Differentiation of some functionals of multidimensional risk processes and determination of optimal reserve allocation

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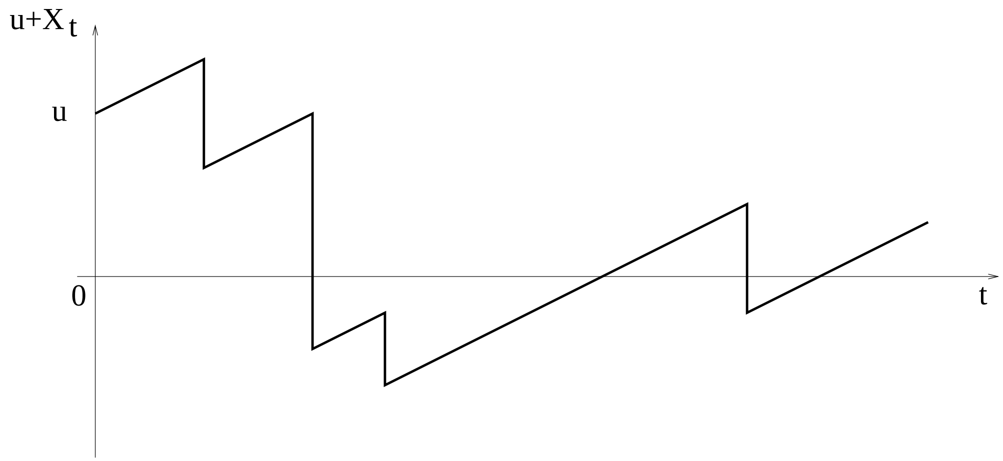
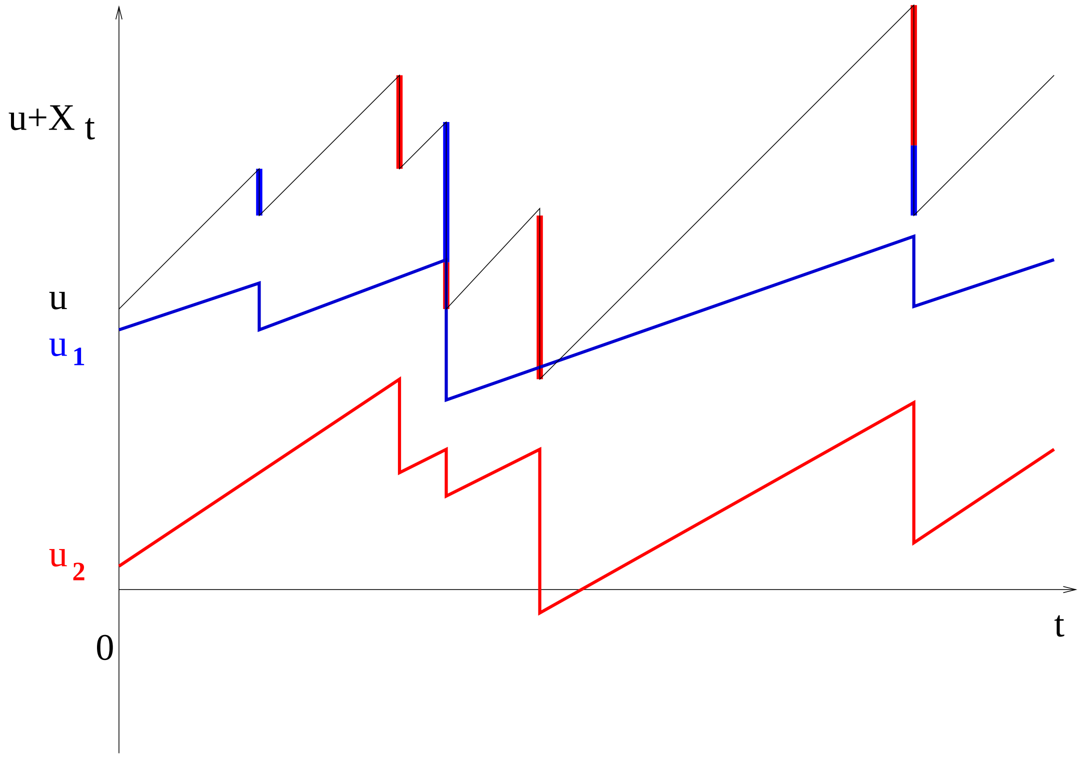


Historical model: for unidimensional risk processes $R_t = u + X_t$,

- with initial reserve u
- and with $X_t = ct - S_t$, where
  - $c > 0$ is the premium income rate,
  - $S_t = \sum_{i=1}^{N(t)} W_i$,
  - the W_i are i.i.d. nonnegative random variables, independent from $(N(t))_{t\geq 0}$,
  - with the convention that the sum is zero if $N(t) = 0$.

Probability of ruin: $\psi(u) = \mathbb{P}\left(\exists t \geq 0, \quad R_t < 0\right)$.



Two lines of business: classical, 1-dimensional surplus process (black),

2-dimensional process (1 for each line of business, blue and red).

$$u = u_1 + u_2$$

3 main directions:

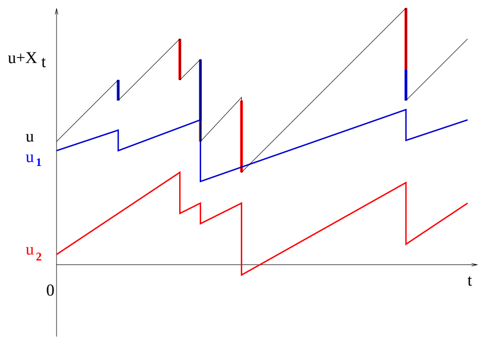


Figure 1: K-dimensional process

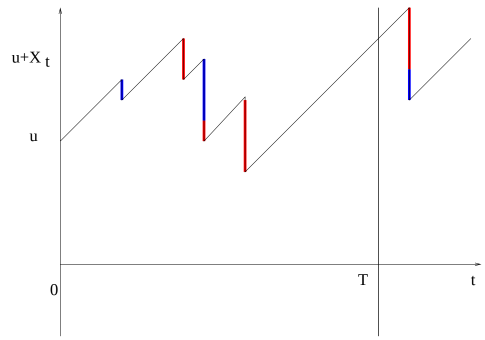


Figure 2: Finite-time.

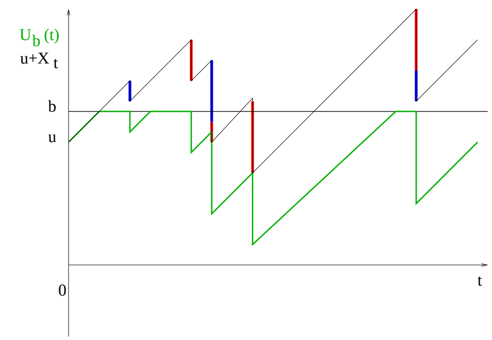


Figure 3: Dividends.

- How to model the stochastic dependence between the K lines of business ?
- Which ruin concept ?
- Within finite or infinite time ?
- Ruin or severity of ruin ?
- Optimal initial reserve allocation ?
- How to measure risk and profit (dividends)?

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- Risk and profit measures for univariate risk processes
- Differentiation of some functionals of risk processes
- A general optimal reserve allocation strategy

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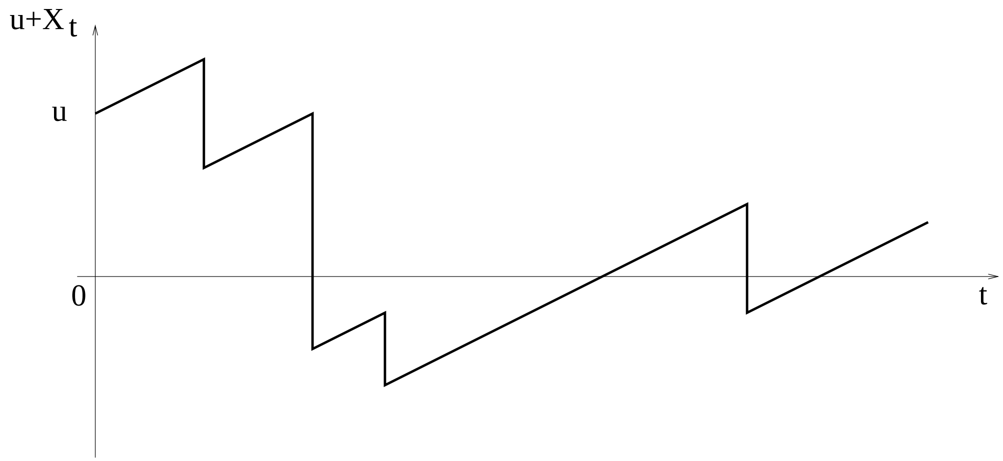
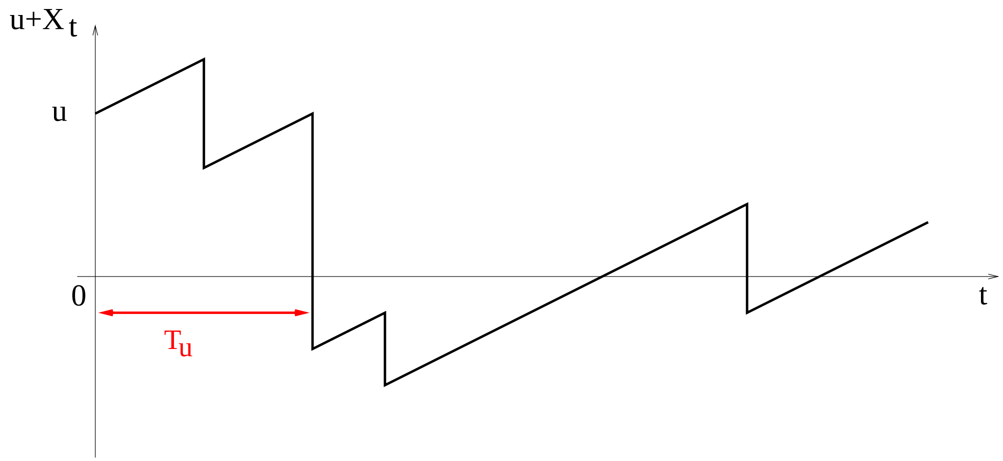


- the time to ruin $T_u = \inf\{t > 0, u + X_t < 0\}$,
- the severity of ruin $|u + X_{T_u}|$, or the couple $(T_u, |u + X_{T_u}|)$,
- the time in the red (below 0) from the first ruin to the first time of recovery $T'_u - T_u$, where

$$T'_u = \inf\{t > T_u, u + X_t = 0\},$$

- the maximal ruin severity $(\inf_{t>0} u + X_t)$,
- the aggregate severity of ruin until recovery $J(u) = \int_{T_u}^{T'_u} |u + X_t| dt$,...
- the total time in the red $\tau(u) = \int_0^{+\infty} 1_{\{u+X_t<0\}} dt$.



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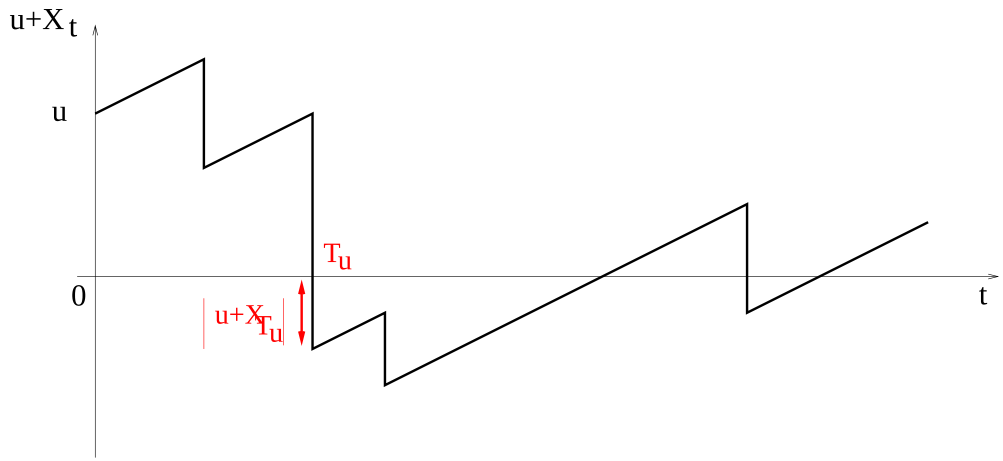

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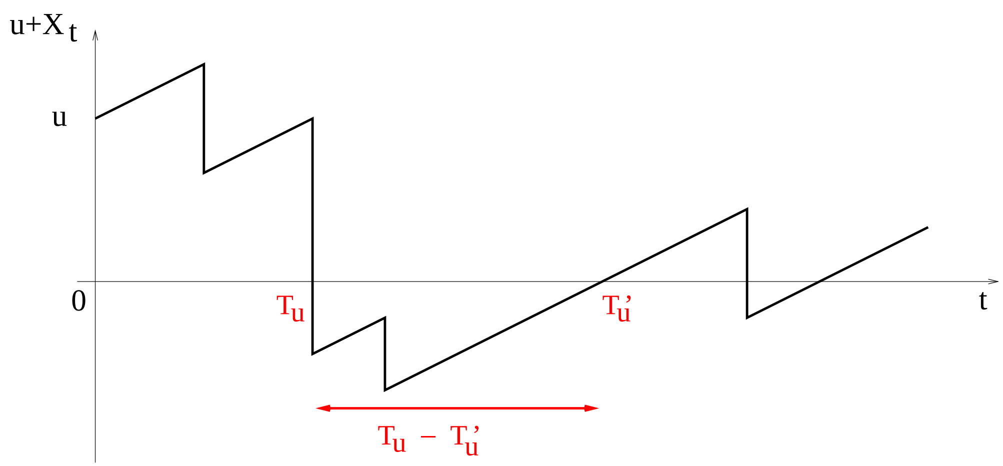

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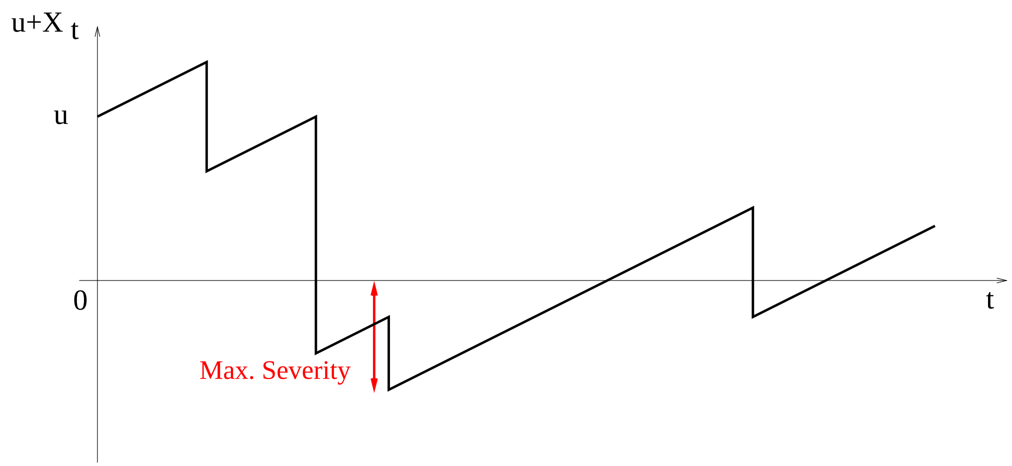

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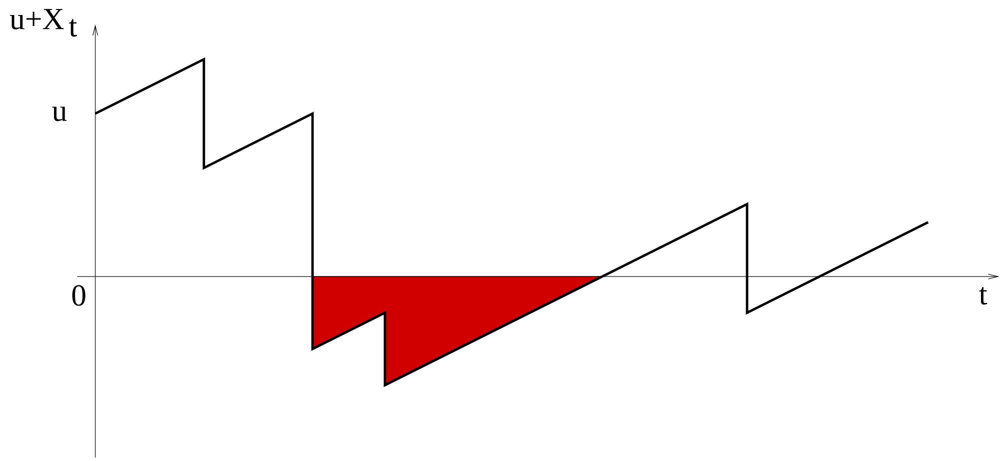

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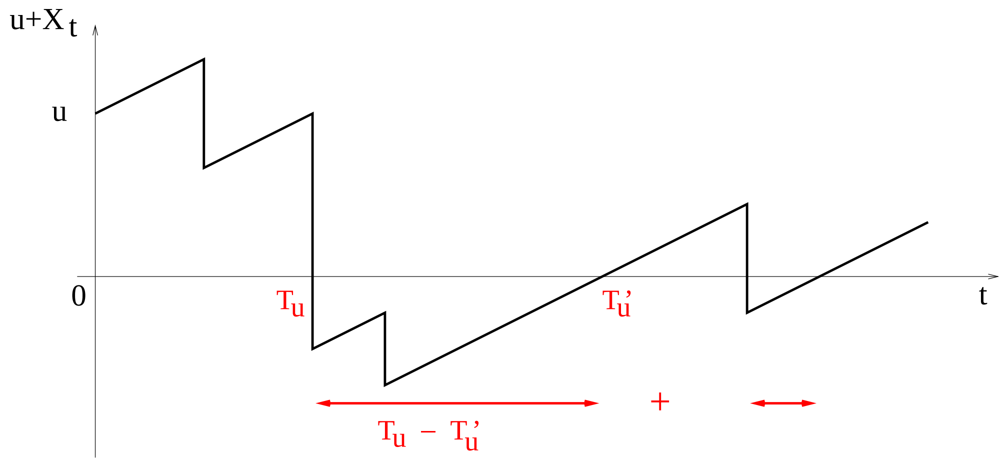

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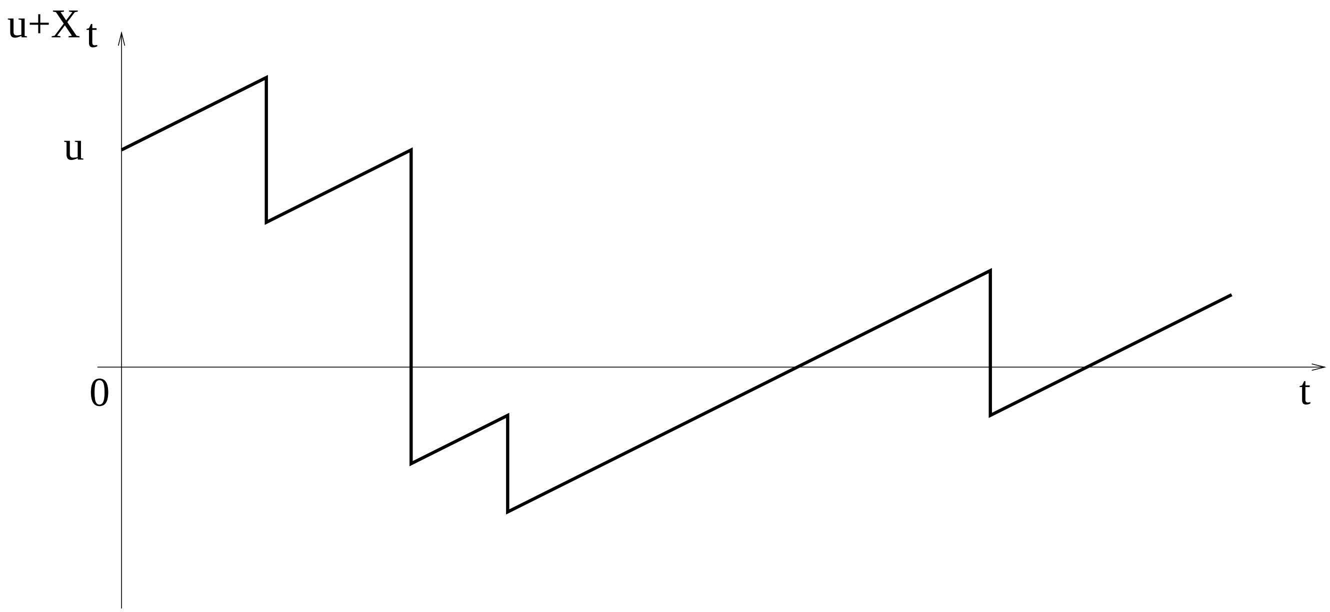
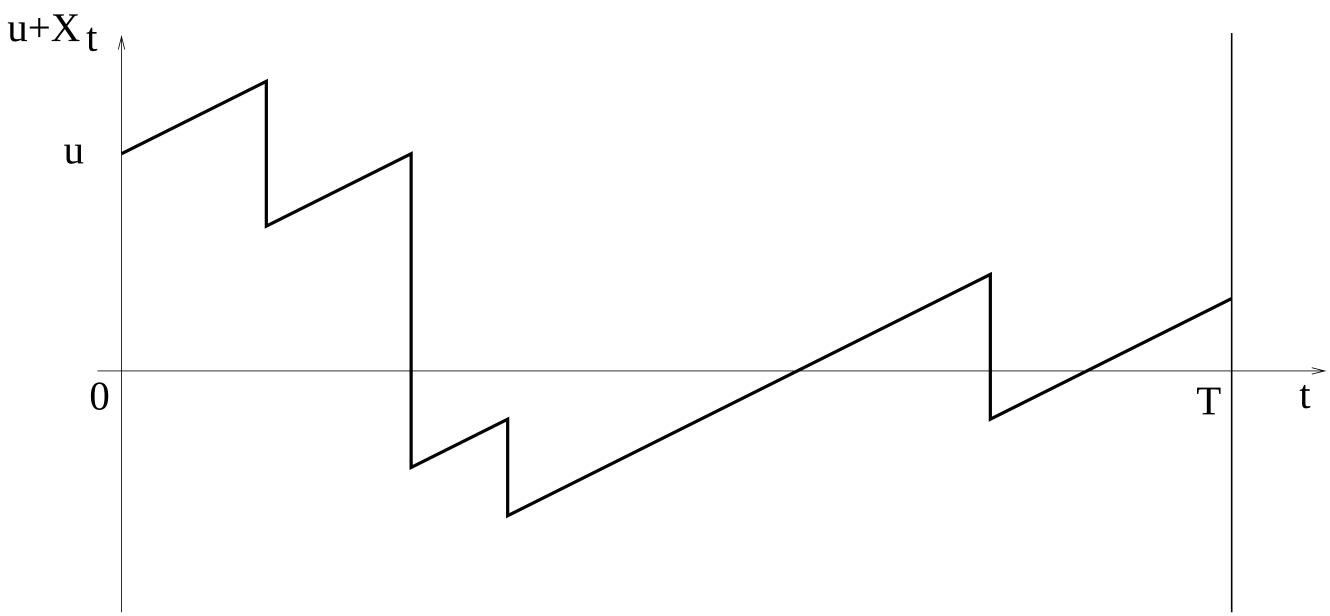


- Consider risk measures based on some fixed time interval $[0, T]$ (T may be infinite).

- Simple penalty function (expected penalty to pay due to insolvency until time horizon T) :

$$\mathbb{E}\left(I_T(u)\right) = \mathbb{E}\left(\int_0^T 1_{\{u+X_t<0\}}|u+X_t|dt\right).$$

- Note that the probability $\mathbb{P}\left(I_T(u)=0\right)$ is the probability of non ruin within finite time T.



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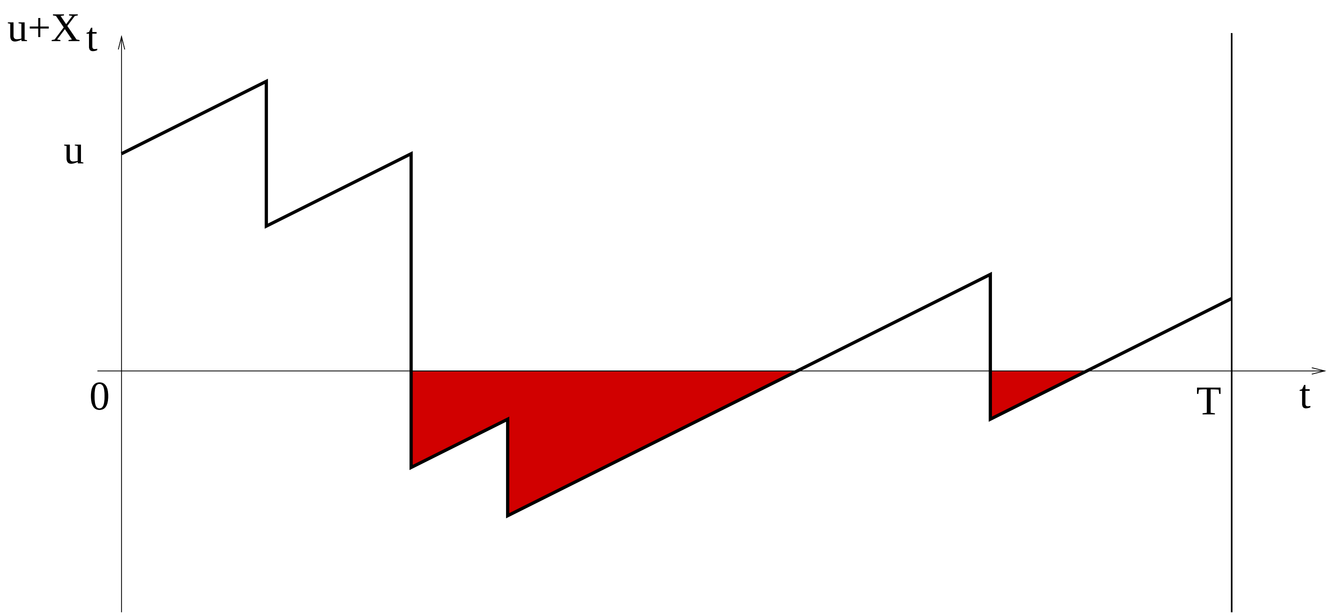

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- From an economical point of view, it seems more consistent to consider

$$\mathbb{E} I_{g,h}(u) = \mathbb{E}\left(\int_0^T \left(1_{\{u+X_t \geq 0\}} g(|u+X_t|) - 1_{\{u+X_t \leq 0\}} h(|u+X_t|)\right) dt\right)$$

  - $0 \leq g \leq h$
  - g corresponds to a reward function for positive reserves,
  - and h is a penalty function in case of insolvency.

- These risk measures may be differentiated with respect to the initial reserve u.

- Fubini's theorem.

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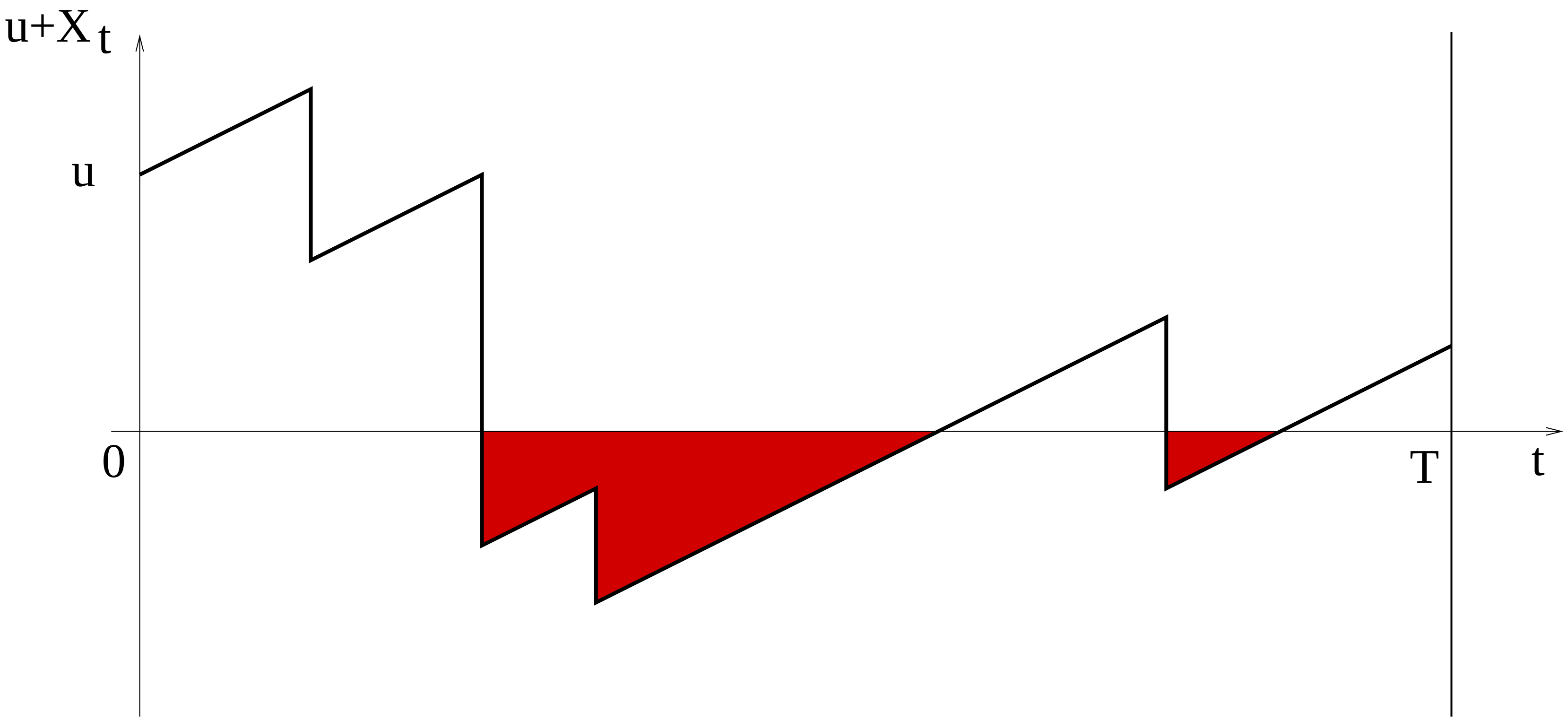

$$I(u) = \int_0^T 1_{\{u+X_t<0\}} |u + X_t| dt$$

Th. (L., 2004b):

Let $(X_t)_{t\in[0,T)}$ be a stochastic process with almost surely time-integrable sample paths. For $u \in \mathbb{R}$, denote by $\tau_0(u)$ the time spent in zero by the process $u + X_t$:

$$\tau_0(u) = \int_0^T 1_{\{u+X_t=0\}} dt.$$

Let f be defined by $f(u) = \mathbb{E}(I_T(u))$ for $u \in \mathbb{R}$, where

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- For $u \in \mathbb{R}$, if $\mathbb{E}\tau_0(u) = 0$, then f is differentiable at u, and $f'(u) = -\mathbb{E}\tau(u)$.

- Th. (L., 2004b): Let $X_t = ct - S_t$, where S_t is a jump process satisfying hypothesis (H1): S_t has a finite expected number of nonnegative jumps in every finite interval, and for each t, the distribution of S_t is absolutely continuous. For example, S_t might be a compound Poisson process with a continuous jump size distribution. Define h by $h(u) = \mathbb{E}(\tau(u))$ for $u \in \mathbb{R}$. h is differentiable on $\mathbb{R}_+^*$, and for $u > 0$,

$$h'(u) = -\frac{1}{c}\mathbb{E}N^0(u),$$

where $N^0(u) = \text{Card}\left(\{t \in [0,T], \quad u + ct - S_t = 0\}\right)$.

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- This remains valid with $T = +\infty$ if X_t has a positive drift and $\tau(u)$ is integrable. In the compound Poisson case, for $u \geq 0$,

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- $\mathbb{E}I_T(.)$ is thus well strictly convex, which will be very important for minimization.

- Theorem: In the Poisson-Exponential($1/\mu$) case, $\psi(u) = \frac{1-\mu R}{\mu R} e^{-Ru}$, with $R = \frac{1}{\mu}\left(1 - \frac{\lambda\mu}{c}\right)$. Hence, for $T = +\infty$,

$$\mathbb{E}\tau(u) = \frac{1-\mu R}{c\mu R^2} e^{-Ru}$$ Gerber, Dos Reis (1993)

and

$$\mathbb{E}I_\infty(u) = \frac{1-\mu R}{c\mu R^3} e^{-Ru}$$ L. (2004b)

- Proof: Integration of the well-known formula for $\psi(u)$. The considered functions tend to 0 as $u \longrightarrow +\infty$.

- It is possible to derive $\mathbb{E}I_\infty(u)$ explicitly for Γ and *phase-type*-distributed claim amounts.

- Th. (L., 2004b): Let g, h be two convex or concave functions in $\mathcal{C}^1(\mathbb{R}^+, \mathbb{R}^+)$, such that for $x \geq 0$, $g(x) \geq g(0)$ and $h(x) \geq h(0)$. Let X_t be a stochastic process such that $t \to g(u + X_t)$ and $t \to h(u + X_t)$ are almost surely integrable on $[0, T]$. Let I_g^+ be the function from $\mathbb{R}$ into the space of nonnegative random variables, and defined by

$$I_g^+(u) = \int_0^T 1_{\{u+X_t \geq 0\}} g(u + X_t)dt$$

for $u \geq 0$ and let $f(.) = \mathbb{E}I_g^+(.) - \mathbb{E}I_h(.)$. Define also

$$L_T(0) = \lim_{\varepsilon \downarrow 0} \left(\frac{1}{2\varepsilon} \int_0^T 1_{\{|u+X_t|<\varepsilon\}} dt \right).$$

If, for $u \in \mathbb{R}$, $\quad \mathbb{E}I_g^+(u), \quad \mathbb{E}I_h(u), \quad \mathbb{E}I_{g'}^+(u), \quad \mathbb{E}I_{h'}(u) \quad < +\infty$,

and if $\mathbb{E}\tau_0(u) = 0$, then f is differentiable on $\mathbb{R}_*^+$, and for $u > 0$,

$$f'(u) = \mathbb{E}I_{g'}^+(u) - \mathbb{E}I_{h'}(u) - (g(0) + h(0))\mathbb{E}L_T(0)$$

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- What has to be minimized is

$$A(u_1, \dots, u_K) = \sum_{k=1}^{K} \mathbb{E} I_T^k(u_k)$$

under the constraint $u_1 + \cdots + u_K = u$, where

$$\mathbb{E} I_T^k(u_k) = \mathbb{E}\left[\int_0^T |R_t^k| 1_{\{R_t^k < 0\}} dt\right]$$

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- This does not depend on the dependence structure.

- From previous differentiation theorems, A is strictly convex. On the compact space

$$\mathcal{S} = \{(u_1, \ldots, u_K) \in (\mathbb{R}^+)^K, \quad u_1 + \cdots + u_K = u\},$$

A admits a unique minimum.

- Lagrange multipliers ➔optimal allocation:

  there is a subset $J \subset [1, K]$ such that

  - for $j \notin J$, $u_j = 0$,
  - and for $j, k \in J$, $\mathbb{E}\tau_j = \mathbb{E}\tau_k$.

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- In the Poisson-Exponential($\frac{1}{\mu}$) case, recall that

$$\mathbb{E}I_u = \frac{1 - \mu R}{c\mu R^3} e^{-Ru}.$$

  Consider a two-line model, with the following parameters:

  $\mu_1 = \mu_2 = 1$, $c_1 = c_2 = 1$, $R_2 = 0.4$ and $u = 10$.

  Three values of R_1 ➔different optimal allocation strategies.

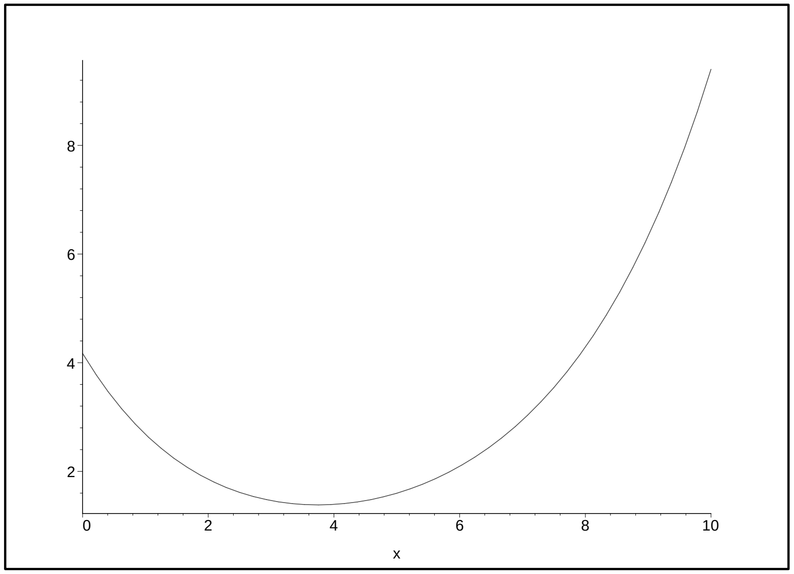

Figure 4: Graph of $A(x, 10-x)$.

When $R_1 = 0.5 > R_2$,

line of business 1 is safer than line 2.

➔ $u_1 < u_2$.

The optimal allocation is about

$(u_1 = 3.5, \quad u_2 = 6.5.)$

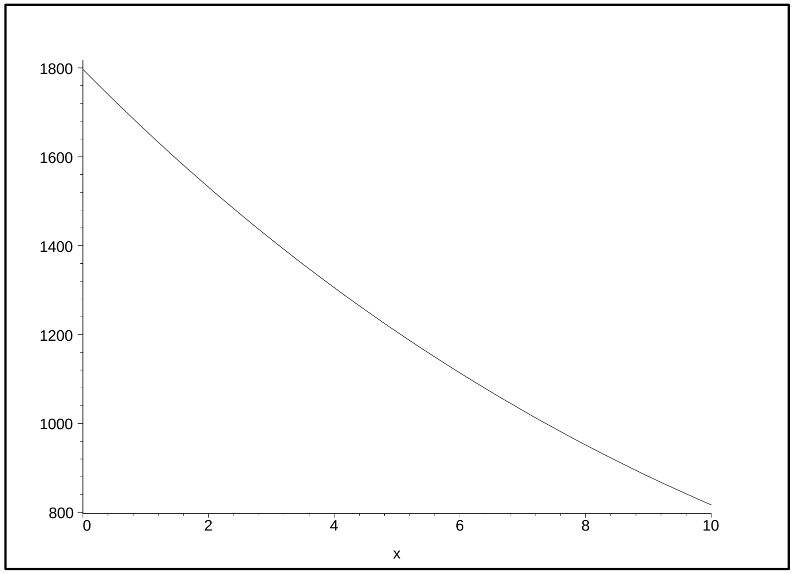

Figure 5: Graph of $A(x, 10-x)$.

When $R_1 = 0.08 < R_2$,

line of business 1 is much riskier than line 2.

➔ $u_1 = u = 10$ and $u_2 = 0$.

Transfer of the whole reserve to line 1.

- Consider the sum

$$B = \sum_{k=1}^{K} \mathbb{E}\tau'_k(u_1, \ldots, u_K)$$

where

$$\mathbb{E}\tau'_k(u_1, \ldots, u_K) = \mathbb{E}\left(\int_0^T 1_{\{R_t^k<0\}} 1_{\{\sum_{j=1}^K R_t^j>0\}} dt\right).$$

B takes dependence into account, and the following proposition prescribes to do the same kind of reasoning:

- Proposition: Let $X_t = ct - S_t$, where S_t satisfies hypothesis (H1). Define B by $B(u_1, \ldots, u_K) = \sum_{k=1}^n \mathbb{E}(\tau'_k(u_1, \ldots, u_K))$ for $u \in \mathbb{R}^K$. B is differentiable on $(\mathbb{R}_+^*)^K$, and for $u_1, \ldots, u_K > 0$,

$$\frac{\partial B}{\partial u_k} = -\frac{1}{c_k}\mathbb{E}N_k^0(u_1, \ldots, u_K),$$

where $N_k^0(u_1, \ldots, u_K) = \text{Card}\left(\{t \in [0,T], \quad (R_t^k = 0) \cap \left(\sum_{j=1}^K R_t^j > 0\right)\}\right)$.

- It is also possible to differentiate with respect to c instead of u.
- Theorem: With the previous notation, consider the case $X_t = ct - S_t$, where S_t satisfies hypothesis (H1), and define $\tilde{f}(c) = \mathbb{E}(I(c))$.
  If for all c, $\mathbb{E}\tau_0(c) = 0$, then $\tilde{f}$ is differentiable on $\mathbb{R}$ and for $c \in \mathbb{R}$,

$$\tilde{f}'(c) = -\int_0^T tP(R_t < 0)dt.$$

- It is interesting to look for the optimal allocation of the global premium $c = c_1 + \cdots + c_n$ because if c_i is small enough to make the safety loading negative, the process R_t^i tends to $-\infty$. Quite often, optimizing with the c_i will be easier than with the u_i for this reason.
- These examples illustrate how these differentiation results may be used.
  The differentiation developed here is quite general and may be useful to solve many problems involving multi-risks models.