The Compound Poisson Risk Model with a Threshold Dividend Strategy

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Abstract

In this paper we discuss a threshold dividend strategy implemented into the classical compound Poisson model. More specifically, we assume that no dividends are paid if the current surplus of the insurance company is below certain threshold level. When the surplus is above this fixed level, dividends are paid at a constant rate that does not exceed the premium rate. This model may also be viewed as the compound Poisson model with the two-step premium rate. Two integro-differential equations for the Gerber-Shiu discounted penalty function are derived and solved. When the initial surplus is below the threshold, the solution is a linear combination of the Gerber-Shiu function with no barrier and the solution of the associated homogeneous integro-differential equation. This latter function is proportional to the product of an exponential function and a compound geometric distribution function. When the initial surplus is above the threshold, the solution involves the respective Gerber-Shiu function with initial surplus lower than the threshold level. These analytic results are utilized to find the probability of ultimate ruin, the time of ruin, the distribution of the first surplus drop below the initial level, and the joint distributions and moments of the surplus immediately before ruin and the deficit at ruin. The special cases where the claim size distribution is exponential and a combination of exponentials are considered in some detail.

Keywords:

compound Poisson model, deficit at ruin, Gerber-Shiu discounted penalty function, surplus immediately before ruin, time of ruin, threshold strategy, two-step premium

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1 Introduction

The classical compound Poisson risk model without constraints and related problems have been studied extensively. See Bowers et al. (1997, Chapter 13), Gerber (1979), Klugman et al. (2004), Panjer and Willmot (1992) and references therein. Recently, Gerber and Shiu (1998) introduced a discounted penalty function with respect to the time of ruin, the surplus immediately before ruin and the deficit at ruin to analyze these and related quantities in a unified manner. The Gerber-Shiu discounted penalty function has proven to be a powerful analytical tool. Various new results on the compound Poisson risk model have been obtained since. See Gerber and Shiu (1998) and Lin and Willmot (1999, 2000). The function has also been employed in Spare Andersen risk models (see Li and Garrido (2004a), Gerber and Shiu (2005b) and references therein) and the compound Poisson risk model with a dividend strategy, which will be discussed below.

Dividend strategies for insurance risk models were first proposed by De Finetti (1957) to reflect more realistically the surplus cash flows in an insurance portfolio. Barrier strategies for the compound Poisson risk model have been studied in a number of papers and books, including Albrecher et al. (2005), Albrecher and Kainhofer (2002), Bühlmann (1970, Section 6.4.9), Dickson and Waters (2004), Gerber (1972), Gerber (1973), Gerber (1979), Gerber (1981), Gerber and Shiu (1998), Gerber and Shiu (2005a), Højgaard (2002), Lin et al. (2003), Paulsen and Gjessing (1997), and Segerdahl (1970).

Two surplus-dependent dividend strategies are of particular interest. The first is the constant barrier strategy under which no dividend is paid when the surplus is below a constant barrier but the total of the surplus above the barrier is paid out as dividends. It is shown in Gerber (1969) that this strategy is optimal for the compound Poisson risk model when the initial surplus is below the barrier. The other is the so-called threshold strategy under which no dividends are paid when the surplus is below a constant barrier and dividends are paid at a rate less than the premium rate when the surplus is above the barrier. Obviously, the former strategy can be considered as a special case of the latter. Gerber and Shiu (2005a) show that the threshold strategy is optimal when the dividend rate is bounded from above and the individual claim distribution is exponential.

In Lin et al. (2003), the constant barrier strategy for the compound Poisson risk model is studied thoroughly. It is shown there that the Gerber-Shiu discounted penalty function can be expressed as a combination of two functions: the Gerber-Shiu discounted penalty function for the compound Poisson model without a barrier and an explicit function of the tail of a particular compound geometric distribution. Results regarding the time of ruin and related quantities are derived. These results are generalized further by Li and Garrido (2004b).

In this paper we consider the Gerber-Shiu discounted penalty function for the compound

Poisson risk model under the threshold strategy. We thus provide a generalization of the model and results in Lin et al. (2003). To introduce the model formally, let $\{Y_1, Y_2, ...\}$ be independent and identically distributed (i.i.d.) positive random variables representing the successive individual claim amounts. These random variables are assumed to have common cumulative distribution function (c.d.f.) $P(y) = 1 - \overline{P}(y)$, $y \geq 0$, with probability density function (p.d.f.) $p(y) = P'(y)$ and Laplace transform $\tilde{p}(s) = \int_0^\infty e^{-sy} dP(y)$. Furthermore, let the total number of claims up to time t, denoted by $N(t)$ and independent of $\{Y_1, Y_2, ...\}$, be a Poisson process with parameter $\lambda > 0$. Consequently, the respective i.i.d. interclaim-time random variables $\{V_1, V_2, ...\}$, independent of $\{Y_1, Y_2, ...\}$, have an exponential distribution with mean $1/\lambda$. The aggregate claims process is defined by $\{S(t) = \sum_{i=1}^{N(t)} Y_i;\, t \geq 0\}$ where $S(t) = 0$ if $N(t) = 0$.

For an insurer's surplus process under the threshold strategy, denote $u \geq 0$ to be the initial surplus, $b > 0$ the constant barrier level, and $c_1 > 0$ the annual premium rate. As usual, we write $c_1 = (1 + \theta_1)\lambda\mathbb{E}\{Y_1\}$, where $\theta_1 > 0$ is the relative security loading. Let $\alpha, 0 < \alpha \leq c_1$, be the annual dividend rate, i.e., when the surplus is above the barrier b, dividends are paid at rate α. In this case, the net premium rate after dividend payments is $c_2 = c_1 - \alpha \geq 0$. Similarly, we set $c_2 = (1 + \theta_2)\lambda\mathbb{E}\{Y_1\}$ and note that the relative security loading θ_2 may be non-positive. The surplus process $\{U_b(t);\, t \geq 0\}$ can be expressed as

$$dU_b(t) = \begin{cases} c_1 dt - dS(t), & U_b(t) \leq b \\ c_2 dt - dS(t), & U_b(t) > b, \end{cases}$$

and the time of (ultimate) ruin is defined as $T_b = \inf\{t \,|\, U_b(t) < 0\}$ where $T_b = \infty$ if ruin does not occur in finite time. See Figure 1 for a graphical representation of a sample path of the surplus process.

Several remarks are now made. If $b = \infty$, the threshold risk model reduces to the classical model without constraints. As discussed in Lin et al. (2003), although the classical Poisson model may be viewed as a limiting case of the present risk model, the results obtained in this paper are not applicable. If $\alpha = c_1$, the threshold model coincides with the compound Poisson model under the constant barrier strategy, studied in more detail by Lin et al. (2003). Finally, this risk model may be interpreted as a compound Poisson model with a two-step premium rate as discussed in Asmussen (2000, Chapter VII, Section 1) and Zhou (2004).

We now introduce the Gerber-Shiu discounted penalty function

$$m(u; b) = \mathbb{E}\{e^{-\delta T_b} w(U_b(T_b-), |U_b(T_b)|) I(T_b < \infty) | U_b(0) = u\},$$

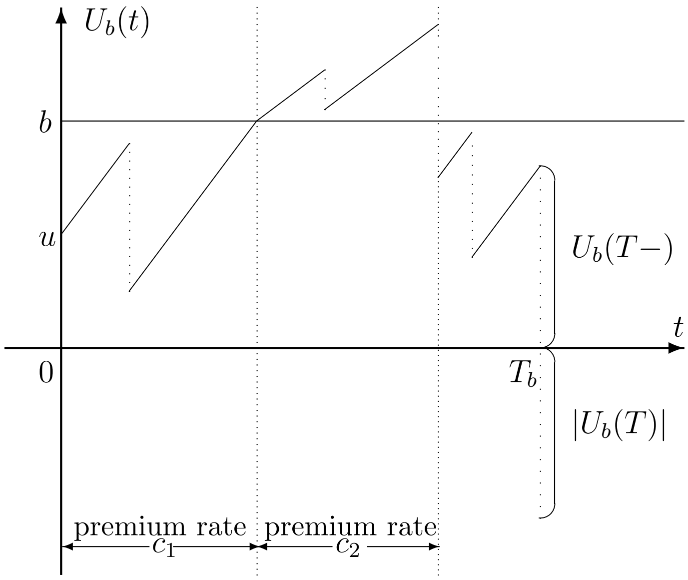

Figure 1: Graphical representation of the surplus process $U_b(t)$

where $\delta \geq 0$ is interpreted as the force of interest or the variable of a Laplace transform, $w(x_1, x_2), x_1 \geq 0, x_2 \geq 0$, is a nonnegative function of the surplus immediately before ruin, x_1, and the deficit at ruin, x_2, and $I(E)$ is the indicator function of an event E.

A number of particular cases of the discounted penalty function lead to important quantities of interest in risk theory. For instance, setting $w(x_1, x_2) = 1$ for all $x_1, x_2 \geq 0$, we obtain the Laplace transform of the time of ruin. When $\delta = 0$ and $w(x_1, x_2) = 1$ for all $x_1, x_2 \geq 0$, the discounted penalty function reduces to the probability of ultimate ruin. Another example is the case $\delta = 0$ and $w(x_1, x_2) = x_1^h x_2^l$ with h and l being nonnegative integers. The discounted penalty function produces then the joint moments of the surplus immediately before ruin and the deficit at ruin.

The aim of this paper is to study the Gerber-Shiu discounted penalty function and its special cases under the threshold strategy. So far, only limited number of results have been obtained. Asmussen (2000, Chapter VII, Section 1) discussed the probability of ruin, and more recently Gerber and Shiu (2005a) provided two intergo-differential equations satisfied by the Laplace transform of the time of ruin. Zhou (2004) derived some analytical expressions related to the joint distribution of the maximal surplus before ruin, the surplus immediately before ruin and the deficit at ruin. In this paper, we derive two integro-differential equations and then the general solution for the discounted penalty function. Utilizing the analytical expression of the solution, we obtain results on the probability of ruin, the time of ruin, and the joint distribution and moments of the surplus immediate before ruin and the deficit at ruin.

The rest of the paper is organized as follows. Section 2 recalls properties of the translation operator, first employed for risk processes by Dickson and Hipp (2001) and recently discussed by Li and Garrido (2004a) and Gerber and Shiu (2005b). A concise review of some results obtained in Lin et al. (2003) is also recalled. In particular, we recall the general solution of an integro-differential equation, which is satisfied by the Gerber-Shiu discounted penalty function under the compound Poisson risk model with a constant dividend barrier strategy. In Section 3, we derive two integro-differential equations for the discounted penalty function under the compound Poisson risk model with a threshold strategy. In Section 4 we obtain a renewal equation satisfied by the discounted penalty function with initial surplus above the threshold level via the translation operator. Analytical solutions of the two integro-differential equations are presented in Section 5. In Sections 6 to 8, we apply the analytical solutions to the probability of ultimate ruin, the Laplace transform of the time of ruin T_b, and the joint distribution and moments of the the surplus before ruin $U_b(T_b-)$, and the deficit at ruin $|U_b(T_b)|$. These applications are also illustrated by examples where the claim size distribution is specified to be exponential or a combination of exponential distributions.

2 Preliminaries

In this section, we first introduce the translation operator T_s and some of its properties. We then discuss the general solution of an integro-differential equation that is satisfied by the Gerber-Shiu discounted penalty function under the compound Poisson risk model with and without a dividend barrier.

Let f be a real-valued (Riemannian) integrable function and s be a nonnegative real number (or a complex number with nonnegative real part). The translation operator T_s on a function f is defined by

$$T_s f(x) = \int_x^\infty e^{-s(y-x)} f(y) dy.$$

The operator T_s is commutative, i.e. $T_r\, T_s = T_s\, T_r$. Moreover,

$$T_s T_r f(x) = T_r T_s f(x) = \frac{T_s f(x) - T_r f(x)}{r - s}, \quad s \neq r. \tag{2.1}$$

For more properties of the translation operator, see Gerber and Shiu (2005b).

As it will be seen in the next two sections, the discounted penalty function under the threshold strategy is closely related to the discounted penalty function under the dividend strategy. We thus need then to employ some of the notation and results obtained in Lin et al. (2003). For convenience, we recall these results and notation here.

Consider the integro-differential equation

$$m'(u) = \frac{\lambda+\delta}{c_1} m(u) - \frac{\lambda}{c_1} \int_0^u m(u-y) dP(y) - \frac{\lambda}{c_1} \zeta(u), \quad u \geq 0, \tag{2.2}$$

where

$$\zeta(t) = \int_t^\infty w(t, y-t) dP(y). \tag{2.3}$$

It is shown in Lin et al. (2003) that the general solution of (2.2) can be expressed as

$$m(u) = m_\infty(u) + \kappa v(u), \quad u \geq 0, \tag{2.4}$$

where κ is an arbitrary constant, m_∞ is the discounted penalty function under the classical compound Poisson risk model with premium rate c_1, and the function v satisfies the integro-differential equation

$$v'(u) = \frac{\lambda+\delta}{c_1} v(u) - \frac{\lambda}{c_1} \int_0^u v(u-y) dP(y), \quad u \geq 0.$$

Furthermore, assume that the surplus process has a positive relative security loading. For $i = 1$ and 2, let ρ_i be the nonnegative root of the *Lundberg fundamental equation*

$$c_i s + \lambda \tilde{p}(s) - (\lambda + \delta) = 0. \tag{2.5}$$

It is then known (see Bühlmann (1970, Section 6.4.9)) that

$$v(u) = \frac{1 - \Psi(u)}{1 - \Psi(0)} e^{\rho_1 u}, \quad u \geq 0, \tag{2.6}$$

where Ψ is the solution of the equation

$$\Psi'(u) = \frac{\lambda}{\hat{c_1}} \int_0^u [1 - \Psi(u-y)] d\hat{P}(y) - \frac{\lambda}{\hat{c_1}} [1 - \Psi(u)], \quad u \geq 0,$$

with

$$\hat{c_1} = \frac{c_1}{\tilde{p}(\rho_1)},$$

and the c.d.f. $\hat{P}$ satisfying

$$d\hat{P}(y) = \frac{e^{-\rho_1 y} dP(y)}{\tilde{p}(\rho_1)}.$$

Therefore, Ψ is the tail of a compound geometric distribution with geometric parameter

$$\Psi(0) = \frac{\lambda}{\hat{c}_1} \int_0^\infty t \, d\hat{P}(t),$$

and claim-size distribution, which has p.d.f. $\overline{\hat{P}}(y)/\int_0^\infty t \, d\hat{P}(t)$, $y \geq 0$.

For $i = 1$ and 2, let A_i be the following c.d.f. defined through its tail

$$\overline{A}_i(y) = 1 - A_i(y) = \frac{\int_y^\infty e^{-\rho_i(t-y)}\overline{P}(t)dt}{\int_0^\infty e^{-\rho_i t}\overline{P}(t)dt} = \frac{T_{\rho_i}\overline{P}(y)}{T_{\rho_i}\overline{P}(0)}, \quad y \geq 0. \tag{2.7}$$

Obviously, it has the Laplace transform

$$\tilde{a}_i(s) = \int_0^\infty e^{-sy} dA_i(y) = \frac{\rho_i}{1-\tilde{p}(\rho_i)} \cdot \frac{\tilde{p}(s)-\tilde{p}(\rho_i)}{\rho_i - s}. \tag{2.8}$$

Let also define the parameter

$$\pi_i = \frac{\lambda[1-\tilde{p}(\rho_i)]}{c_i\rho_i} = \frac{\lambda}{c_i} T_{\rho_i}\overline{P}(0), \tag{2.9}$$

and observe that $0 < \pi_i < 1$ and when $\rho \to 0$, $\pi_i \to 1/(1+\theta_i)$. Then m_∞ satisfies the defective renewal equation

$$m_\infty(u) = \pi_1 \int_0^u m_\infty(u-y)dA_1(y) + \frac{\lambda}{c_1} T_{\rho_1}\zeta(u), \quad u \geq 0. \tag{2.10}$$

See Gerber and Shiu (1998) or Lin and Willmot (1999) for more details concerning the derivations.

3 Integro-Differential Equations for the Discounted Penalty Function

In this section, we derive two integro-differential equations for the discounted penalty function: one for the initial surplus below the barrier and the other for the initial surplus above the barrier. As it will be seen later, the discounted penalty function with initial surplus above the barrier depends on the respective function with initial surplus below the barrier. The reverse relationship though, does not apply. We also identify a continuation condition on the barrier.

Clearly, the discounted penalty function $m(u;b)$ behaves differently, depending on whether its initial surplus u is below or above the barrier level b. Hence, for notational convenience, we write

$$m(u;b) = \begin{cases} m_1(u), & 0 \leq u \leq b \\ m_2(u), & u > b. \end{cases}$$

The following theorem provides integro-differential equations for the function $m(u;b)$.

Theorem 3.1. *The discounted penalty function $m(u;b)$ satisfies the integro-differential equations*

$$m'(u;b) = \begin{cases} m_1'(u) = \dfrac{\lambda+\delta}{c_1} m_1(u) - \dfrac{\lambda}{c_1}\displaystyle\int_0^u m_1(u-y)dP(y) - \dfrac{\lambda}{c_1}\zeta(u), & 0 \leq u \leq b \\ m_2'(u) = \dfrac{\lambda+\delta}{c_2} m_2(u) - \dfrac{\lambda}{c_2}\Big[\displaystyle\int_0^{u-b} m_2(u-y)dP(y) \\ \qquad + \displaystyle\int_{u-b}^{u} m_1(u-y)dP(y)\Big] - \dfrac{\lambda}{c_2}\zeta(u), & u > b. \end{cases} \tag{3.1}$$

Proof. If we condition on the time and the amount of the first claim when $0 \leq u \leq b$, contingent on this time, there are two options: the first claim occurs before the surplus has attained the barrier level or it occurs after attaining the barrier. When we consider the amount of the first claim, there are two possibilities as well: after it the process starts all over again with new initial surplus or the first claim leads to ruin. Implementing these considerations we obtain

$$\begin{aligned} m(u;b) &= m_1(u) \\ &= \int_0^{\frac{b-u}{c_1}} e^{-\delta t}\left[\int_0^{u+c_1t} m(u+c_1t-y;b)dP(y) + \int_{u+c_1t}^{\infty} w(u+c_1t, y-u-c_1t)dP(y)\right]\lambda e^{-\lambda t}dt \\ &\quad + \int_{\frac{b-u}{c_1}}^{\infty} e^{-\delta t}\Bigg[\int_0^{b+c_2\left(t-\frac{b-u}{c_1}\right)} m\left(b+c_2\left(t-\frac{b-u}{c_1}\right)-y;b\right)dP(y) \\ &\quad + \int_{b+c_2\left(t-\frac{b-u}{c_1}\right)}^{\infty} w\left(b+c_2\left(t-\frac{b-u}{c_1}\right), y-b-c_2\left(t-\frac{b-u}{c_1}\right)\right)dP(y)\Bigg]\lambda e^{-\lambda t}dt \\ &= \lambda\int_0^{\frac{b-u}{c_1}} e^{-(\lambda+\delta)t}\gamma(u+c_1t;b)dt + \lambda\int_{\frac{b-u}{c_1}}^{\infty} e^{-(\lambda+\delta)t}\gamma\left(b+c_2\left(t-\frac{b-u}{c_1}\right);b\right)dt, \end{aligned} \tag{3.2}$$

where $\gamma(t;b) = \int_0^t m(t-y;b)dP(y) + \zeta(t)$.

Now, a change of variables in (3.2) results in

$$m_1(u) = \frac{\lambda}{c_1} e^{(\lambda+\delta)u/c_1} \int_u^b e^{-(\lambda+\delta)t/c_1} \gamma(t;b)\, dt \\ + \frac{\lambda}{c_2} e^{(\lambda+\delta)u/c_1} \int_b^\infty e^{-(\lambda+\delta)[t-(c_1-c_2)b/c_1]/c_2} \gamma(t;b)\, dt, \quad 0 \le u \le b. \tag{3.3}$$

By differentiating (3.3) with respect to u we achieve

$$\begin{aligned} m_1'(u) =& \frac{\lambda}{c_1} \cdot \frac{\lambda+\delta}{c_1} e^{(\lambda+\delta)u/c_1} \int_u^b e^{-(\lambda+\delta)t/c_1} \gamma(t;b)\, dt - \frac{\lambda}{c_1}\gamma(u;b) \\ &+ \frac{\lambda}{c_2} \cdot \frac{\lambda+\delta}{c_1} e^{(\lambda+\delta)u/c_1} \int_b^\infty e^{-(\lambda+\delta)[t-(c_1-c_2)b/c_1]/c_2} \gamma(t;b)\, dt \\ =& \frac{\lambda+\delta}{c_1} m_1(u) - \frac{\lambda}{c_1}\gamma(u;b), \quad 0 \le u \le b, \end{aligned}$$

or equivalently,

$$m_1'(u) = \frac{\lambda+\delta}{c_1} m_1(u) - \frac{\lambda}{c_1} \int_0^u m_1(u-y) dP(y) - \frac{\lambda}{c_1}\zeta(u), \quad 0 \le u \le b.$$

Similarly, when $u \ge b$,

$$m(u;b) = m_2(u) = \frac{\lambda}{c_2} e^{(\lambda+\delta)u/c_2} \int_u^\infty e^{-(\lambda+\delta)t/c_2} \gamma(t;b) dt, \tag{3.4}$$

and

$$\begin{aligned} m_2'(u) =& \frac{\lambda+\delta}{c_2} m_2(u) - \frac{\lambda}{c_2}\gamma(u;b) \\ =& \frac{\lambda+\delta}{c_2} m_2(u) - \frac{\lambda}{c_2} \left[\int_0^{u-b} m_2(u-y) dP(y) + \int_{u-b}^u m_1(u-y) dP(y) \right] - \frac{\lambda}{c_2}\zeta(u), \end{aligned}$$

by which the proof is concluded. □

Theorem 3.1 may be linked to previously obtained results in the literature. Namely, both pairs of equations (5.1), (5.2) and (10.2), (10.3) in Gerber and Shiu (2005a) are special cases of the integro-differential equations (3.1) above, although the functions (5.1) and (5.2) are not a special case of the discounted penalty function. Another instance of an already known result is the case when $c_1 = c$ and $c_2 = 0$. Equation (3.2) above reduces then to equation (2.1) in Lin et al. (2003).

We note that equation (3.1) for m_1 coincides with (2.2). Consequently, the general analytical expression (2.4) for the solution of (2.2) applies. What remains is the parameter κ that will be determined in the next section.

Observe also that the integro-differential equation (3.1) for m_1 does not involve m_2 but the integro-differential equation (3.1) for m_2 incorporates m_1.

In conclusion of this section, we examine $m(u;b)$ when $u=b$. Equations (3.3) and (3.4) show that $m(u;b)$ is continuous for $u=b$, i.e. $m_1(b)=m_2(b):=\lim_{u\to b^+} m_2(u)$. However, it is not true for $m'(u;b)$ at $u=b$. To see this, let $u\to b^+$ in the expression for $m_2(u)$ in (3.1) and employ the integro-differential form of $m_1(b)$ in (3.1) afterwards. We then have

$$\begin{aligned} m_2'(b) := \lim_{u\to b^+} m_2'(u) =& \frac{\lambda+\delta}{c_2} m_2(b) - \frac{\lambda}{c_2}\int_0^b m_1(b-y)dP(y) - \frac{\lambda}{c_2}\zeta(b) \\ =& \frac{\lambda+\delta}{c_2} m_1(b) + \frac{c_1}{c_2}\left[m_1'(b) - \frac{\lambda+\delta}{c_1} m_1(b)\right]. \end{aligned}$$

This results in an interesting identity:

$$c_1 m_1'(b) = c_2 m_2'(b),$$

where the derivative $m_1'(b)$ is a left-derivative and the derivative $m_2'(b)$ is a right-derivative. Thus, the discounted penalty function is continuous but not differentiable.

4 A Renewal Equation for the Discounted Penalty Function $\mathbf{m_2(u)}$

In this section, utilizing the translation operator introduced in Section 2 we derive a renewal equation for the discounted penalty function $m_2(u)$, $u>b$. It is assumed that the relative security loading for the net premium rate is positive, i.e., $\theta_2>0$. For a non-positive relative security loading, the time to ruin is finite and the case will be discusses elsewhere.

Theorem 4.1. *The function m_2 satisfies*

$$m_2(u) = \pi_2\left[\int_0^{u-b} m_2(u-y)dA_2(y) + \int_{u-b}^{u} m_1(u-y)dA_2(y)\right] + \frac{\lambda}{c_2}T_{\rho_2}\zeta(u), \tag{4.1}$$

where π_2, A_2, and ζ are as defined in (2.9), (2.7), *and* (2.3) *respectively.*

Proof. We adopt a similar approach to the one utilized in Section 4 of Willmot and Dickson (2003). For a fixed $s>0$, we multiply the second equation of (3.1) by $c_2e^{-s(u-b)}$ and integrate with respect to u from b to ∞. By integration by parts, we obtain

$$c_2\int_b^\infty e^{-s(u-b)}m_2'(u)du = c_2\left[-m_2(b) + s\int_b^\infty e^{-s(u-b)}m_2(u)du\right]$$

$$
\begin{aligned}
=&c_2 s\ T_s m_2(b) - c_2 m_2(b)\\
=&(\lambda+\delta)T_s m_2(b) - \lambda \int_b^\infty e^{-s(u-b)} \int_0^{u-b} m_2(u-y)dP(y)du\\
&-\lambda \int_b^\infty e^{-s(u-b)} \int_0^b m_1(y)p(u-y)dydu - \lambda T_s\zeta(b)\\
=&(\lambda+\delta)T_s m_2(b) - \lambda \int_0^\infty e^{-sy} \int_{y+b}^\infty e^{-s(u-y-b)} m_2(u-y)dudP(y)\\
&-\lambda \int_0^b m_1(y) \int_b^\infty e^{-s(u-b)} p(u-y)dudy - \lambda T_s\zeta(b)\\
=&(\lambda+\delta)T_s m_2(b) - \lambda\tilde{p}(s)T_s m_2(b) - \lambda \int_0^b m_1(y)T_s p(b-y)dy - \lambda T_s\zeta(b).
\end{aligned}
$$

Rearranging the above terms leads to

$$
[c_2 s - (\lambda+\delta) + \lambda\tilde{p}(s)]T_s m_2(b) = c_2 m_2(b) - \lambda \int_0^b m_1(y)T_s p(b-y)dy - \lambda T_s\zeta(b). \tag{4.2}
$$

To determine the constant term $c_2 m_2(b)$ in (4.2), we substitute s there with the nonnegative solution ρ_2 to the Lundberg fundamental equation (2.5). Hence

$$
c_2 m_2(b) = \lambda \int_0^b m_1(y)T_{\rho_2} p(b-y)dy + \lambda T_{\rho_2}\zeta(b).
$$

Consequently, equation (4.2) reduces to

$$
\begin{aligned}
[c_2(s-\rho_2) + \lambda\tilde{p}(s) - \lambda\tilde{p}(\rho_2)]T_s m_2(b) =&\lambda \left[\int_0^b m_1(y)T_{\rho_2} p(b-y)dy - \int_0^b m_1(y)T_s p(b-y)dy\right]\\
&+\lambda[T_{\rho_2}\zeta(b) - T_s\zeta(b)],
\end{aligned}
$$

which, divided by $s-\rho_2$, produces

$$
\begin{aligned}
\left[c_2 - \lambda\frac{\tilde{p}(s)-\tilde{p}(\rho_2)}{\rho_2 - s}\right] T_s m_2(b) = &\left[c_2 - \lambda\frac{1-\tilde{p}(\rho_2)}{\rho_2}\tilde{a}_2(s)\right] T_s m_2(b)\\
=&\lambda \int_0^b m_1(y) \frac{T_{\rho_2} p(b-y) - T_s p(b-y)}{s-\rho_2}\, dy + \lambda \frac{T_{\rho_2}\zeta(b) - T_s\zeta(b)}{s-\rho_2},
\end{aligned}
$$

where $\tilde{a}_2(s)$ is given by (2.8) with $i=2$. Slight rearrangements in the above equation along with implementation of formula (2.1) lead to

$$
c_2 T_s m_2(b) = \lambda\frac{1-\tilde{p}(\rho_2)}{\rho_2}\,\tilde{a}_2(s)T_s m_2(b) + \lambda \int_0^b m_1(y)T_s T_{\rho_2} p(b-y)dy + \lambda T_s T_{\rho_2}\zeta(b).
$$

We invert the operators to obtain

$$m_2(u) = \pi_2 \int_0^{u-b} m_2(u-y)dA_2(y) + \frac{\lambda}{c_2}\int_{u-b}^{u} m_1(u-y)T_{\rho_2}p(y)dy + \frac{\lambda}{c_2}T_{\rho_2}\zeta(u), \quad u > b,$$

where π_2 is as defined in (2.9) with $i = 2$. The above expression may be slightly simplified by noticing that through integration by parts

$$T_{\rho_2}p(y)dy = \left[\int_0^\infty e^{-\rho_2 t}\overline{P}(t)dt\right] dA_2(y)$$

and that

$$\int_0^\infty e^{-\rho_2 t}\overline{P}(t)dt = \frac{1-\tilde{p}(\rho_2)}{\rho_2}.$$

Then we finally have, for $u > b$,

$$m_2(u) = \pi_2 \left[\int_0^{u-b} m_2(u-y)dA_2(y) + \int_{u-b}^{u} m_1(u-y)dA_2(y)\right] + \frac{\lambda}{c_2}T_{\rho_2}\zeta(u),$$

which is the required result. □

We remark that the result in Theorem 4.1 may be restated as

$$m(u;b) = \pi_2 \int_0^u m(u-y;b)dA_2(y) + \frac{\lambda}{c_2}T_{\rho_2}\zeta(u), \quad u > b.$$

5 Analytical Expressions for the Discounted Penalty Function

In this section, we obtain an analytical expression for the discounted penalty function. Part of its derivation relies on results obtained in Lin and Willmot (1999) and Lin et al. (2003). As will be seen in later sections, several results in Asmussen (2000) and Gerber and Shiu (2005a) are special cases.

We begin with m_1. Since equation (3.1) for m_1 has the exact same form as equation (2.9) in Lin et al. (2003) with c there replaced by c_1 here, we may apply the same approach as far as we obtain an initial condition for m_1.

The solution m_1 is then of the form

$$m_1(u) = m_\infty(u) + \kappa v(u), \tag{5.1}$$

where κ is a constant, which we specify by implementing equation (5.1) and returning to (4.1). Thus, for $u = b$ we have

$$m_\infty(b) + \kappa v(b) = m_1(b) = m_2(b)$$

$$= \pi_2 \int_0^b m_\infty(b-y)dA_2(y) + \kappa\, \pi_2 \int_0^b v(b-y)dA_2(y) + \frac{\lambda}{c_2} T_{\rho_2}\zeta(b).$$

Thus,

$$\kappa = \frac{\pi_2 \int_0^b m_\infty(b-y)dA_2(y) - m_\infty(b) + \frac{\lambda}{c_2} T_{\rho_2}\zeta(b)}{v(b) - \pi_2 \int_0^b v(b-y)dA_2(y)}.$$

Next, we express m_∞ in terms of a compound geometric distribution. For $i = 1$ and 2, introduce the following compound geometric c.d.f.

$$K_i(y) = 1 - \overline{K}_i(y) = \sum_{n=0}^{\infty} (1-\pi_i)\pi_i^n \; A_i^{*n}(y), \quad u \geq 0, \tag{5.2}$$

where A_i^{*n} is the n-fold convolution of A_i with itself. Then by Theorem 2.1 in Lin and Willmot (1999),

$$\begin{aligned} m_\infty(u) =& \frac{\lambda}{c_1(1-\pi_1)} \int_0^u T_{\rho_1}\zeta(u-y)dK_1(y) + \frac{\lambda}{c_1} T_{\rho_1}\zeta(u) \\ =& \frac{\lambda}{c_1(1-\pi_1)} \left[- \int_0^u \overline{K}_1(u-y)dT_{\rho_1}\zeta(y) - \overline{K}_1(u)T_{\rho_1}\zeta(0) + T_{\rho_1}\zeta(u) \right], \quad u \geq 0. \end{aligned} \tag{5.3}$$

Lastly, we derive an expression for m_2. Let $x = u - b$ and $g(x) = m_2(x+b)$, $x > 0$. We can then rewrite (4.1) as

$$g(x) = \pi_2 \int_0^x g(x-y)dA_2(y) + h(x+b), \quad x > 0.$$

where

$$h(x+b) = h(u) = \pi_2 \int_{u-b}^u m_1(u-y)dA_2(y) + \frac{\lambda}{c_2} T_{\rho_2}\zeta(u), \quad u > b.$$

Hence, Theorem 2.1 in Lin and Willmot (1999) applies to $g(x)$ and we may express $m_2(u)$ via $g(x)$ as

$$\begin{aligned} m_2(u) =& g(x) \\ =& \frac{1}{1-\pi_2} \int_0^x h(x+b-y)dK_2(y) + h(x+b) \\ =& \frac{1}{1-\pi_2} \int_0^{u-b} h(u-y)dK_2(y) + h(u) \\ =& -\frac{1}{1-\pi_2} \left[\int_b^u \overline{K}_2(u-y)dh(y) + \overline{K}_2(u-b)h(b) \right] + h(u). \end{aligned} \tag{5.4}$$

All above derivations are summarized by the theorem below.

Theorem 5.1. *The discounted penalty function $m(u;b)$ may be expressed analytically in the following steps:*

$$\text{(i)}\quad m_\infty(u) = \frac{\lambda}{c_1(1-\pi_1)}\int_0^u T_{\rho_1}\zeta(u-y)dK_1(y) + \frac{\lambda}{c_1}T_{\rho_1}\zeta(u); \tag{5.5}$$

$$\text{(ii)}\quad \kappa = \frac{\pi_2\int_0^b m_\infty(b-y)dA_2(y) - m_\infty(b) + \frac{\lambda}{c_2}T_{\rho_2}\zeta(b)}{v(b) - \pi_2\int_0^b v(b-y)dA_2(y)}; \tag{5.6}$$

$$\text{(iii)}\quad m_1(u) = m_\infty(u) + \kappa\,\frac{1-\Psi(u)}{1-\Psi(0)}\,e^{\rho_1 u}, \quad 0\le u\le b; \tag{5.7}$$

$$\text{(vi)}\quad h(u) = \pi_2\int_{u-b}^u m_1(u-t)dA_2(t) + \frac{\lambda}{c_2}T_{\rho_2}\zeta(u), \quad u>b, \tag{5.8}$$

$$\text{(v)}\quad m_2(u) = \frac{1}{1-\pi_2}\int_0^{u-b} h(u-y)dK_2(y) + h(u), \quad u>b, \tag{5.9}$$

where $K_i, i=1,2,$ are given in (5.2)*.*

The definitions of the remaining functions and parameters utilized in Theorem 5.1 above are provided in Sections 2 and 3. Also note that equations (5.5) and (5.8) have alternative forms provided by (5.3) and (5.4) respectively.

We now proceed with some important special cases in the rest of the paper.

6 The Probability of Ultimate Ruin

This section discusses the probability that the time of ruin is finite, or the probability of ultimate ruin, and the probability of a drop below the initial surplus level as a special case.

As it is well known, many particular cases of the discounted penalty function, which result in important quantities of interest in risk theory, are produced when δ is set to be 0. The nonnegative solutions of the respective Lundberg fundamental equations (2.5) are then $\rho_1=\rho_2=0$. On one hand, this implies that $\pi_i = 1/(1+\theta_i),\ i=1,2,$ as noted after the definition in (2.9). On the other hand, by (2.7) both A_1 and A_2 reduce to the first order equilibrium distribution P_e of P, namely,

$$P_e(y) = 1 - \overline{P}_e(y) = \int_0^y \frac{\overline{P}(t)dt}{\mathbb{E}\{Y_1\}} = \int_0^y p_e(t)dt, \quad y \geq 0,$$

where $p_e = \overline{P}(t)/\mathbb{E}\{Y_1\}$ is the equilibrium p.d.f.

Furthermore, $T_{\rho_1}\zeta(u) = T_{\rho_2}\zeta(u) = \int_u^\infty \zeta(t)dt$ and the definition of Ψ in Section 2 shows that $\Psi(u) = \psi_{1,\infty}(u),\ u \geq 0$, where $\psi_{1,\infty}$ is the probability of ultimate ruin under the classical compound Poisson model with premium rate c_1. Also, with the changes in π_i and $A_i,\ i = 1, 2$, it is clear that $\overline{K}_1$ and $\overline{K}_2$, defined by (5.2), coincide with the probabilities of ultimate ruin $\psi_{1,\infty}$ and $\psi_{2,\infty}$ respectively. Here $\psi_{2,\infty}$ is the probability of ultimate ruin under the ordinary compound Poisson model with premium rate c_2. It is well known that $\psi_{1,\infty}(0) = 1/(1 + \theta_1)$ and $\psi_{2,\infty}(0) = 1/(1 + \theta_2)$.

After setting $\delta = 0$, let also assume that $w(x_1, x_2) = 1$ for all $x_1, x_2 \geq 0$. Then $m(u; b)$ becomes the probability of ultimate ruin, which we denote by ψ_b. Also, by (2.3) ζ reduces to $\overline{P}$.

From Theorem 5.1 we have the following result.

Theorem 6.1. *The probability of ultimate ruin for the compound Poisson risk model with a threshold strategy is given by*

$$\psi(u;b) = \begin{cases} \psi_1(u) = 1 - q(b) + q(b)\psi_{1,\infty}(u), & 0 \leq u \leq b \\ \\ \psi_2(u) = -\dfrac{1+\theta_2}{\theta_2} \displaystyle\int_0^{u-b} h(u-y)d\psi_{2,\infty}(y) + h(u), & u > b, \end{cases} \tag{6.1}$$

with

$$q(b) = \frac{\theta_2}{(\theta_1 - \theta_2)\psi_{1,\infty}(b) + \theta_2} \tag{6.2}$$

belonging to the interval [0,1] and

$$h(u) = \frac{1}{1+\theta_2} \int_{u-b}^{u} \psi_1(u-t)dP_e(t) + \frac{1}{1+\theta_2}\overline{P}_e(u), \quad u > b. \tag{6.3}$$

Proof. Equation (5.6) yields

$$\kappa = \frac{\dfrac{1}{1+\theta_2}\displaystyle\int_0^b \psi_{1,\infty}(b-y)dP_e(y) - \psi_{1,\infty}(b) + \dfrac{1}{1+\theta_2}\overline{P}_e(b)}{\dfrac{1-\psi_{1,\infty}(b)}{1-\psi_{1,\infty}(0)} - \dfrac{1}{1+\theta_2}\displaystyle\int_0^b \dfrac{1-\psi_{1,\infty}(b-y)}{1-\psi_{1,\infty}(0)}dP_e(y)}$$

$$
\begin{aligned}
=&[1-\psi_{1,\infty}(0)]\frac{\frac{1}{1+\theta_2}[(1+\theta_1)\psi_{1,\infty}(b)-\overline{P}_e(b)]-\psi_{1,\infty}(b)+\frac{1}{1+\theta_2}\overline{P}_e(b)}{1-\psi_{1,\infty}(b)-\frac{1}{1+\theta_2}\left[P_e(b)-\int_0^b\psi_{1,\infty}(b-y)dP_e(y)\right]}\\
=&[1-\psi_{1,\infty}(0)]\frac{\left(\frac{1+\theta_1}{1+\theta_2}-1\right)\psi_{1,\infty}(b)}{1-\psi_{1,\infty}(b)-\frac{1}{1+\theta_2}P_e(b)+\frac{1+\theta_1}{1+\theta_2}\psi_{1,\infty}(b)-\frac{1}{1+\theta_2}\overline{P}_e(b)}\\
=&[1-\psi_{1,\infty}(0)]\frac{(\theta_1-\theta_2)\psi_{1,\infty}(b)}{(\theta_1-\theta_2)\psi_{1,\infty}(b)+\theta_2}.
\end{aligned}
$$

By equation (5.7) we have

$$
\begin{aligned}
\psi_1(u)=&\psi_{1,\infty}(u)+\frac{(\theta_1-\theta_2)\psi_{1,\infty}(b)}{(\theta_1-\theta_2)\psi_{1,\infty}(b)+\theta_2}[1-\psi_{1,\infty}(u)]\\
=&\frac{(\theta_1-\theta_2)\psi_{1,\infty}(b)}{(\theta_1-\theta_2)\psi_{1,\infty}(b)+\theta_2}+\frac{\theta_2}{(\theta_1-\theta_2)\psi_{1,\infty}(b)+\theta_2}\psi_{1,\infty}(u),\quad 0\le u\le b,
\end{aligned}
$$

which produces (6.1). The remaining part of the proof follows directly from Theorem 5.1 along with the assumptions about δ and w. □

We remark that formula (6.1) implies that ψ_1 is a weighted average of the probability of ruin in the dividend case and the probability of ruin in the ordinary Poisson model.

Also, as a particular case of (6.1), we obtain the following.

Corollary 6.2. *For $0\le u\le b$ the probability of a drop below the initial surplus level u, is provided by*

$$
\psi(0;b-u)=\frac{(1+\theta_1)(\theta_1-\theta_2)\psi_{1,\infty}(b-u)+\theta_2}{(1+\theta_1)\left[(\theta_1-\theta_2)\psi_{1,\infty}(b-u)+\theta_2\right]}. \tag{6.4}
$$

Proof. We find

$$
\begin{aligned}
\psi(0;b-u)=&1-q(b-u)+q(b-u)\psi_{1,\infty}(0)=1-\frac{\theta_1}{1+\theta_1}q(b-u)\\
=&\frac{(1+\theta_1)(\theta_1-\theta_2)\psi_{1,\infty}(b-u)+\theta_2}{(1+\theta_1)\left[(\theta_1-\theta_2)\psi_{1,\infty}(b-u)+\theta_2\right]},\quad 0\le u\le b,
\end{aligned}
$$

by employing (6.1) and (6.2). □

Note that when $u>b$, the probability of a drop below the initial level, $\psi_0(0)$ under the present model, reduces to the same probability under the classical compound Poisson model with premium rate c_2. The distribution of the first drop below the initial level will be discussed in Section 8.

An advantage of formula (6.1) over the result of Proposition 1.10 in Asmussen (2000, Chapter VII, Section 1) is that all functions involved in Theorem 6.1 have a known form and hence can be computed straightforwardly. This becomes obvious when illustrated in the following two examples.

Example 6.1. (Exponential Claim Amounts)

When the individual claim amounts are exponentially distributed with mean $1/\mu,\ \mu > 0$, upon a little algebra it is seen that

$$P(y) = 1 - e^{-\mu y} = P_e(y), \quad y \geq 0.$$

Let $\beta_i = \frac{\theta_i}{1+\theta_i}\mu,\ i = 1, 2$, be the Lundberg adjustment coefficient for the compound Poisson risk model with premium rate c_i. Then, it is well known that

$$\psi_{i,\infty}(u) = \frac{1}{1+\theta_i} e^{-\beta_i u}, \quad u \geq 0,\ i = 1, 2.$$

Hence, by (6.4) the probability of a drop below the initial surplus level is given by

$$\psi(0; b-u) = \frac{(\theta_1 - \theta_2)e^{-\beta_1(b-u)} + \theta_2}{(\theta_1 - \theta_2)e^{-\beta_1(b-u)} + (1+\theta_1)\theta_2}, \quad 0 \leq u \leq b.$$

Also, it follows from (6.1) that

$$\psi_1(u) = 1 - q(b) + \frac{q(b)}{1+\theta_1} e^{-\beta_1 u}, \quad 0 \leq u \leq b, \tag{6.5}$$

with

$$q(b) = \frac{(1+\theta_1)\theta_2}{(\theta_1 - \theta_2)e^{-\beta_1 b} + (1+\theta_1)\theta_2} \tag{6.6}$$

by (6.2). Equation (6.5) yields further that (6.3) reduces to

$$\begin{aligned} h(u) =& \frac{1}{1+\theta_2} \left\{ -\int_{u-b}^{u} [1-q(b)] de^{-\mu t} - \frac{q(b)}{1+\theta_1} \int_{u-b}^{u} e^{-\beta_1(u-t)} de^{-\mu t} + e^{-\mu u} \right\} \\ =& \frac{1}{1+\theta_2} \left\{ [1-q(b)]e^{-\mu(u-b)} - [1-q(b)]e^{-\mu u} + \frac{\mu q(b)}{1+\theta_1} e^{-\beta_1 u} \int_{u-b}^{u} e^{(\beta_1-\mu)t} dt + e^{-\mu u} \right\} \\ =& \frac{1}{1+\theta_2} \left\{ [1-q(b)]e^{-\mu(u-b)} + q(b)e^{-\mu u} + \frac{\mu q(b)}{1+\theta_1} \cdot \frac{1}{\beta_1 - \mu} e^{-\beta_1 u} \left[1 - e^{-(\beta_1-\mu)b}\right] e^{(\beta_1-\mu)u} \right\} \\ =& \frac{1}{1+\theta_2} \left\{ [1-q(b)]e^{-\mu(u-b)} + q(b)e^{-\mu u} - q(b)\left[1 - e^{-(\beta_1-\mu)b}\right] e^{-\mu u} \right\} \\ =& \frac{1}{1+\theta_2} \left[1 - q(b) + q(b)e^{-\beta_1 b}\right] e^{-\mu(u-b)} \end{aligned}$$

$$=\frac{1}{1+\theta_2}Q(b)e^{-\mu(u-b)}, \quad u>b, \tag{6.7}$$

where $Q(b)=1-q(b)+q(b)e^{-\beta_1 b}$. Therefore, (6.1) produces

$$\begin{aligned}
\psi_2(u) =&\frac{1}{1+\theta_2}\left\{-\frac{1}{\theta_2}\int_0^{u-b} Q(b)e^{-\mu(u-y-b)}de^{-\beta_2 y}+Q(b)e^{-\mu(u-b)}\right\}\\
=&\frac{1}{1+\theta_2}\left\{Q(b)\frac{\beta_2}{\theta_2}\int_0^{u-b} e^{-(\beta_2-\mu)y}dy+Q(b)\right\}e^{-\mu(u-b)}\\
=&\frac{1}{1+\theta_2}\left\{-Q(b)\frac{\beta_2}{\theta_2}\cdot\frac{1}{\beta_2-\mu}\left[e^{-(\beta_2-\mu)(u-b)}-1\right]+Q(b)\right\}e^{-\mu(u-b)}\\
=&\frac{1}{1+\theta_2}\left\{Q(b)\left[e^{-(\beta_2-\mu)(u-b)}-1\right]+Q(b)\right\}e^{-\mu(u-b)}\\
=&\frac{1}{1+\theta_2}Q(b)e^{-\beta_2(u-b)}, \quad u>b.
\end{aligned} \tag{6.8}$$

To summarize,

$$\psi(u;b)=\begin{cases}\psi_1(u)=1-q(b)+\dfrac{q(b)}{1+\theta_1}e^{-\beta_1 u}, & 0\le u\le b\\[2ex] \psi_2(u)=\dfrac{1}{1+\theta_2}\left[1-q(b)+q(b)e^{-\beta_1 b}\right]e^{-\beta_2(u-b)}, & u>b,\end{cases}$$

with $q(b)$ specified by (6.6) and $\beta_i=\frac{\theta_i}{1+\theta_i}\mu,\ i=1,2$, which coincides with the result obtained in Example 1.11 in Asmussen (2000, Chapter VII, Section 1).

It is easy to see that $0\le Q(b)\le 1$. Hence we obtain a Lundberg-type upper bound for $\psi_2(u)$:

$$\psi_2(u)\le\frac{1}{1+\theta_2}e^{-\beta_2(u-b)}.$$

□

Example 6.2. (Combination of Exponentials Claim Amounts)

Consider now the individual claim amounts have a combination of exponentials distribution. More specifically,

$$\overline{P}(y)=\sum_{j=1}^{n}\omega_j\,e^{-\mu_j y}, \quad y\ge 0,\ \mu_j>0,\ j=1,2,\dots,n,$$

with $\sum_{j=1}^{n}\omega_j=1$ for a positive integer number n.

As noted in Gerber et al. (1987),

$$\overline{P}_e(y)=\sum_{j=1}^{n}\omega_j^*\,e^{-\mu_j y}, \quad y\ge 0,$$

where

$$\omega_j^* = \frac{\omega_j/\mu_j}{\sum_{l=1}^n \omega_l/\mu_l}, \quad j = 1, 2, \ldots, n,$$

and

$$\psi_{i,\infty}(u) = \sum_{j=1}^n C_{ij} e^{-\beta_{ij} u}, \quad u \geq 0, \ i = 1, 2,$$

where $0 < \beta_{i1} < \beta_{i2} < \cdots < \beta_{in}$ are the roots of $\sum_{j=1}^n [\omega_j^* \mu_j/(\mu_j - \beta)] = 1 + \theta_i, \ i = 1, 2,$ assumed to be real valued and distinct, and

$$C_{ij} = \left[\sum_{l=1}^n \frac{\omega_l^*}{\mu_l - \beta_{ij}} \right] \Bigg/ \left[\sum_{l=1}^n \frac{\omega_l^* \mu_l}{(\mu_l - \beta_{ij})^2} \right], \quad i = 1, 2, \ j = 1, 2, \ldots, n.$$

We remark that $\beta_{i1}, \ i = 1, 2,$ are the Lundberg adjustment coefficients.

It follows from Corollary 6.2 that the probability of a drop below the initial surplus level is given by

$$\psi(0; b - u) = \frac{(1 + \theta_1)(\theta_1 - \theta_2) \sum_{j=1}^n C_{1j} e^{-\beta_{1j}(b-u)} + \theta_2}{(1 + \theta_1) \left[(\theta_1 - \theta_2) \sum_{j=1}^n C_{1j} e^{-\beta_{1j}(b-u)} + \theta_2 \right]}, \quad 0 \leq u \leq b.$$

Furthermore, it is easy to see

$$\psi_1(u) = 1 - q(b) + q(b) \sum_{j=1}^n C_{1j} e^{-\beta_{1j} u}, \quad 0 \leq u \leq b,$$

where

$$q(b) = \frac{\theta_2}{(\theta_1 - \theta_2) \sum_{j=1}^n C_{1j} \, e^{-\beta_{1j} b} + \theta_2}.$$

Before we derive an analytical expression for $\psi_2(u)$, two useful identities are presented. Slightly changing the notation, formula (10.4.13) in Willmot and Lin (2001) becomes

$$\sum_{l=1}^n \frac{\beta_{il} C_{il}}{\mu_j - \beta_{il}} = \frac{\theta_i}{1 + \theta_i}, \quad i = 1, 2. \tag{6.9}$$

Since $\sum_{l=1}^n C_{il} = \psi_{i,\infty}(0) = 1/(1 + \theta_i)$, we have

$$\begin{aligned} \sum_{l=1}^n \frac{\mu_j C_{il}}{\mu_j - \beta_{il}} &= \sum_{l=1}^n C_{il} + \sum_{l=1}^n \frac{\beta_{il} C_{il}}{\mu_j - \beta_{il}} \\ &= \frac{1}{1 + \theta_i} + \frac{\theta_i}{1 + \theta_i} = 1, \quad i = 1, 2. \end{aligned} \tag{6.10}$$

Analogously to equations (6.7) and (6.8) we find respectively

$$
\begin{aligned}
h(u) =& \frac{1}{1+\theta_2}\sum_{j=1}^{n}\omega_j^*\left\{\mu_j\int_{u-b}^{u}\psi_1(u-t)e^{-\mu_j t}dt + e^{-\mu_j u}\right\} \\
=& \frac{1}{1+\theta_2}\sum_{j=1}^{n}\omega_j^*\Bigg\{[1-q(b)]e^{-\mu_j(u-b)} + q(b)e^{-\mu_j u} \\
& + q(b)\sum_{l=1}^{n}\frac{\mu_j C_{1l}}{\mu_j-\beta_{1l}}\left[e^{-\beta_{1l}b}e^{-\mu_j(u-b)} - e^{-\mu_j u}\right]\Bigg\}.
\end{aligned}
$$

Using identity (6.10) with $i = 1$, we have

$$
h(u) = \sum_{j=1}^{n}\omega_j^* Q_j(b)e^{-\mu_j(u-b)}, \tag{6.11}
$$

where

$$
Q_j(b) = 1 - q(b) + q(b)\sum_{l=1}^{n}\frac{\mu_j C_{1l}}{\mu_j-\beta_{1l}}e^{-\beta_{1l}b}, \quad j = 1, \cdots, n.
$$

Thus,

$$
\begin{aligned}
\psi_2(u) =& \frac{1}{\theta_2}\sum_{j=1}^{n}\omega_j^* Q_j(b)\sum_{l=1}^{n}\frac{\beta_{2l}C_{2l}}{\mu_j-\beta_{2l}}e^{-\beta_{2l}(u-b)} \\
& + \sum_{j=1}^{n}\omega_j^* Q_j(b)\left\{\frac{1}{1+\theta_2} - \frac{1}{\theta_2}\sum_{l=1}^{n}\frac{\beta_{2l}C_{2l}}{\mu_j-\beta_{2l}}\right\}e^{-\mu_j(u-b)} \\
=& \frac{1}{\theta_2}\sum_{j=1}^{n}\omega_j^* Q_j(b)\sum_{l=1}^{n}\frac{\beta_{2l}C_{2l}}{\mu_j-\beta_{2l}}e^{-\beta_{2l}(u-b)} \\
=& \frac{1}{\theta_2}\sum_{j=1}^{n}\beta_{2j}C_{2j}\left\{\sum_{l=1}^{n}\frac{\omega_l^* Q_l(b)}{\mu_l-\beta_{2j}}\right\}e^{-\beta_{2j}(u-b)}, \quad u > b,
\end{aligned}
$$

due to identity (6.9) with $i = 2$. Therefore,

$$
\psi(u;b) = \begin{cases}
\psi_1(u) = 1 - q(b) + q(b)\sum_{j=1}^{n} C_{1j}\, e^{-\beta_{1j}u}, & 0 \le u \le b \\[2ex]
\psi_2(u) = \dfrac{1}{\theta_2}\sum_{j=1}^{n}\beta_{2j}C_{2j}\left[\sum_{l=1}^{n}\dfrac{\omega_l^* Q_l(b)}{\mu_l-\beta_{2j}}\right]e^{-\beta_{2j}(u-b)}, & u > b.
\end{cases}
$$

If the individual claim distribution is a mixture, i.e., $\omega_j \ge 0$ for all j, then $C_{il} \ge 0$ for all l and $i = 1, 2$. Hence, $0 \le Q_j(b) \le 1$ for all j. We can again obtain a Lundberg-type upper bound for $\psi_2(u)$ in this case:

$$
\begin{aligned}
\psi_2(u) =& \frac{1}{\theta_2}\sum_{j=1}^{n}\omega_j^* Q_j(b)\sum_{l=1}^{n}\frac{\beta_{2l}C_{2l}}{\mu_j-\beta_{2l}}e^{-\beta_{2l}(u-b)} \\
\leq& \frac{1}{\theta_2}\sum_{j=1}^{n}\omega_j^* Q_j(b)\sum_{l=1}^{n}\frac{\beta_{2l}C_{2l}}{\mu_j-\beta_{2l}}e^{-\beta_{21}(u-b)} \\
=& \frac{1}{1+\theta_2}\sum_{j=1}^{n}\omega_j^* Q_j(b)e^{-\beta_{21}(u-b)} \leq \frac{1}{1+\theta_2}e^{-\beta_{21}(u-b)}, \quad u>b.
\end{aligned}
$$

□

7 The time of ruin

In this section we turn our attention to another particular case of $m(u;b)$. We consider the defective Laplace transform of the ruin time $\mathbb{E}\{e^{-\delta T_b}I(T_b<\infty)|U_b(0)=u\}$ by letting $\delta\geq 0$ to be arbitrary and setting $w(x_1,x_2)=1$ for all $x_1,x_2\geq 0$. Recall that in this case ζ reduces to $\overline{P}$. Also, when $\delta>0$, without loss of generality we may consider $\mathbb{E}\{e^{-\delta T_b}|U_b(0)=u\}$ instead of the above-mentioned Laplace transform. For $\delta=0$ we have the probability of ultimate ruin, discussed in Section 6. Once the Laplace transform of the time of ultimate ruin T_b is identified, its distribution may be obtained using an inversion method. Its moments may be obtained by differentiation with respect to δ. Although such tasks are theoretically possible, they are practically tedious and challenging as demonstrated in Drekic et al. (2004). It is partly due to the fact that the discounted penalty function $m(u;b)$ given in Theorem 5.1 is not explicitly expressed in terms of δ. Instead, it is expressed via $\rho_1=\rho_1(\delta)$ and $\rho_2=\rho_2(\delta)$.

Let

$$\mathcal{L}(u;b)=m(u;b)\Big|_{w\equiv 1},$$

$$\mathcal{L}_1(u)=m_1(u)\Big|_{w\equiv 1},\qquad \mathcal{L}_2(u)=m_2(u)\Big|_{w\equiv 1},$$

and

$$\mathcal{L}_\infty(u)=m_\infty(u)\Big|_{w\equiv 1}.$$

We note that the above notation is somewhat misleading as the true variable is δ that is implicitly included in these functions. However, expressing them in terms of u is necessary for our derivation purposes. We now utilize our main result to derive the Laplace transform of T_b.

Theorem 7.1. *The Laplace transform of the time of ruin for the compound Poisson risk model with a threshold strategy satisfies*

$$\mathcal{L}(u;b) = \begin{cases} \mathcal{L}_1(u) = \mathcal{L}_\infty(u) + \kappa \dfrac{1-\Psi(u)}{1-\Psi(0)} e^{\rho_1 u}, & 0 \le u \le b \\ \\ \mathcal{L}_2(u) = \dfrac{1}{1-\pi_2} \displaystyle\int_0^{u-b} h(u-y) dK_2(y) + h(u), & u > b, \end{cases} \tag{7.1}$$

where

$$\mathcal{L}_\infty(u) = \frac{\pi_1}{1-\pi_1}\left[\int_0^u \overline{K}_1(u-y)dA_1(y) - \overline{K}_1(u) + \overline{A}_1(u)\right], \quad u \ge 0,$$

$$\kappa = \frac{\pi_2 \int_0^b \mathcal{L}_\infty(b-y)dA_2(y) - \mathcal{L}_\infty(b) + \pi_2\overline{A}_2(b)}{v(b) - \pi_2 \int_0^b v(b-y)dA_2(y)}, \tag{7.2}$$

and

$$h(u) = \pi_2 \int_{u-b}^{u} \mathcal{L}_1(u-t)dA_2(t) + \pi_2\overline{A}_2(u), \quad u > b. \tag{7.3}$$

Proof. A direct application of Theorem 5.1 yields equation (7.1) with κ and h as specified in (7.2) and (7.3) respectively, where the latter two equations are obtained with the help of (2.7) and (2.9). Moreover, slightly simplifying equation (5.3), we deduce for $\mathcal{L}_\infty$ that

$$\mathcal{L}_\infty(u) = \frac{\pi_1}{1-\pi_1}\left[\int_0^u \overline{K}_1(u-y)dA_1(y) - \overline{K}_1(u) + \overline{A}_1(u)\right], \quad u \ge 0,$$

by which the result is demonstrated. □

For certain claim-size distributions, exact results may be derived. Such are the exponential and the combination of exponentials. We illustrate the idea through the following example.

Example 7.1. (Exponential Claim Amounts) When the claim amounts are exponentially distributed, namely, $P(y) = 1 - e^{-\mu y}, y \ge 0, \mu > 0$, explicit formulae for all participating elements are available. More specifically, $A_1(y) = A_2(y) = 1 - e^{-\mu y} = P(y)$. Since the Laplace transform of Y_1 is $\tilde{p}(s) = \mu/(s+\mu)$, we find by (2.9) that $\pi_i = \lambda/[c_i(\rho_i+\mu)]$, $i = 1, 2$.

By Lin et al. (2003), pp. 561-562, we have that $\overline{K}_i(u) = \pi_i e^{-\tau_i u}$, $i = 1, 2$, where $\tau_i = (1-\pi_i)\mu$. Moreover, $\mathcal{L}_\infty(u) = \overline{K}_1(u) = \pi_1 e^{-\tau_1 u}$ and $\Psi(u) = \frac{\mu-\tau_1}{\mu+\rho_1} e^{-(\rho_1+\tau_1)u}$. Note that $-\tau_i$ and ρ_i are the respective negative and the nonnegative roots of the Lundberg fundamental equation (2.5) with $i = 1$ and 2.

By determining Ψ and $\mathcal{L}_\infty$ we may obtain exact formulae for v and κ, which leads to an exact expression for $\mathcal{L}(u;b)$. More specifically, employing the above formulea along with equations (2.10) and (2.6) into (7.2), we obtain

$$\begin{aligned}
\kappa =& \frac{\left(\frac{\pi_2}{\pi_1}-1\right)\mathcal{L}_\infty(b)}{\frac{1-\Psi(b)}{1-\Psi(0)}e^{\rho_1 b} - \pi_2\int_0^b \frac{1-\Psi(b-y)}{1-\Psi(0)}e^{\rho_1(b-y)}dA_2(y)} \\
=& \frac{\left(1-\frac{\mu-\tau_1}{\mu+\rho_1}\right)\left(\frac{\pi_2}{\pi_1}-1\right)\pi_1 e^{-\tau_1 b}}{\left[1-\frac{\mu-\tau_1}{\mu+\rho_1}e^{-(\rho_1+\tau_1)b}\right]e^{\rho_1 b} + \pi_2 e^{\rho_1 b}\int_0^b\left[1-\frac{\mu-\tau_1}{\mu}e^{-(\rho_1+\tau_1)(b-y)}\right]e^{-\rho_1 y}de^{-\mu y}} \\
=& \frac{\rho_1+\tau_1}{\mu+\rho_1}\cdot\frac{(\pi_2-\pi_1)e^{-(\rho_1+\tau_1)b}}{1-\frac{\mu-\tau_1}{\mu+\rho_1}e^{-(\rho_1+\tau_1)b}+\frac{\mu\pi_2}{\mu+\rho_1}\left[e^{-(\mu+\rho_1)b}-1\right]+\mu\pi_2\frac{\mu-\tau_1}{\mu+\rho_1}e^{-(\rho_1+\tau_1)b}\int_0^b e^{(\tau_1-\mu)y}dy} \\
=& \frac{\rho_1+\tau_1}{\mu+\rho_1}\cdot\frac{(\pi_2-\pi_1)e^{-(\rho_1+\tau_1)b}}{1-\frac{\mu-\tau_1}{\mu+\rho_1}e^{-(\rho_1+\tau_1)b}+\frac{\mu\pi_2}{\mu+\rho_1}\left[e^{-(\mu+\rho_1)b}-1\right]+\frac{\mu\pi_2}{\mu+\rho_1}\left[e^{-(\rho_1+\tau_1)b}-e^{-(\mu+\rho_1)b}\right]} \\
=& \frac{(\pi_2-\pi_1)(\rho_1+\tau_1)e^{-(\rho_1+\tau_1)b}}{\rho_1+\tau_2+(\tau_1-\tau_2)e^{-(\rho_1+\tau_1)b}}.
\end{aligned}$$

Hence, Theorem 7.1 yields

$$\begin{aligned}
\mathcal{L}_1(u) =& \pi_1 e^{-(1-\pi_1)\mu u} + \frac{\kappa}{1-\frac{\mu-\tau_1}{\mu+\rho_1}}\left[1-\frac{\mu-\tau_1}{\mu+\rho_1}e^{-(\rho_1+\tau_1)u}\right]e^{\rho_1 u} \\
=& \left\{\pi_1 - \frac{\kappa(\mu-\tau_1)}{\rho_1+\tau_1}\right\}e^{-\tau_1 u} + \frac{\kappa(\mu-\tau_1)}{\rho_1+\tau_1}e^{\rho_1 u} \\
=& \pi_1\left\{1-\frac{\mu(\pi_2-\pi_1)e^{-(\rho_1+\tau_1)b}}{\rho_1+\tau_2+(\tau_1-\tau_2)e^{-(\rho_1+\tau_1)b}}\right\}e^{-\tau_1 u} + \frac{\mu(\pi_2-\pi_1)e^{-(\rho_1+\tau_1)b}}{\rho_1+\tau_2+(\tau_1-\tau_2)e^{-(\rho_1+\tau_1)b}}e^{\rho_1 u} \\
=& \pi_1 r(b)e^{-\tau_1 u} + [1-r(b)]e^{\rho_1 u}, \quad 0 \le u \le b,
\end{aligned}$$

where

$$r(b) = 1 - \frac{\mu(\pi_2-\pi_1)e^{-(\rho_1+\tau_1)b}}{\rho_1+\tau_2+(\tau_1-\tau_2)e^{-(\rho_1+\tau_1)b}} \tag{7.4}$$

is introduced to facilitate the further calculations by making the analogy with $q(b)$ in Example 6.1. Therefore, if we set

$$R(b) = \frac{\rho_1}{\mu+\rho_1}[1-r(b)]e^{-\mu b} + \frac{\mu}{\mu+\rho_1}[1-r(b)]e^{\rho_1 b} + r(b)e^{-\tau_1 b},$$

in a similar way as we obtained equation (6.7), we have

$$h(u) = \pi_2 R(b) e^{-\mu(u-b)}, \quad u > b.$$

Furthermore, it becomes obvious that the relation between equations (6.7) and (6.8) yields here

$$\mathcal{L}_2(u) = \pi_2 R(b) e^{-\tau_2(u-b)}, \quad u > b.$$

Finally,

$$\mathcal{L}(u;b) = \begin{cases} \mathcal{L}_1(u) = [1-r(b)]e^{\rho_1 u} + \pi_1 r(b) e^{-\tau_1 u}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le u \le b \\ \\ \mathcal{L}_2(u) = \pi_2 \left\{ \dfrac{\rho_1}{\mu+\rho_1}[1-r(b)]e^{-\mu b} + \dfrac{\mu}{\mu+\rho_1}[1-r(b)]e^{\rho_1 b} + r(b)e^{-\tau_1 b} \right\} e^{-\tau_2(u-b)}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u > b, \end{cases}$$

where $r(b)$ is provided by equation (7.4).

The above result is very similar to that in Example 6.1 as expected. □

The Laplace transform of the time of ruin when the individual claim amounts follow a combination of exponentials can be derived similarly to that in Example 6.2. Using the same notation as in Example 6.2, we have for $i = 1$ and 2,

$$\overline{A}_i(y) = \sum_{j=1}^{n} \omega_{ij}^* \, e^{-\mu_j y}, \quad y \ge 0,$$

where

$$\omega_{ij}^* = \frac{\omega_j/(\mu_j+\rho_i)}{\sum_{l=1}^n \omega_l/(\mu_l+\rho_i)}, \quad j = 1, 2, \cdots, n.$$

See Example 3.1 of Lin and Willmot (1999).

Furthermore,

$$\overline{K}_i(u) = \sum_{j=1}^{n} C_{ij} e^{-\beta_{ij} u}, \quad u \ge 0,$$

where $0 < \beta_{i1} < \beta_{i2} < \cdots < \beta_{in}$ are the roots of $\sum_{j=1}^n [\omega_{ij}^* \mu_j/(\mu_j - \beta)] = 1 + \theta_i,\ i = 1, 2$, and

$$C_{ij} = \left[\sum_{l=1}^{n} \frac{\omega_{il}^*}{\mu_l - \beta_{ij}} \right] \Bigg/ \left[\sum_{l=1}^{n} \frac{\omega_{il}^* \mu_l}{(\mu_l - \beta_{ij})^2} \right], \quad i = 1, 2,\ j = 1, 2, \ldots, n.$$

Identities (6.9) and (6.10) still hold for β_{ij} and C_{ij} as in Example 6.2. See also Example 4.1 of Lin and Willmot (1999). Thus, the rest of the derivation is exactly the same as in Example 6.2 and we omit the details.

8 The surplus before ruin and the deficit at ruin

In this section we discuss the defective joint and marginal cumulative distribution functions of the surplus before ruin and the deficit at ruin, as well as the distribution of the amount of the first surplus drop below the initial level u.

We set $\delta = 0$ and $w(X_1, X_2) = I(X_1 \leq x_1)I(X_2 \leq x_2)$ for fixed $x_1, x_2 \geq 0$. The reader may refer to Section 7 to recall implications of the discount free assumption. Also, we denote the respective defective cumulative distribution functions resulting from $m(u; b)$, $m_1(u)$, $m_2(u)$, and $m_\infty(u)$ by $F(x_1, x_2|u; b)$, $F_1(x_1, x_2|u; b)$, $F_2(x_1, x_2|u; b)$, and $F_\infty(x_1, x_2|u)$.

By the definition of ζ we have $\zeta(u) = I(u \leq x_1)[\overline{P}(u) - \overline{P}(u + x_2)]$. Therefore, it can be easily verified that equation (5.5) reduces to the result provided by Corollary 5.1 in Lin and Willmot (1999). Furthermore, the marginal distributions of the surplus immediately before ruin and the deficit at ruin provided by equations (8.1) and (8.2) below coincide with (5.6) and (5.7) in Lin and Willmot (1999). We recall these results with only notational changes. The joint distribution satisfies

$$F_\infty(x_1, x_2|u) = \begin{cases} \dfrac{1}{\theta_1}\Big\{[P_e(x_1 + x_2) - P_e(x_1) - P_e(x_2)]\psi_{1,\infty}(u) \\ \quad + \dfrac{1}{\mathbb{E}\{Y_1\}}\displaystyle\int_0^{x_1} \psi_{1,\infty}(u - y)[\overline{P}(y) - \overline{P}(y + x_2)]dy\Big\}, & 0 \leq x_1 \leq u \\ \dfrac{1}{\theta_1}\Big\{(1 + \theta_1)[\psi_{1,\infty}(u) - \psi_{1,\infty}(u + x_2)] \\ \quad + \psi_{1,\infty}(u)[P_e(x_1 + x_2) - P_e(x_1) - P_e(x_2)] \\ \quad + \dfrac{1}{\mathbb{E}\{Y_1\}}\displaystyle\int_0^{x_2} \psi_{1,\infty}(u + x_2 - y)\overline{P}(y)dy \\ \quad - [P_e(x_1 + x_2) - P_e(x_1)]\Big\}, & x_1 > u. \end{cases}$$

The respective marginal distributions are

$$F_{\infty,X_1}(x_1|u) = \begin{cases} \dfrac{1}{\theta_1}\Big\{ - P_e(x_1)\psi_{1,\infty}(u) + \dfrac{1}{\mathbb{E}\{Y_1\}}\displaystyle\int_0^{x_1} \psi_{1,\infty}(u - y)\overline{P}(y)dy\Big\}, & 0 \leq x_1 \leq u \\ \dfrac{1}{\theta_1}\left\{\theta_1\psi_{1,\infty}(u) + [1 + \psi_{1,\infty}(u)]\overline{P}_e(x_1)\right\}, & x_1 > u, \end{cases} \tag{8.1}$$

and

$$
\begin{aligned}
F_{\infty,X_2}(x_2|u) = &\frac{1}{\theta_1}\Big\{(1+\theta_1)[\psi_{1,\infty}(u) - \psi_{1,\infty}(u+x_2)] - \psi_{1,\infty}(u)P_e(x_2) \\
&+ \frac{1}{\mathbb{E}\{Y_1\}}\int_0^{x_2} \psi_{1,\infty}(u+x_2-y)\overline{P}_e(y)dy\Big\}.
\end{aligned} \tag{8.2}
$$

Hence, equations (5.4) to (5.8) produce analytical expressions for $F(x_1, x_2|u; b)$.

Theorem 8.1. *Let*

$$
\sigma(x_1, x_2|u) = \overline{P}_e(u) + \overline{P}_e(x_1+x_2) - \overline{P}_e(x_1) - \overline{P}_e(u+x_2), \quad u \geq 0.
$$

Then $F(x_1, x_2|u; b)$ *is given by*

$$
\begin{cases}
F_1(x_1, x_2|u; b) = F_\infty(x_1, x_2|b) + \kappa[1 - \psi_{1,\infty}(u)], & 0 \leq u \leq b, \\
F_2(x_1, x_2|u; b) = -\dfrac{1+\theta_2}{\theta_2}\left[\displaystyle\int_b^u \psi_{2,\infty}(u-y)dh(y) + \psi_{2,\infty}(u-b)h(b)\right] + h(u), & u > b,
\end{cases}
$$

with

$$
\kappa = \frac{(\theta_1 - \theta_2)F_\infty(x_1, x_2|b) - \overline{P}_e(b) + I(b \leq x_1)\sigma(x_1, x_2|b)}{(\theta_1 - \theta_2)\psi_{1,\infty}(b) + \theta_2},
$$

and

$$
h(u) = \frac{1}{1+\theta_2}\int_{u-b}^u F_1(x_1, x_2|u-y; b)dP_e(y) + \frac{I(u \leq x_1)}{1+\theta_2}\sigma(x_1, x_2|u), \quad u > b.
$$

Letting either $x_2 \to \infty$, or $x_1 \to \infty$ we derive either the defective marginal c.d.f. of the surplus before ruin, or the defective marginal c.d.f. of the deficit at ruin, respectively.

Corollary 8.2. *The defective marginal c.d.f. of the surplus immediately before ruin* $F_{X_1}(x_1|u; b)$ *is given by*

$$
\begin{cases}
F_{1,X_1}(x_1|u; b) = F_{\infty,X_1}(x_1|b) + \kappa[1 - \psi_{1,\infty}(u)], & 0 \leq u \leq b, \\
F_{2,X_1}(x_1|u; b) = -\dfrac{1+\theta_2}{\theta_2}\left[\displaystyle\int_b^u \psi_{2,\infty}(u-y)dh(y) + \psi_{2,\infty}(u-b)h(b)\right] + h(u), & u > b,
\end{cases}
$$

where

$$
\kappa = \frac{(\theta_1 - \theta_2)F_{\infty,X_1}(x_1|b) - \overline{P}_e(b) + I(u \leq x_1)[\overline{P}_e(b) - \overline{P}_e(x_1)]}{(\theta_1 - \theta_2)\psi_{1,\infty}(b) + \theta_2},
$$

and

$$
h(u) = \frac{1}{1+\theta_2}\int_{u-b}^u F_{1,X_1}(x_1|u-y; b)dP_e(y) + \frac{I(u \leq x_1)}{1+\theta_2}[\overline{P}_e(u) - \overline{P}_e(x_1)], \quad u > b.
$$

In a similar manner, we obtain a respective result regarding the severity at ruin.

Corollary 8.3. *The defective marginal c.d.f. of the deficit at ruin $F_{X_2}(x_2|u;b)$ may be obtained through*

$$\begin{cases} F_{1,X_2}(x_2|u;b) = F_{\infty,X_2}(x_2|b) + \kappa[1-\psi_{1,\infty}(u)], & 0 \le u \le b, \\ F_{2,X_2}(x_2|u;b) = -\dfrac{1+\theta_2}{\theta_2}\left[\displaystyle\int_b^u \psi_{2,\infty}(u-y)dh(y) + \psi_{2,\infty}(u-b)h(b)\right] + h(u), & u > b, \end{cases}$$

with

$$\kappa = \frac{(\theta_1-\theta_2)F_{\infty,X_2}(x_2|b) - \overline{P}_e(b+x_2)}{(\theta_1-\theta_2)\psi_{1,\infty}(b)+\theta_2},$$

and

$$h(u) = \frac{1}{1+\theta_2}\int_{u-b}^u F_{1,X_2}(x_2|u-y;b)dP_e(y) + \frac{1}{1+\theta_2}[\overline{P}_e(u) - \overline{P}_e(u+x_2)], \quad u > b.$$

When the barrier level is $b-u$ and the initial surplus is 0, Corollary 8.3 yields the distribution of the first surplus drop below the initial level.

Corollary 8.4. *The distribution of the first surplus drop below the initial level is given by*

$$G(x_2|0) = F_{\infty,X_2}(x_2|b-u) + \frac{\theta_1}{1+\theta_1}\kappa, \quad 0 \le u \le b,$$

with

$$\kappa = \frac{(\theta_1-\theta_2)F_{\infty,X_2}(x_2|b-u) - \overline{P}_e(b-u+x_2)}{(\theta_1-\theta_2)\psi_{1,\infty}(b-u)+\theta_2}.$$

It is possible to obtain joint and marginal moments of the surplus before ruin and the deficit at ruin. We set $w(x_1,x_2) = x_1^h x_2^l$ for an arbitrary pair of nonnegative integers h and l. Denote then by $F^{h,l}(u;b)$, $F_1^{h,l}(u)$, $F_2^{h,l}(u)$, and $F_\infty^{h,l}(u)$ the joint moments resulting from $m(u;b)$, $m_1(u)$, $m_2(u)$, and $m_\infty(u)$ respectively with the specific choice of w.

In order to simplify the expressions to follow, let n be a nonnegative integer and define recursively the nth order equilibrium tail of P to be

$$\overline{P}_{e,n}(y) = 1 - P_{e,n}(y) = \frac{\int_y^\infty \overline{P}_{e,n-1}(t)dt}{\int_0^\infty \overline{P}_{e,n-1}(t)dt},$$

with $\overline{P}_{0,e}(t) = \overline{P}(t)$ and $\overline{P}_{1,e}(t) = \overline{P}_e(t)$. If in addition, we define $p_n = \int_0^\infty y^n dP(y)$ to be the nth moment of Y_1, we can apply formula (2.4) in Lin and Willmot (2000), namely,

$$\overline{P}_{e,n}(y) = \frac{1}{p_n}\int_y^\infty (t-y)^n dP(t)$$

to find

$$\int_u^\infty \zeta(t)dt = p_l \int_u^\infty t^h \overline{P}_{e,l}(t)dt$$

for this particular form of w. Therefore, equations (5.4) to (5.8) yield the following.

Theorem 8.5.

$$F^{h,l}(u;b) = \begin{cases} F_1^{h,l}(u) = F_\infty^{h,l}(u) + \kappa[1 - \psi_{1,\infty}(u)], & 0 \le u \le b, \\ F_2^{h,l}(u) = -\dfrac{1+\theta_2}{\theta_2}\left[\displaystyle\int_b^u \psi_{2,\infty}(u-y)dh(y) + \psi_{1,\infty}(u-b)h(b)\right] + h(u), & u > b, \end{cases} \tag{8.3}$$

where

$$F_\infty^{h,l}(u) = \frac{\theta_1 p_l}{p_1}\left\{\int_0^u y^h[\psi_{1,\infty}(u-y) - \psi_{1,\infty}(u)]\overline{P}_{e,l}(y)dy + \int_u^\infty y^h[1-\psi_{1,\infty}(u)]\overline{P}_{e,l}(y)dy\right\},$$

$$\kappa = \frac{(\theta_2 - \theta_1)F_\infty^{h,l}(b) - \overline{P}_e(b) + \frac{p_l}{p_1}\int_b^\infty t^h \overline{P}_{e,l}(t)dt}{(\theta_2-\theta_1)\psi_{1,\infty}(b) + \theta_2},$$

and

$$h(u) = \frac{1}{1+\theta_2}\int_{u-b}^u F_1^{h,l}(u-y)dP_e(y) + \frac{\lambda p_l}{c_2}\int_u^\infty t^h \overline{P}_{e,l}(t)dt, \quad u > b. \tag{8.4}$$

To obtain marginal moments of the surplus immediately before ruin or the deficit at ruin, simply set $l = 0$ or $h = 0$ in equations (8.3) to (8.4).

We want to point out that further simplification for the results in Theorems 8.1 and 8.5, Corollaries 8.2, and 8.3 is possible. For example we may use the explicit expressions obtained in Lin and Willmot (2000), pp. 29 and 32 for the joint and marginal moments $F_\infty^{h,l}, F_\infty^h$ and F_∞^l in Theorem 8.5. However, since the simplification does not provide new insight, we omit it in this paper.

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