The Compound Poisson Risk Model with a Threshold Dividend Strategy

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36^{th} ASTIN Colloquium
September 4-7, 2005

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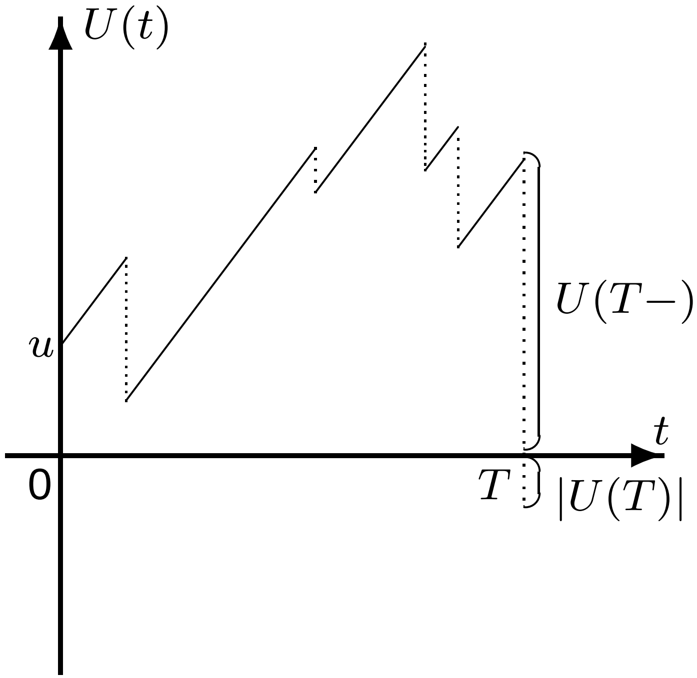
- $\{S(t) = \sum_{i=1}^{N(t)} Y_i;\, t \geq 0\}$ - aggregate-claim process

Introduction



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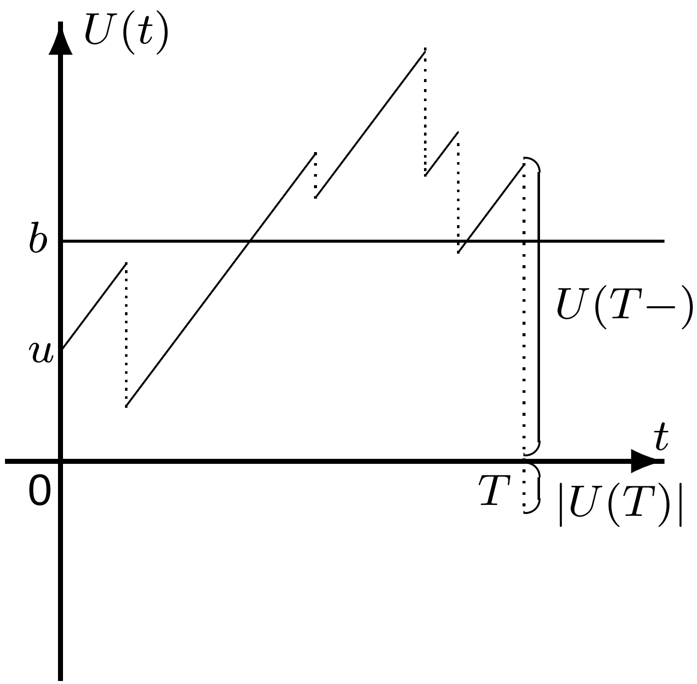

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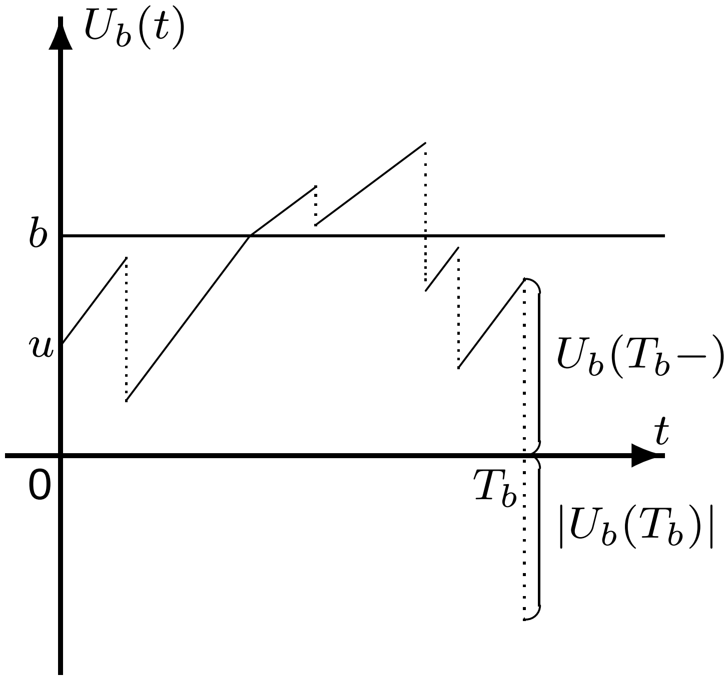

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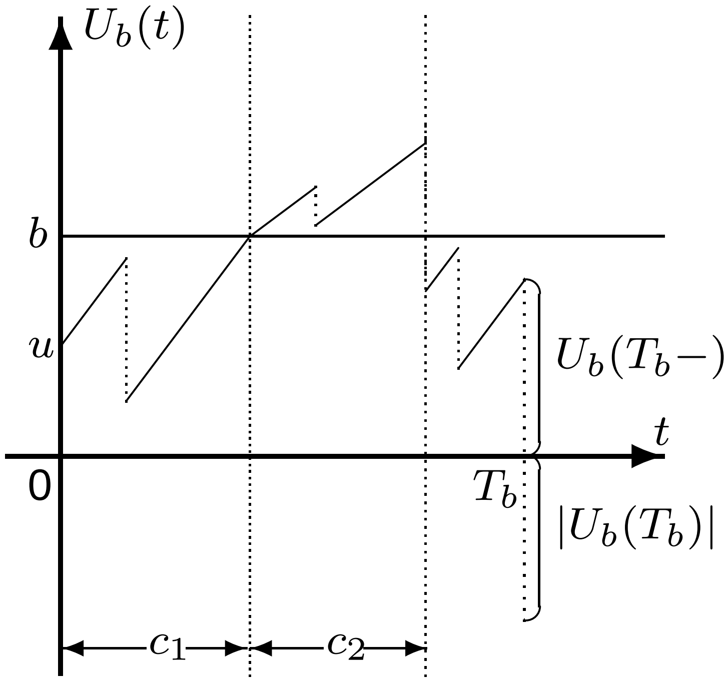

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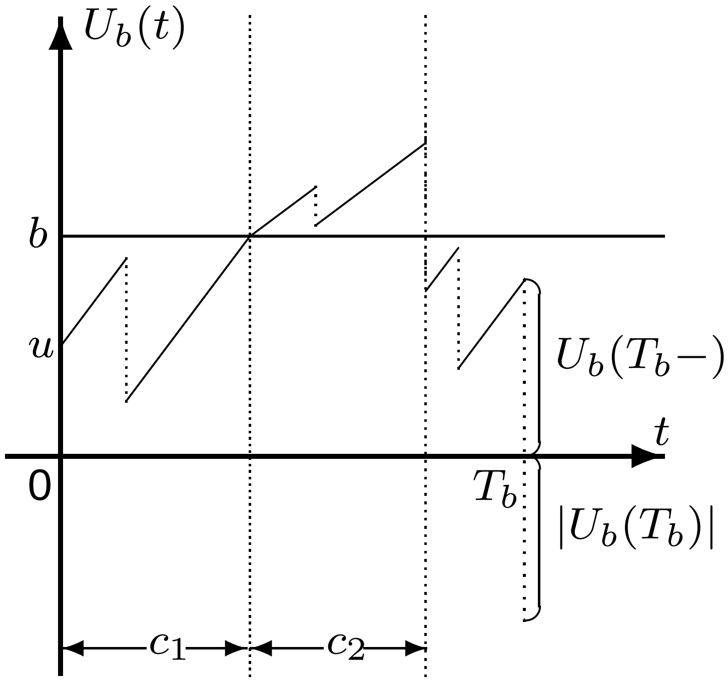

- $dU_b(t) = \begin{cases} c_1 dt - dS(t), & U_b(t) \le b \\ c_2 dt - dS(t), & U_b(t) > b \end{cases}$ – insurer's surplus at time t

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Problem description

Similarly to the classical compound Poisson model, we define the Gerber-Shiu discounted penalty function by

$$m(u;b) = \mathbb{E}\left\{e^{-\delta T_b} w(U_b(T_b-), |U_b(T_b)|) I(T_b < \infty) | U_b(0) = u\right\}$$

where $\delta \geq 0$ is the force of interest and I is the indicator function.

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where $\delta \geq 0$ is the force of interest and I is the indicator function.
Then, for notational clarity, we set

$$m(u;b) = \begin{cases} m_1(u), & 0 \leq u \leq b \\ m_2(u), & u > b. \end{cases}$$

Problem description

We demonstrate that the discounted penalty function satisfies

$$m'(u;b) = \frac{\lambda+\delta}{c_i} m(u;b) - \frac{\lambda}{c_i}\int_0^u m(u-y;b)dP(y) - \frac{\lambda}{c_i}\zeta(u),$$

where $i = 1$ or 2 depending on the interval for u and $\zeta(u) = \int_u^\infty w(u, y-u)dP(y)$.

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$$\int_0^{u-b} m_2(u-y)dP(y) + \int_{u-b}^{u} m_1(u-y)dP(y)$$

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where $i = 1$ or 2 depending on the interval for u and $\zeta(u) = \int_u^\infty w(u, y-u)\,dP(y)$.
We are interested in finding an analytic solution to the above equation.

Analytic solution

To do so, we fist need to secure an initial condition. As such may serve the continuity of m. More specifically, it may be shown that $m_1(b) = m_2(b) := \lim_{u \longrightarrow b^+} m_2(u)$.

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The integro-differential equation for m_2 yields

$$m_2(u) = \pi_2 \left[\int_0^{u-b} m_2(u-y) dA_2(y) + \int_{u-b}^{u} m_1(u-y) dA_2(y) \right] + \frac{\lambda}{c_2} T_{\rho_2} \zeta(u), \quad u > b,$$

where $T_{\rho_2}\zeta(u) = \int_0^\infty e^{-\rho_2 y} \zeta(u+y) dy$.

Analytic solution

Therefore, Theorem 2.1 in Lin and Willmot (1999) yields

$$m_2(u) = \frac{1}{1-\pi_2}\int_0^{u-b} h(u-y)dK_2(y) + h(u), \qquad u > b,$$

where h is a function depending on m_1 and K_2 is a compound geometric c.d.f.

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and a solution of the nonhomogeneous equation

$$m'_\infty(u) = \frac{\lambda + \delta}{c_1} m_\infty(u) - \frac{\lambda}{c_1} \int_0^u m_\infty(u-y) dP(y) - \frac{\lambda}{c_1} \zeta(u).$$

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For more detail about the approach, see Lin et al. (2003).

Analytic solution

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where Ψ is a compound geometric tale and Gerber and Shiu (1998) find that the solution in the classical compound Poisson case satisfies

$$m_\infty(u) = \pi_1 \int_0^u m_\infty(u - y) dA_1(y) + \frac{\lambda}{c_1} T_{\rho_1} \zeta(u), \qquad u \geq 0,$$

which by Theorem 2.1 in Lin and Willmot (1999) may be analyticaly expressed as

$$m_\infty(u) = \frac{\lambda}{c_1(1-\pi_1)} \int_0^u T_{\rho_1}\zeta(u-y)dK_1(y) + \frac{\lambda}{c_1} T_{\rho_1}\zeta(u),$$

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where K_1 is a compound geometric c.d.f.
Finally, the solution of the integro-differential equation for m_1 is of the form

$$m_1(u) = m_\infty(u) + \kappa v(u), \quad u \geq 0,$$

where κ is a constant to be determined through the initial condition.

Applications

The probability of ultimate ruin $\psi(u;b)$ equals

$$\psi_1(u) = 1 - q(b) + q(b)\psi_{1,\infty}(u), \qquad 0 \le u \le b$$

$$\psi_2(u) = -\frac{1+\theta_2}{\theta_2}\int_0^{u-b} h(u-y)\,d\psi_{2,\infty}(y) + h(u), \quad u > b,$$

where $q(b) \in [0,1]$ is a constant, $\psi_{i,\infty}$, $i = 1, 2$, is the probability of ultimate ruin under the classical compound Poisson model with premium rate c_i, h is a function depending on ψ_1, and θ_2 is the relative security loading when the surplus is above the barrier b.

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A related result is obtained in Asmussen (2000).

Applications

The Laplace transform of the time of ruin $\mathcal{L}(u;b)$ equals

$$\mathcal{L}_1(u) = \mathcal{L}_\infty(u) + \kappa \, \frac{1-\Psi(u)}{1-\Psi(0)} \, e^{\rho_1 u}, \qquad 0 \le u \le b$$

$$\mathcal{L}_2(u) = \frac{1}{1-\pi_2} \int_0^{u-b} h(u-y) dK_2(y) + h(u), \; u > b,$$

where κ is known and h is a function depending on $\mathcal{L}_1$.

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where κ is known and h is a function depending on $\mathcal{L}_1$. Both the joint and the marginal defective distributions, along with the moments, of the surplus before ruin and the deficit at ruin may be expressed analytically.

Examples

If the claim amounts have exponential distribution
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- the probability of ultimate ruin $\psi(u;b)$ becomes

$$\psi_1(u) = 1 - q(b) + \frac{q(b)}{1+\theta_1} e^{-\beta_1 u}, \qquad 0 \leq u \leq b$$

$$\psi_2(u) = \frac{1}{1+\theta_2} \left[1 - q(b) + q(b) e^{-\beta_1 b} \right] e^{-\beta_2 (u-b)},\ u > b,$$

where $q(b) \in [0,1]$ is a known constant and $\beta_i = \frac{\theta_i}{1+\theta_i}\mu,\ i = 1, 2.$

Examples

- the Laplace transform of the time of ruin $\mathcal{L}(u; b)$ becomes

$$\mathcal{L}_1(u) = [1 - r(b)]e^{\rho_1 u} + \pi_1 r(b) e^{-\tau_1 u}, \; 0 \le u \le b$$

$$\mathcal{L}_2(u) = \pi_2 \left\{ \frac{\rho_1}{\mu + \rho_1} [1 - r(b)] e^{-\mu b} + \frac{\mu}{\mu + \rho_1} [1 - r(b)] e^{\rho_1 b} + r(b) e^{-\tau_1 b} \right\} e^{-\tau_2 (u - b)}, \quad u > b,$$

where $r(b)$ is a known constant.

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