from C_i to C_j for a policyholder with claim frequency λ is given by

$$p_{ij}(\lambda) = \sum_{k=0}^{\infty} p_k(\lambda) t_{ij}^{(k)},$$

where $p_k(\lambda)$ is the probability that a driver with claim frequency λ has k claims in one period. Under the assumption that the claim frequency of an insured is stationary in time, a finite homogeneous Markov chain with the state space $S = \{1, 2, ..., r\}$ and transition matrix

$$\mathbf{P}(\lambda) = [p_{ij}(\lambda)] = \sum_{k=0}^{\infty} p_k(\lambda) \mathbf{T}_k$$

is a model of a bonus-malus system (see Lemaire [1995]). In this paper we restrict our attention merely to these bonus-malus systems, whose models are irreducible ergodic finite and homogeneous Markov chains. It is worth mentioning that most existing bonus-malus systems form such Markov chains.

The assumption of the constant claim frequency of a driver, which is indispensable for the analysis of a bonus-malus system within the framework of the homogeneous Markov chain theory, seems unrealistic. In fact, the claim frequency may change over time due to insurance companies' actions, changes in the driving abilities and behaviour of a policyholder as well as external factors such as weather conditions or a state of roads. Irrespective of the reason for these changes, the need for their evaluation arises. An ergodic Markov set-chain proves to be helpful in this respect.

In setting up the model of a bonus-malus system, being the Markov set-chain, which enables us to determine the consequences of changes in the claim frequency of a policyholder, we need the following assumptions:

(1) the number of claims of a policyholder characterised by λ conforms to a Poisson distribution;

(2) $\lambda^{(1)}$ and $\lambda^{(2)}$ such that

$$0<\lambda^{(1)}<\lambda^{(2)}<1 \qquad (1)$$

are the lower and upper bounds on the interval of the claim frequency variability.

Since the actual claim frequency hardly ever exceeds the value 1, condition (1) is not restrictive in the analysis of existing bonus-malus systems. However, it is necessary that the following relationship be satisfied

$$p_k(\lambda)\in[min\{p_k(\lambda^{(1)});p_k(\lambda^{(2)})\},max\{p_k(\lambda^{(1)});p_k(\lambda^{(2)})\}], \qquad (2)$$

where $\lambda\in[\lambda^{(1)},\lambda^{(2)}]$ and $p_k(\lambda)$ for $k=0,1,2,...$ is the probability from the Poisson distribution. It is easy to verify that $p_0(\lambda)$ and $p_k(\lambda)$ for $k=1,2,...$ are respectively a decreasing and increasing function of λ in the interval (0,1), which ensures that relation (2) is valid.

Under assumptions (1)–(2) we can determine the matrix interval that comprises all transition matrices of the Markov set-chain, the model of a bonus-malus system. Lower and upper bounds on that interval can be expressed as

$$\mathbf{K}=\sum_{k=0}^{\infty}min\{p_k^{(1)},p_k^{(2)}\}\mathbf{T}_k=[min\{p_{ij}^{(1)},p_{ij}^{(2)}\}], \qquad (3)$$

$$\mathbf{Q}=\sum_{k=0}^{\infty}max\{p_k^{(1)},p_k^{(2)}\}\mathbf{T}_k=[max\{p_{ij}^{(1)},p_{ij}^{(2)}\}], \qquad (4)$$

where upper indices $^{(1)}$ and $^{(2)}$ indicate that a given probability has been calculated for $\lambda^{(1)}$ and $\lambda^{(2)}$ respectively. Note that **K** and **Q** are nonnegative $r\times r$ matrices such that $\mathbf{K}\le\mathbf{Q}$. From relation (2) and formulas (3) and (4) we obtain

$$\mathbf{P}(\lambda)\in[\mathbf{K},\mathbf{Q}] \quad \text{for } \lambda\in[\lambda^{(1)},\lambda^{(2)}].$$

Hence, the interval $[\mathbf{K},\mathbf{Q}]$ contains all, corresponding to each $\lambda\in[\lambda^{(1)},\lambda^{(2)}]$, transition matrices of irreducible ergodic finite homogeneous Markov chains that are models of the same bonus-malus system and differ merely in the assumed value of the claim frequency of a policyholder.

Theorem 1

Let a Markov set-chain be a model of a bonus-malus system under assumptions (1)-(2). Let $[\mathbf{K},\mathbf{Q}]$, where $\mathbf{K}$ and $\mathbf{Q}$ are given by formulas (3) and (4), be its transition matrix interval. Then the Markov set-chain is ergodic.

Proof: Note that the arrangement of all nonzero elements in each matrix from the interval $[\mathbf{K},\mathbf{Q}]$ is identical and depends on the transition rules expressed in matrices $\mathbf{T}_k$. The interval $[\mathbf{K},\mathbf{Q}]$ comprises one-step transition matrices of irreducible ergodic finite and homogeneous Markov chains. Each of these matrices is both irreducible and primitive and in their canonical form the arrangement of zero and positive elements is identical. As the same arrangement of zero and positive elements concerns any matrix from the interval $[\mathbf{K},\mathbf{Q}]$, including $\mathbf{K}$ and $\mathbf{Q}$, all matrices belonging to this interval are irreducible and primitive. This fact is a sufficient condition that the Markov set-chain is ergodic (see Definitions A.5-A.6 and Theorem A.1 in Appendix).

The important feature of the matrix interval is its tightness. This property is particularly required while applying the Hi-Lo method, an algorithm for computing bounds on each step transition probabilities, stationary distribution as well as mean first passage times (see Hartfiel [1991], [1998]). It can be proved that the transition matrix interval $[\mathbf{K},\mathbf{Q}]$ of the considered model is tight. To show that let us define two sets of indices. Suppose that for $i, j \in S$

$$A_i = \{ j : min\{ p_{ij}^{(1)}, p_{ij}^{(2)} \} = p_{ij}^{(1)} \} ,$$

$$B_i = \{ j : min\{ p_{ij}^{(1)}, p_{ij}^{(2)} \} = p_{ij}^{(2)} \} .$$

It is easily seen that for the above sets the conditions hold:

$$A_i = \{ j : max\{ p_{ij}^{(1)}, p_{ij}^{(2)} \} = p_{ij}^{(2)} \} ,$$

$$B_i = \{ j : max\{ p_{ij}^{(1)}, p_{ij}^{(2)} \} = p_{ij}^{(1)} \} .$$

The sets A_i and B_i are disjoint. Their definition and the fact that $\mathbf{K} \le \mathbf{Q}$ implicate the following relationships:

- if $t \in A_i$ for $i \in S$ then $p_{it}^{(1)}$ is an element of the matrix $\mathbf{K}$ and $p_{it}^{(2)}$ is an element of the matrix $\mathbf{Q}$;

- if $t \in B_i$ for $i \in S$ then $p_{it}^{(2)}$ is an element of the matrix **K** and $p_{it}^{(1)}$ is an element of the matrix **Q**.

As an immediate consequence of the above implications we obtain

$$p_{it}^{(1)} \le p_{it}^{(2)} \text{ for } t \in A_i \text{ and } i,t \in S, \tag{5}$$

$$p_{it}^{(2)} \le p_{it}^{(1)} \text{ for } t \in B_i \text{ and } i,t \in S. \tag{6}$$

Furthermore, referring to the stochastic property of the transition matrices $\mathbf{P}(\lambda^{(1)}) = [p_{ij}^{(1)}]$ and $\mathbf{P}(\lambda^{(2)}) = [p_{ij}^{(2)}]$, we have

$$\sum_{t \in A_i} p_{it}^{(1)} + \sum_{t \in B_i} p_{it}^{(1)} = 1 \text{ and } \sum_{t \in A_i} p_{it}^{(2)} + \sum_{t \in B_i} p_{it}^{(2)} = 1 \quad \text{for } \; i,t \in S. \tag{7}$$

<u>Theorem 2</u>

Let a Markov set-chain be a model of a bonus-malus system under assumptions (1)-(2). Let $[\mathbf{K},\mathbf{Q}]$, where **K** and **Q** are given by formulas (3) and (4), be its transition matrix interval. Then the interval $[\mathbf{K},\mathbf{Q}]$ is tight.

Proof: Since $[\mathbf{K},\mathbf{Q}] = \{\mathbf{P} : \mathbf{P}_i \in [\mathbf{K}_i, \mathbf{Q}_i] \text{ for } i = 1, 2,.., r\}$, where $\mathbf{P}_i$, $\mathbf{K}_i$, $\mathbf{Q}_i$ are *i*-th rows of **P**, **K**, **Q** respectively, it suffices to show that for all $i \in S$ *i*-th rows of **K** and **Q** are bounds on tight vector intervals. Hence, we have to prove that for each vector interval $[\mathbf{K}_i, \mathbf{Q}_i]$, where $\mathbf{K}_i = [k_j]$ and $\mathbf{Q}_i = [q_j]$, the following conditions hold:

$$k_j + \sum_{t \ne j} q_t \ge 1 \text{ and } q_j + \sum_{t \ne j} k_t \le 1 \text{ for all } j. \tag{8}$$

If the probability p_{ib} taken from the matrix **K** is a function of the claim frequency $\lambda^{(1)}$ then using relations (5) and (7) we get

$$p_{ib}^{(1)} + \sum_{\substack{t \in A_i \\ t \ne b}} p_{it}^{(2)} + \sum_{\substack{t \in B_i \\ t \ne b}} p_{it}^{(1)} \ge p_{ib}^{(1)} + \sum_{\substack{t \in A_i \\ t \ne b}} p_{it}^{(1)} + \sum_{\substack{t \in B_i \\ t \ne b}} p_{it}^{(1)} = 1 \text{ for all } i,b,t \in S,$$

and if it depends on $\lambda^{(2)}$ then from (6) and (7) we have

$$p_{ib}^{(2)} + \sum_{\substack{t \in A_i \\ t \ne b}} p_{it}^{(2)} + \sum_{\substack{t \in B_i \\ t \ne b}} p_{it}^{(1)} \ge p_{ib}^{(2)} + \sum_{\substack{t \in A_i \\ t \ne b}} p_{it}^{(2)} + \sum_{\substack{t \in B_i \\ t \ne b}} p_{it}^{(2)} = 1 \text{ for all } i,b,t \in S.$$

If the probability p_{ib} taken from the matrix **Q** is a function of the claim frequency $\lambda^{(1)}$ then relations (6) and (7) imply that

$$p_{ib}^{(1)} + \sum_{\substack{t \in A_i \\ t \ne b}} p_{it}^{(1)} + \sum_{\substack{t \in B_i \\ t \ne b}} p_{it}^{(2)} \le p_{ib}^{(1)} + \sum_{\substack{t \in A_i \\ t \ne b}} p_{it}^{(1)} + \sum_{\substack{t \in B_i \\ t \ne b}} p_{it}^{(1)} = 1 \text{ for all } i,b,t \in S,$$

and if it depends on $\lambda^{(2)}$ then relationships (5) and (7) lead us to

$$p_{ib}^{(2)} + \sum_{\substack{t\in A_i\\ t\neq b}} p_{it}^{(1)} + \sum_{\substack{t\in B_i\\ t\neq b}} p_{it}^{(2)} \le p_{ib}^{(2)} + \sum_{\substack{t\in A_i\\ t\neq b}} p_{it}^{(2)} + \sum_{\substack{t\in B_i\\ t\neq b}} p_{it}^{(2)} = 1 \text{ for all } i,b,t \in S .$$

Thus, conditions (8) are fulfilled, which means that the vector intervals bounded by i-th rows of **K** and **Q** and, consequently, the matrix interval $[\mathbf{K}, \mathbf{Q}]$ are tight.

3. A numerical example

As mentioned in the preceding Section, the Markov set-chain theory enables us to examine the consequences of the claim frequency changes within a given interval. In order to illustrate the application of the model described in Section 2, we analyse the bonus-malus system currently employed in first-party coverage insurance by Powszechny Zaklad Ubezpieczen SA (PZU), the Polish insurance company. Most calculations presented in this Section were obtained using our own programmes devised in MATLAB 6.0.

The bonus-malus system of PZU consists of 13 classes. New policyholders enter the system in class C_5. The specific premium levels for each class as well as transition rules are provided in Table 1. The properties of the system allow for modelling it as an irreducible ergodic finite and homogeneous Markov chain and therefore - as a Markov set-chain.

Table 1. Bonus-malus system of PZU

Class number	Premium level (in percentage)	Class number after						
		0	1	2	3	4	5	6 or more
		claims						
1	200	2	1	1	1	1	1	1
2	150	3	1	1	1	1	1	1
3	130	4	1	1	1	1	1	1
4	115	5	2	1	1	1	1	1
5	100	6	3	1	1	1	1	1
6	90	7	4	2	1	1	1	1
7	80	8	5	3	1	1	1	1
8	80	9	6	4	2	1	1	1
9	70	10	7	5	3	1	1	1
10	60	11	8	6	4	2	1	1
11	50	12	9	7	5	3	1	1
12	50	13	10	8	6	4	2	1
13	40	13	11	9	7	5	3	1

Source: General Conditions for First-Party Coverage Insurance of Powszechny Zaklad Ubezpieczen SA established on 25th April 2003

Since the average claim frequency in automobile first-party coverage insurance in Poland has been close to 0.15 over the recent years, let us consider a policyholder with the claim frequency varying from 0.1 to 0.2. Having assumed that the number of claims follows the Poisson distribution, by the application of formulas (3) and (4) we get bounds **K** and **Q** on the interval comprising all possible transition matrices of the ergodic Markov set-chain, the model of the bonus-malus system for the considered policyholder. By Theorem 2 the obtained interval is tight, which is essential to apply the Hi-Lo method. Using the method we calculate lower and upper limit bounds $\mathbf{L}^{\infty}$ and $\mathbf{H}^{\infty}$, which are rank one matrices with the following rows:

$$\mathbf{l}^{\infty} = \begin{bmatrix} 0.00002 \\ 0.00004 \\ 0.00011 \\ 0.00022 \\ 0.00056 \\ 0.00108 \\ 0.00297 \\ 0.00506 \\ 0.01621 \\ 0.02180 \\ 0.08555 \\ 0.06722 \\ 0.51409 \end{bmatrix}' \qquad \mathbf{h}^{\infty} = \begin{bmatrix} 0.00246 \\ 0.00358 \\ 0.00534 \\ 0.00771 \\ 0.01174 \\ 0.01652 \\ 0.02633 \\ 0.03497 \\ 0.06154 \\ 0.07160 \\ 0.15314 \\ 0.13363 \\ 0.77898 \end{bmatrix}' .$$

The above vectors are bounds on the interval of all possible stationary probability distributions, whose elements can be interpreted as the probabilities that the policyholder belongs to a given class, once full stationarity has been reached. The variability range of these probabilities for the driver with the claim frequency in the interval <0.1, 0.2> is diversified. Generally, in higher classes (with lower premiums) the steady–state probabilities are higher and more sensitive to the claim frequency changes. Thus, the driver has a better chance of being in a high-discount class, but the probability of this event is subject to larger variations than the probability of being in a low-discount class.

The analysis of bounds on mean first passage times may also provide valuable information. In the context of the model of a bonus-malus system each of these times, denoted by $\overline{m}_{ij}$, indicates an average time needed by the policyholder from class C_i to reach C_j for the first time. For our numerical example, the matrices of lower and upper bounds on these times $\overline{\mathbf{M}}^{l}$ and $\overline{\mathbf{M}}^{h}$ are as follows:

$$\overline{\mathbf{M}}^{l}=\left[\begin{array}{rrrrrrrrrrrrr}
407.33 & 1.11 & 2.33 & 3.68 & 5.06 & 6.46 & 7.87 & 9.29 & 10.71 & 12.13 & 13.54 & 14.96 & 16.38\\
496.29 & 279.46 & 1.22 & 2.57 & 3.95 & 5.36 & 6.77 & 8.18 & 9.60 & 11.02 & 12.44 & 13.86 & 15.28\\
604.95 & 339.87 & 187.12 & 1.35 & 2.73 & 4.14 & 5.55 & 6.96 & 8.38 & 9.80 & 11.22 & 12.64 & 14.05\\
737.67 & 413.65 & 226.82 & 129.76 & 1.38 & 2.79 & 4.20 & 5.61 & 7.03 & 8.45 & 9.87 & 11.29 & 12.70\\
800.51 & 503.98 & 275.52 & 156.67 & 85.15 & 1.40 & 2.82 & 4.23 & 5.65 & 7.07 & 8.49 & 9.90 & 11.32\\
855.54 & 546.35 & 335.25 & 189.79 & 102.13 & 60.53 & 1.41 & 2.83 & 4.24 & 5.66 & 7.08 & 8.50 & 9.92\\
886.27 & 583.36 & 362.86 & 230.53 & 123.16 & 72.04 & 37.98 & 1.42 & 2.83 & 4.25 & 5.67 & 7.09 & 8.51\\
909.07 & 603.70 & 386.87 & 248.98 & 149.14 & 86.39 & 44.48 & 28.59 & 1.42 & 2.84 & 4.25 & 5.67 & 7.09\\
922.60 & 618.60 & 399.72 & 264.93 & 160.48 & 104.23 & 52.73 & 33.01 & 16.25 & 1.42 & 2.84 & 4.26 & 5.67\\
931.58 & 627.13 & 408.91 & 273.12 & 170.15 & 111.65 & 63.12 & 38.72 & 17.93 & 13.97 & 1.42 & 2.84 & 4.26\\
936.66 & 632.54 & 413.84 & 278.77 & 174.73 & 117.86 & 66.95 & 46.01 & 20.30 & 15.15 & 6.53 & 1.42 & 2.84\\
939.47 & 635.28 & 416.68 & 281.47 & 177.64 & 120.45 & 70.00 & 48.34 & 23.51 & 16.90 & 6.06 & 7.48 & 1.42\\
940.56 & 636.46 & 417.73 & 282.71 & 178.60 & 121.84 & 70.76 & 50.07 & 23.88 & 19.35 & 5.81 & 7.23 & 1.28
\end{array}\right]$$

$$\overline{\mathbf{M}}^{h}=\left[\begin{array}{rrrrrrrrrrrrr}
48039.25 & 1.22 & 2.71 & 4.54 & 6.52 & 8.64 & 10.84 & 13.10 & 15.41 & 17.74 & 21.39 & 23.99 & 24.82\\
53090.48 & 22425.35 & 1.49 & 3.31 & 5.30 & 7.42 & 9.62 & 11.88 & 14.19 & 16.52 & 20.17 & 22.77 & 23.60\\
58672.94 & 24782.61 & 9314.63 & 1.82 & 3.80 & 5.92 & 8.13 & 10.39 & 12.69 & 15.02 & 18.68 & 21.28 & 22.11\\
64842.53 & 27387.79 & 10292.87 & 4518.94 & 1.98 & 4.10 & 6.31 & 8.57 & 10.87 & 13.20 & 16.85 & 19.46 & 20.28\\
66351.92 & 30267.07 & 11374.11 & 4992.74 & 1785.76 & 2.12 & 4.32 & 6.59 & 8.89 & 11.22 & 14.87 & 17.47 & 18.30\\
67461.81 & 30970.92 & 12569.22 & 5516.52 & 1972.05 & 927.84 & 2.20 & 4.47 & 6.77 & 9.10 & 12.75 & 15.35 & 16.18\\
67806.03 & 31488.27 & 12860.73 & 6095.58 & 2178.12 & 1023.87 & 336.15 & 2.26 & 4.57 & 6.90 & 10.55 & 13.15 & 13.98\\
68007.59 & 31648.20 & 13074.79 & 6236.26 & 2406.07 & 1130.20 & 369.92 & 197.78 & 2.30 & 4.63 & 8.29 & 10.89 & 11.71\\
68079.66 & 31741.53 & 13140.39 & 6339.38 & 2460.80 & 1247.94 & 407.47 & 216.98 & 61.71 & 2.33 & 5.98 & 8.58 & 9.41\\
68116.41 & 31774.42 & 13178.33 & 6370.47 & 2500.68 & 1275.68 & 449.19 & 238.43 & 66.59 & 45.87 & 3.65 & 6.25 & 7.08\\
68130.08 & 31790.82 & 13191.16 & 6388.15 & 2512.11 & 1295.72 & 458.33 & 262.38 & 72.22 & 49.08 & 11.69 & 3.90 & 4.73\\
68135.98 & 31796.44 & 13197.14 & 6393.64 & 2518.24 & 1300.98 & 464.68 & 267.16 & 78.69 & 52.87 & 11.16 & 14.88 & 2.37\\
68137.60 & 31798.44 & 13198.65 & 6395.81 & 2519.55 & 1303.48 & 465.67 & 270.31 & 79.19 & 57.30 & 10.81 & 14.65 & 1.95
\end{array}\right]$$

It can be seen that the variability range of the mean first passage times is strongly diversified. It is relatively narrow for the mean times of the first promotion to high-discount classes as well as of moving between classes: C_9, C_{10}, C_{11}, C_{12}, C_{13}. For the rest of times the range is wide, its spread exceeds 50 years and in some cases even 67 000 years. Such diversity of the variability ranges of the mean times indicates a different level of their sensitivity to the changes in the claim frequency. The fluctuation of the claim frequency in the interval <0.1, 0.2> can result in maximal change of the mean time equal only to 0.12 (in the case of transfer from C_1 to C_2) as well as to over 67 197 years (in the case of transfer from C_{13} to C_1).

It is worthwhile paying attention to the values of the mean first passage times. It should be noted that some transfers of the policyholder with the claim frequency in the interval <0.1, 0.2> are practically impossible. It is hard to expect that the policyholder from one class will move to the other, if the lower bound on the mean time of such a transfer exceeds 50 years. Such a value is taken by approximately 40% of the elements of the matrix $\overline{\mathbf{M}}^{l}$. These are mainly lower bounds on the mean times of downgrading in the class hierarchy. It means that in the PZU system it is comparatively difficult for the policyholder to reach the class with higher premium. On the other hand, in most cases the expected times for lowering a premium are significantly shorter and hence the transfer to higher-discount classes is feasible. For instance, the mean time of the first passage from the initial class C_5 to the best one C_{13} amounts to about 11 years at best and to 18 years at worst, which is usually shorter than the whole period of having and insuring a car.

4. Conclusions

In this paper we propose the application of an ergodic Markov set-chain to the analysis of a bonus-malus system. We relax the assumption of the constant claim frequency of a policyholder, which is necessary while modelling the system with the use of a homogeneous Markov chain. It is shown that the theory of the presented chain broadens the scope of studies carried out in the framework of the classical Markov chain theory. It enables to examine the consequences of claim frequency changes within a given interval. It provides tools for determining the variability range of each step transition probabilities, stationary distribution as well as mean first passage times. Therefore, we can analyse various characteristics' sensitivity and intensity of their reaction to the changes in claim frequency. The obtained information may be crucial to insurance companies having interest not only in system evaluation but also in predicting changes in its performance caused by the factor which they can influence to a limited extent.

Appendix

The description of a Markov set-chain presented in this Appendix is based on Hartfiel's [1998] monograph.

Definition A.1

Let N^1 be a compact set of $r \times r$ stochastic matrices. Let consider Markov chains with the state space $S = \{1, 2, ..., r\}$, having all their transition matrices in N^1. A Markov set-chain is the sequence

$$N^1,\ N^2,\ N^3,\ ...,$$

where $N^k = \{\mathbf{P} : \mathbf{P} = \mathbf{P}_1 \mathbf{P}_2 \cdots \mathbf{P}_k$, where $\mathbf{P}_i \in N^1$ for all $i = 1, 2, ..., k\}$ for each *k*.

The set N^k contains all possible *k*-step transition matrices obtained provided that transition matrices at first step belong to the set N^1.

According to Definition A.1 a Markov set-chain may be treated as a nonhomogeneous Markov chain having each transition matrix in N^1. Note that its transition matrices at each step are not determined uniquely. Since the set N^1 is complex, it is closed and bounded. Therefore, in a particular case it can be defined as an interval.

<u>Definition A.2</u>

The matrix interval is an interval

$$[\mathbf{K}, \mathbf{Q}] = \{\mathbf{P} : \mathbf{K} \leq \mathbf{P} \leq \mathbf{Q}\},$$

where $\mathbf{P} = [p_{ij}]$ denotes a $r \times r$ stochastic matrix and $\mathbf{K} = [k_{ij}]$ and $\mathbf{Q} = [q_{ij}]$ are nonnegative $r \times r$ matrices such that $\mathbf{K} \leq \mathbf{Q}$.

As the interval $[\mathbf{K}, \mathbf{Q}]$ can be constructed by rows, it is useful to define also a vector interval.

<u>Definition A.3</u>

The vector interval is an interval

$$[\mathbf{k}, \mathbf{q}] = \{\mathbf{x} : \mathbf{k} \leq \mathbf{x} \leq \mathbf{q}\},$$

where $\mathbf{x} = [x_j]$ is a $1 \times r$ stochastic vector and $\mathbf{k} = [k_j]$ and $\mathbf{q} = [q_j]$ are nonnegative $1 \times r$ vectors such that $\mathbf{k} \leq \mathbf{q}$.

The important feature of the described intervals is their tightness.

<u>Definition A.4</u>

A matrix interval $[\mathbf{K}, \mathbf{Q}]$ is tight if $k_{ij} = \min_{\mathbf{P} \in [\mathbf{K}, \mathbf{Q}]} p_{ij}$ and $q_{ij} = \max_{\mathbf{P} \in [\mathbf{K}, \mathbf{Q}]} p_{ij}$ for all *i* and *j*.

A vector interval $[\mathbf{k}, \mathbf{q}]$ is tight if $k_j = \min_{\mathbf{x} \in [\mathbf{k}, \mathbf{q}]} x_j$ and $q_j = \max_{\mathbf{x} \in [\mathbf{k}, \mathbf{q}]} x_j$ for all *j*.

It is easy to show that for a tight vector interval the following conditions hold:

$$k_j + \sum_{t \neq j} q_t \geq 1 \text{ and } q_j + \sum_{t \neq j} k_t \leq 1 \text{ for all } j.$$

If N^1 is a tight interval $[\mathbf{K}, \mathbf{Q}]$ then **K** and **Q** are column tight component bounds on N^1.

Henceforth we restrict our attention merely to Markov set-chains determined by a matrix interval $[\mathbf{K}, \mathbf{Q}]$.

Markov set-chains are classified into ergodic, regular and absorbing, analogously to classical Markov chains. Taking into account the properties of most bonus-malus systems, in this paper we focus only on an ergodic Markov set-chain. So as to define this type of chains, we need first to introduce the term of an ergodic class.

The decomposition of the state space $S = \{1, 2,..., r\}$ of a Markov set-chain is based on the structure of the upper bound of the matrix interval $[\mathbf{K}, \mathbf{Q}]$. Through simultaneous permutations of rows and columns, $\mathbf{Q}$ can be put into the canonical form:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{22} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{Q}_{nn} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{Q}_{n+1,1} & \mathbf{Q}_{n+1,2} & \cdots & \mathbf{Q}_{n+1,n} & \mathbf{Q}_{n+1,n+1} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{Q}_{s1} & \mathbf{Q}_{s2} & \cdots & \mathbf{Q}_{sn} & \mathbf{Q}_{s,n+1} & \cdots & \mathbf{Q}_{s,s-1} & \mathbf{Q}_{ss} \end{bmatrix},$$

where $n \geq 1$, $\mathbf{Q}_{kk}$ is a $k \times k$ irreducible matrix for all $k = 1, 2, ..., n$ and if $t > n$ then $\mathbf{Q}_{tk} \neq \mathbf{0}$ for some $k = 1, 2, ..., t-1$. It is also assumed that all matrices in the interval $[\mathbf{K}, \mathbf{Q}]$ have undergone the same simultaneous row and column permutations. The definition of an ergodic class and state follows.

Definition A.5

Let consider a Markov set-chain determined by a matrix interval $[\mathbf{K}, \mathbf{Q}]$. Let S_i be the class of states corresponding to $\mathbf{Q}_{ii}$ and, consequently, to the interval of submatrices $[\mathbf{K}_{ii}, \mathbf{Q}_{ii}]$. Then class S_i for $i \leq n$ and each of its states are called ergodic if $\lim_{k\to\infty}[\mathbf{K}_{ii}, \mathbf{Q}_{ii}]^k$ exists and each matrix in the limit is rank 1.

While determining if the class is ergodic, it is convenient to refer to the following theorem.

Theorem A.1

Under the assumptions of Definition A.5, class S_i for $i \leq n$ is ergodic if $\mathbf{K}_{ii}$ is primitive.

Now we are in a position to present the following definition.

Definition A.6

A Markov set-chain is ergodic if it has only one class and that class is ergodic.

One of the most important properties of an ergodic Markov set-chain is its convergence.

Theorem A.2

If a Markov set-chain with the compact set of transition matrices N^1 is ergodic, then

$$\lim_{k\to\infty} N^k = N^\infty ,$$

where N^∞ is a compact set of rank one matrices.

If the set N^1 converges to the set N^∞ and $\mathbf{L}^k$ and $\mathbf{H}^k$ are lower and upper bounds on N^k respectively, then the sequences $\{\mathbf{L}^k\}_{k\geq 0}$, $\{\mathbf{H}^k\}_{k\geq 0}$ are convergent: $\lim_{k\to\infty}\mathbf{L}^k = \mathbf{L}^\infty$, $\lim_{k\to\infty}\mathbf{H}^k = \mathbf{H}^\infty$. We call $\mathbf{L}^\infty$ and $\mathbf{H}^\infty$ lower and upper limit bounds on N^1. Note that $\mathbf{L}^\infty$ and $\mathbf{H}^\infty$ are rank one matrices and their rows constitute bounds on the set of stationary probability distributions.

In order to compute bounds on sets of transition matrices at each step, Hartfiel [1991, 1998] proposed the application of the Hi-Lo method. It is an approximate iterative algorithm that consists in finding column tight component bounds on N^k on the basis of column tight component bounds on N^{k-1}, where $k = 2, 3, \ldots$ Hence, column tight component bounds on N^1 produce column tight component bounds on N^2, which give column tight component bounds on N^3, and so on.

For an ergodic Markov set-chain it is also possible to find bounds on mean first passage time, defined as

$$\overline{m}_{ij} = [\mathbf{P}_1 + 2\mathbf{P}_1\dddot{\mathbf{P}}_2^j + \ldots + n\mathbf{P}_1\dddot{\mathbf{P}}_2^j \cdots \dddot{\mathbf{P}}_n^j + \ldots]_{ij} , \qquad \text{(A.1)}$$

where $\mathbf{P}_n \in N^1$ and $\dddot{\mathbf{P}}_n^j$ is the matrix formed from $\mathbf{P}_n$ by replacing its *j*-th row by the row of 0's. The matrices of lower and upper bounds on mean first passage times are denoted by $\overline{\mathbf{M}}^l$ and $\overline{\mathbf{M}}^h$ respectively. As Hartfiel and Seneta [1994] proved, the sum in (A.1) converges and its lower and upper bound can be obtained by applying the algorithm based on the Hi-Lo method.

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