On the Optimality of Proportional Reinsurance

I. Lampaert* and J.F. Walhin*$

*Secura
$ Institut des Sciences Actuarielles, Université Catholique de Louvain

Secura
Avenue des Nerviens, 9-31
B-1040 Bruxelles
Belgium
phone : + 32 2 504 82 22
fax : + 32 2 504 82 00
e-mail : jfw@secura-re.com

Abstract:
Proportional reinsurance is often thought to be a very simple method of covering the portfolio of an insurer. Theoreticians have not been particularly interested in analysing the optimality properties of these types of reinsurance covers. In this paper, we will use a real-life insurance portfolio in order to compare four proportional structures: quota share reinsurance, variable quota share reinsurance, surplus reinsurance and surplus reinsurance with a table of lines. We adopt the point of view of the ceding company and propose ways to optimize the proportional covers of the primary insurer.

Keywords: Proportional Reinsurance, Quota Share Reinsurance, Variable Quota Share, Surplus Reinsurance, Table of Lines, Optimality, RORAC, de Finetti, Individual Risk Model.

1. INTRODUCTION

It is well-known in literature that non-proportional reinsurance is more efficient compared to proportional reinsurance. See e.g. Vermandele and Denuit (1998) where it is proved that the retention of an insurer covered by an excess of loss treaty is smaller in the stop-loss order than the retention covered by any other reinsurance of the individual type (i.e. compensation on a claim by claim basis) under the hypothesis that the expected retained loss is the same in both situations as well as the loading of the reinsurer. Vermandele and Denuit (1998) also show that the retention of an insurer covered by a stop-loss treaty is smaller in the stop-loss order than the retention covered by any other reinsurance treaty, under the hypothesis that the expected retained loss is the same in both situations as well as the loading of the reinsurer.

At first sight, it therefore seems that proportional reinsurance is less efficient than excess of loss and stop-loss covers, which are of the non proportional type.

In practice this is not the case for multiple reasons such as:

1. stop-loss covers are difficult to obtain due to the possible moral hazard behaviour that the ceding company may adopt after buying such a cover
2. stop-loss covers are extremely difficult to price by reinsurers
3. the loading for a stop-loss cover will clearly differ from a proportional cover (e.g. due to the first two points)
4. excess of loss covers are sometimes difficult to price
5. the loading for an excess of loss cover will also differ from a proportional cover.

Proportional covers can be quite desirable and it is worth analysing their optimality properties.

The main objective of this paper is to illustrate by means of a numerical example that the traditional belief that surplus treaties with a table of lines are better (more efficient) than standard surplus treaties is wrong. We will take this opportunity to compare all the proportional types of reinsurance.

The rest of the paper is organized as follows. Section 2 describes the data we will use for the numerical application. Section 3 explains how the individual risk model will be used as well as approximations of the aggregate claims distribution within the individual risk model. Section 4 describes the four types of proportional reinsurance to be compared in section 5 where we will look for optimal reinsurance structures. Section 6 concludes.

2. DATA

For the calculations a real-life data set will be used. It is obtained from one of the leading Belgian insurance companies and contains $27\,551$ fire policies, covering industrial risks.

The $27\,551$ policies are divided into four classes ($j=1,2,3,4$), depending on their frequency (q_{ij}) as well as their relative claims severity (X_{ij}), $i=1,\ldots,n_j$ where n_j is the number of policies in class j. Knowing the sum insured SI_{ij}, we can obtain the loss amount: $L_{ij}=SI_{ij}\times X_{ij}$. We will assume the X_{ij} to be identically distributed within a given risk class ($j=1,2,3,4$) : $X_{ij}\approx X_j$, $i=1,\ldots,n_j$, $j=1,2,3,4$. We also assume that the probability of having a loss is identical within a class : $q_{ij}=q_j$, $i=1,\ldots,n_j$, $j=1,2,3,4$.

For the density of X_j we will use the MBBEFD distribution class introduced by Bernegger (1997). Using the following notations

$$b(c)=e^{3.1-0.15c(1+c)}$$
$$g(c)=e^{c(0.78+0.12c)}$$

we assume the density function of X_j to be

$$f(x)=\frac{(b-1)(g-1)\ln(b)b^{1-x}}{\left((g-1)b^{1-x}+(1-gb)\right)^2},0\le x<1$$
$$f(1)=\frac{1}{g}.$$

We then have a family of distributions indexed by the parameter c. According to Bernegger (1997), $c=2,3,4,5$ corresponds to the Swiss Re exposure curves 2, 3, 4 and the Lloyd's industrial exposure curve respectively. We will assume that we have the following characteristics for our portfolio:

Class	q	c
1	0.75%	2
2	1.00%	3
3	1.25%	4
4	1.50%	5

Table 2.1 Claims characteristics of the portfolio

Regarding the sum insured, we have the following information:

$Class(j)$	n	$\mu_j(SI)$	$\sigma_j(SI)$	$\gamma_j(SI)$
1	3933	13457022	10752926	8.51
2	17472	12034729	7960092	2.23
3	3121	11826858	9119825	4.62
4	3025	10879648	7826747	11.98

Table 2.2 Characteristics of the sums insured

where

$$\mu_j(SI) = \frac{\sum_{i=1}^{n_j} SI_{ij}}{n_j}$$

= expected insured sum

$$\sigma_j(SI) = \sqrt{\frac{\sum_{i=1}^{n_j} (SI_{ij} - \mu_j(SI))^2}{n_j}}$$

=standard deviation of insured sum

$$\gamma_j(SI) = \frac{\sum_{i=1}^{n_j} (SI_{ij} - \mu_j(SI))^3}{\sigma_j^3(SI)}$$

= skewness insured sum.

3. INDIVIDUAL RISK MODEL AND APPROXIMATIONS

Clearly our portfolio fits into the definition of the individual risk model (see e.g. Klugman et al. (1998)). We have $n = \sum_{j=1}^{4} n_j$ policies with a different sum insured, which are divided into four classes according to their claims behaviour (frequency and severity).

Therefore the aggregate claims amount is given by

$$S^{ind} = \sum_{j=1}^{4} \sum_{i=1}^{n_j} D_{ij} L_{ij}$$

where

1. D_{ij} is the indicator function taking value 1 when there is a claim and 0 when there is no claim. We have $P[D_{ij} = 1] = P[D_j = 1] = q_j$.
2. $L_{ij} = SI_{ij} X_{ij}$ is the conditional loss value.
3. $S_{ij} = D_{ij} L_{ij}$ is the loss associated to policy ij .

Obtaining the exact distribution of S^{ind} is possible by using recursive formulae (see e.g. Dhaene and Vandebroek (1995)) but the computing time will be very long due to the size of the portfolio. Moreover a discretization of distribution of L_{ij} is required.

An approximation of the individual risk model is provided by the collective risk model (see e.g. Klugman et al. (1998)) leading to the use of the Panjer recursive formula (see Panjer (1981)). Once again the computing time will be long and discretization will be required.

In this paper, as the porfolio is large, and its skewness less than 2 (see further for the calculations) we will concentrate on a parametric approximation, namely the shifted gamma distribution, that will reproduce the first three moments of the original distribution. We therefore need to obtain the first three moments of S^{ind} .

The shifted gamma distribution (S) (see e.g. Dufresne and Niederhauser (1997)) has the form

$$S \approx Z + x_0$$

where $Z \approx Gam(\alpha,\beta)$, i.e.

$$f_Z(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \quad , \quad x > 0$$

$$F_Z(x) = \int_0^x f_Z(s)ds$$

where $\Gamma(x)$ is the gamma function. By abuse of notation, we will also write $F(\alpha,\beta,x)$ the cumulative density function of Z.

Central moments are given by

$$\mu = \sum_{j=1}^{4} \left[q_j EX_j \right] \sum_{i=1}^{n_j} SI_{ij}$$

$$\mu_2 = \sum_{j=1}^{4} \left[q_j VarX_j + q_j(1-q_j)(EX_j)^2 \right] \sum_{i=1}^{n_j} SI_{ij}^2$$

$$\mu_3 = \sum_{j=1}^{4} \left[q_j EX_j^3 - 3q_j^2 EX_j EX_j^2 + 2q_j^3 (EX_j)^3 \right] \sum_{i=1}^{n_j} SI_{ij}^3$$

Using numerical integration, it is possible to obtain the first three moments of X_j, as a function of the parameter c:

Class	EX_j	EX_j^2	EX_j^3
1	0.2260909	0.1623865	0.1474579
2	0.0871796	0.0479373	0.0407141
3	0.031852	0.0123161	0.0094975
4	0.0121457	0.0030479	0.0020178

Table 3.1 Moments of X_j

An analytical formula exists for EX : $EX = \frac{\ln(gb)(1-b)}{\ln(b)(1-gb)}$ but not for higher moments. The $\sum_{i=1}^{n_j} SI_{ij}^x, x = 1,2,3$ terms are easily obtained from table 2.2.

From this we can obtain the mean (μ), the standard deviation (σ), the coefficient of variation ($CV = \frac{\sigma}{\mu}$) and the skewness ($\gamma = \frac{E(S-\mu)^3}{\sigma^3}$) of S^{ind} :

$$\mu = 293\,751\,934$$
$$\sigma = 57\,364\,022$$
$$CV = 0.20$$
$$\gamma = 0.6$$

The corresponding shifted gamma approximation has the following parameters:

$$\alpha = \frac{4}{\gamma^2} = 10.44$$
$$\beta = \frac{2}{\gamma\sigma} = 5.63 10^{-8}$$
$$x_0 = \mu - \frac{2\sigma}{\gamma} = 108\,404\,392$$

4. PROPORTIONAL REINSURANCE

Proportional reinsurance is the easiest way of covering an insurance portfolio. In proportional reinsurance, the ceding company and the reinsurer agree on a cession percentage, say τ_i, for each policy in portfolio. The premium corresponding to the policy i, say P_i, is then shared proportionally between the insurer and the reinsurer. The reinsurer receives $\tau_i P_i$ whereas the insurer keeps the premium $(1-\tau_i)P_i$. If S_i is a claim hitting policy i, the reinsurer is liable for $\tau_i S_i$ whereas the insurer retains $(1-\tau_i)S_i$.

Clearly the way a proportional reinsurance works is extremely simple. Moving to the way of fixing the cession percentage τ_i, we can distinguish between four subtypes of proportional reinsurance: quota share reinsurance, variable quota share reinsurance, surplus reinsurance and surplus reinsurance with a table of lines.

Note that proportional reinsurance is sometimes called pro-rata reinsurance.

We will use the following notations:

- S_{ij} is the loss associated with policy ij.

- $S = \sum_{j=1}^{4} \sum_{i=1}^{n_j} S_{ij}$ is the aggregate loss of the insurer.
- $S^{Re} = \sum_{j=1}^{4} \sum_{i=1}^{n_j} \tau_{ij} S_{ij}$ is the aggregate liability of the reinsurer.
- $S^{R} = \sum_{j=1}^{4} \sum_{i=1}^{n_j} (1-\tau_{ij}) S_{ij}$ is the aggregate loss in retention when a reinsurance cover is bought.
- P_{ij} is the premium associated with policy ij.
- $P = \sum_{j=1}^{4} \sum_{i=1}^{n_j} P_{ij}$ is the total premium of the insurer.
- $P^{Re} = \sum_{j=1}^{4} \sum_{i=1}^{n_j} \tau_{ij} P_{ij}$ is the total ceded premium.
- $P^{R} = \sum_{j=1}^{4} \sum_{i=1}^{n_j} (1-\tau_{ij}) P_{ij}$ is the total retained premium.

It is clear that only the risk premium has to be considered. In practice the insurer cedes on the basis of the commercial premium and the reinsurer pays a reinsurance commission representing the management expenses and acquisition costs of the ceding company. To keep things simple, we will always refer to the risk premium in the following and not to the reinsurance commission.

4.1 Quota Share Reinsurance

In quota share reinsurance τ_i is the same for the whole insurance portfolio. Quota share reinsurance is therefore extremely simple as the cession percentage does not vary among policies: we note it as τ. As a consequence the administration of a quota share treaty is straightforward: it suffices to obtain the total premium and the total claims in order to share the premium and the claims with the reinsurer. Quota share reinsurance is of the individual type (i.e. the reinsurance compensation applies claim by claim) and of the global type (i.e. the reinsurance compensation applies on the yearly aggregate loss) at the same type:

$$S^{Re} = \sum_{j=1}^{4} \sum_{i=1}^{n_j} \tau S_{ij} = \tau \sum_{j=1}^{4} \sum_{i=1}^{n_j} S_{ij} = \tau S.$$

Quota share reinsurance has a nice property if we compare its use to the use of the allocated capital (u).

Let ε be the ruin probability without quota share reinsurance:

$$\varepsilon = P[S > u + P].$$

Let ε_R be the ruin probability after quota share reinsurance:

$$\begin{aligned}\varepsilon_R &= P[(1-\tau)S > u + (1-\tau)P] \\ &= P[S > \frac{u}{1-\tau} + P] \\ &< \varepsilon.\end{aligned}$$

We observe that buying a quota share treaty has the same effect as increasing the economic capital in the same proportion as the cession percentage.

Now let us analyse the retained risk of a portfolio covered by a quota share treaty

$$\begin{aligned}ES^R &= (1-\tau)ES \\ VarS^R &= (1-\tau)^2 VarS \\ \sigma(S^R) &= (1-\tau)\sigma(S) \\ CV(S^R) &= CV(S) \\ E(S^R - ES^R)^3 &= (1-\tau)^3 E(S-ES)^3 \\ \gamma(S^R) &= \gamma(S)\end{aligned}$$

Here we can observe that the variability and the skewness of the retained risk is the same as if there were no quota share reinsurance. Obviously quota share reinsurance does not provide a reduction in the relative homogeneity of the portfolio.

It is nevertheless very much used for multiple reasons such as

- Financing management and acquisition costs by means of the reinsurance commission (in case of a new product or a start-up insurance company).
- Reinsurance against underpricing (new classes of business). It limits the danger of new (unknown) risks.
- Reduction of the required solvency margin.
- Compensation for less balanced treaties of the cedant.

Note that quota share reinsurance is sometimes referred to as participating reinsurance.

4.2 Variable Quota-Share Reinsurance

Sometimes, the cession percentage may vary within the portfolio. This is called variable quota share reinsurance. In our example, we will assume that the percentage may vary in function of the class of risk. This is equivalent to analysing four different quota share treaties.

We then have the following relations:

$$ES^R = \sum_{j=1}^{4} (1-\tau_j) ES_j$$

$$VarS^R = \sum_{j=1}^{4} (1-\tau_j)^2 VarS_j$$

$$\sigma(S^R) = \sqrt{\sum_{j=1}^{4} (1-\tau_j)^2 VarS_j}$$

$$CV(S^R) \neq CV(S)$$

$$E(S^R - ES^R)^3 = \sum_{j=1}^{4} (1-\tau_j)^3 E(S_j - ES_j)^3$$

$$\gamma(S^R) \neq \gamma(S)$$

It becomes impossible to compare the coefficient of variation and the skewness analytically. This will be done numerically.

4.3 Surplus Reinsurance

In surplus reinsurance the cession percentage is a function of both the sum insured and the line, or retention, chosen by the ceding company.

The line (R) is the maximal amount that the insurer wants to pay in case of a loss. If one wants to make use of proportional reinsurance and of the property that the maximal loss will never be larger than the line, the cession percentage must be defined as

$$\tau_{ij} = \max\left(0, 1 - \frac{R}{SI_{ij}}\right).$$

The retention percentage is

$$(1-\tau_{ij}) = \min\left(1, \frac{R}{SI_{ij}}\right).$$

In case of a total loss, the retained loss is

$$\min\left(1, \frac{R}{SI_{ij}}\right) \times SI_{ij} = SI_{ij} if SI_{ij} < R$$

$$\min\left(1, \frac{R}{SI_{ij}}\right) \times SI_{ij} = R if SI_{ij} > R$$

It is clear that surplus reinsurance is appealing from an optimality point of view. In surplus reinsurance, the loss amount may not exceed the line. Furthermore, the smallest risks are not reinsured. Therefore, one feels that the retained risk will be more homogeneous than it is in case of a quota share reinsurance.

The retained risk has the following central moments:

$$S^R = \sum_{j=1}^{4} \sum_{i=1}^{n_j} (1-\tau_{ij}) D_{ij} L_{ij}$$

$$\mu = \sum_{j=1}^{4} \left[q_j EX_j \right] \sum_{i=1}^{n_j} (1-\tau_i) SI_{ij}$$

$$\mu_2 = \sum_{j=1}^{4} \left[q_j VarX_j + q_j (1-q_j)(EX_j)^2 \right] \sum_{i=1}^{n_j} (1-\tau_i)^2 SI_{ij}^2$$

$$\mu_3 = \sum_{i=1}^{4} \left[q_j EX_j^3 - 3q_j^2 EX_j EX_j^2 + 2q_j^3 (EX_j)^3 \right] \sum_{i=1}^{n_j} (1-\tau_i)^3 SI_{ij}^3$$

It is not possible to make analytical comparisons with these formulae. We will therefore concentrate on the numerical application in order to make further comments.

One should note that surplus reinsurance is far more expensive from an administrative point of view since each policy must be closely examined in order to compute the ceded premium and the possible recovery from the reinsurer, based on its own cession percentage, which is a function of the insured sum.

Note that surplus reinsurance is sometimes referred to as surplus share reinsurance.

4.4 Surplus Reinsurance with a Table of Lines

We now move on to surplus reinsurance with a table of lines. In the above definition of surplus reinsurance, the same retention R is used for the whole portfolio. In practice however, it may happen that a surplus programme is presented with a table of lines. This means that a retention is fixed per group of similar risks. In this way the portfolio that the ceding company retains is qualitatively more homogeneous. It is especially the fire risks in an insurer's portfolio that may differ in quality. Determining factors are the location of the risk, the building's construction, its use, the loss prevention and protection measures, ... The quality of the risk is translated into a frequency and severity distribution: the better the risk, the smaller the frequency and the less dangerous the claims severity. So we have four classes of risks with different characteristics. If we choose the same retention for the entire portfolio as we described above, the expected loss per risk would not be homogeneous. With the same retention the yearly expected loss of the ceding company would depend upon the kind of risk that has been affected. We therefore choose a different retention per class, in order to make the expected loss per risk independent of the kind of risk. As a consequence the insurer is able to retain more of the good risks and less of the bad risks. For the reinsurer however there is always the risk that only the dangerous risks are transferred. When the cedant's rate is wrong, this implies a danger to the reinsurer. This phenomenon is called antiselection.

Thus, in surplus reinsurance with a table of lines, the cession percentage is

$$\tau_{ij} = \max\left(0,1-\frac{R_j}{SI_{ij}}\right)$$

where the line R_i may vary among the policies.

In order to fix the lines, certain practitioners use one of the following methods with no real justification.

A first method to construct a table of lines is to determine a retention for each class of business by aiming at an equal maximum loss throughout the entire portfolio. This means that the lines will be such that

$$R_1 \times q_1 = R_2 \times q_2 = R_3 \times q_3 = R_4 \times q_4 .$$

This is the method of the inverse claim frequency.

A second method takes into account not only the frequency but also the claims severity. This table of lines is constructed in order to reach the same average loss for all policies, contrary to the same maximum loss of the first method. This means that the lines will be such that

$$R_1 \times rate_1 = R_2 \times rate_2 = R_3 \times rate_3 = R_4 \times rate_4 .$$

where $rate_j = q_j EX_j$. This is the method of the inverse rate.

5. OPTIMAL REINSURANCE

In this section we will compare the original portfolio with the retained portfolio after a proportional cession of the four types described in the previous sections. We will use two criteria:

1. a de Finetti criterion, i.e. we will minimize the variance of the retained loss under the constraint that the expected gain is fixed.
2. a RORAC criterion, i.e. we will maximize the return on risk adjusted capital of the retained risk.

We will assume that the insurer is using a loading ξ . That loading contains only the capital charge. All administrative expenses must be charged on top of that loading. We will also assume that the reinsurer is using a loading ξ^{Re} . That loading includes the capital charge of the reinsurer as well as the administrative expenses. It is clear that the insurer pays for the administrative expenses of the reinsurer in the reinsurance premium. For the numerical application, we will use $\xi = 5\%$ and $\xi^{Re} = 7\%$.

5.1 de Finetti's Type Results

Following de Finetti (1940), we will minimize the variance of the gain of the retained portfolio by assuming that the four subportfolios are covered by a quota-share with a possible different cession rate. The gain of the retained portfolio is

$$Z(\tau) = \sum_{j=1}^{4} \sum_{i=1}^{n_j} ((1+\xi_{ij})ES_{ij} - (1+\xi_{ij}^{Re})\tau_{ij} ES_{ij} - (1-\tau_{ij})S_i).$$

where τ is the vector of cession percentages $\{\tau_{11},\ldots,\tau_{n_1 1},\tau_{12}\ldots,\tau_{n_2 2},\tau_{13},\ldots,\tau_{n_3 3},\tau_{14},\ldots,\tau_{n_4 4}\}$. The de Finetti problem is the following:

$$\min_{\tau} VarZ(\tau)$$

under the constraint that

$$EZ(\tau) = k.$$

de Finetti (1940) showed that the solution is given by

$$\tau_{ij} = \max\left(0, 1 - \frac{b\xi_{ij}^{Re} ES_{ij}}{VarS_{ij}}\right) \quad , \quad j = 1, \cdots, 4 \quad , \quad i = 1, \ldots, n_j,$$

where b is a constant given by the condition $EZ(\tau) = k$.

Assuming that we want to keep an expected gain equal to $5\,000\,000$, the solution provided by de Finetti is the following:

Case	τ_1	τ_2	τ_3	τ_4	E	σ
1	64.98%	41.75%	24.05%	0.00%	5 000 000	29 173 126

Table 5.1. Optimal variable quota share treaty with expected gain = $5\,000\,000$

If we cover the whole portfolio by a uniform quota share, i.e. with the same cession percentage for all risks in all four classes, we obtain

Case	τ_1	τ_2	τ_3	τ_4	E	σ
2	47.11%	47.11%	47.11%	47.11%	5 000 000	30 338 327

Table 5.2 Quota share treaty with expected gain = $5\,000\,000$

The volatility is larger than it is in case of the variable quota share treaty, which shows the optimality of de Finetti's result.

Now let us move on to surplus reinsurance. We will analyse the following cases:

3. surplus with one line
4. surplus with table of lines corresponding to the quota share treaty
5. surplus with table of lines corresponding to the variable quota share (the lines are chosen such that the global cession for the subportfolio is the same for both covers)
6. surplus with table of lines obtained by the inverse rate method
7. surplus with table of lines obtained by the inverse frequency method

Case	R_1	R_2	R_3	R_4	E	σ
3	7304175	7304175	7304175	7304175	5000000	24858743
4	7989249	7065148	6963402	6167660	5000000	25111701
5	4886924	8036122	12533770	333398280	5000000	24700617
6	4246111	8258874	18083771	39520454	5000000	24913398
7	9084700	6813525	5450820	4542350	5000000	25693734

Table 5.3 Comparison of surplus treaties with expected gain = 5000000

We can make the following comments:

1. the surplus treaty (case 3) is optimal compared to the surplus treaty with table of lines obtained by the practitioners method (cases 6 and 7). This is clearly against the traditional belief.
2. the surplus treaty corresponding to the cessions of the variable quota share treaty (case 5) is the best treaty. This is a sign that building the table of lines according to the shares of de Finetti's solution is probably more sensible than using the practitioners formula which has no theoretical justification.

By minimizing the objective function numerically, we were able to obtain two situations that are more efficient than the previous ones:

Case	R_1	R_2	R_3	R_4	E	σ
8	6001860	7940027	7514596	7483510	5000000	24689131
9	5819865	7592990	10774593	333398280	5000000	24597666

Table 5.4. Trying to obtain the optimal table of lines with expected gain = 5000000 **numerically**

Note that the objective function seems to be very flat. It is therefore difficult to make sure that the global minimum has been achieved. We observe that the two proposed solutions are very different.

In fact, it is not difficult to write the de Finetti's formulae for a surplus treaty or a surplus treaty with a table of lines.

Indeed, we have the following results:

a. Surplus treaty

$$Z(\mathbf{R}) = \sum_{j=1}^{4} \sum_{i=1}^{n_j} ((1+\xi_{ij}) ES_{ij} - (1+\xi_{ij}^{Re}) \tau_{ij} ES_{ij} - (1-\tau_{ij}) S_{ij})$$

with

$$\tau_{ij} = \max\left(0, 1 - \frac{R_{ij}}{SI_{ij}}\right).$$

Glineur and Walhin (2004) have used convex optimization to prove that the optimal lines are

$$R_{ij} = b\xi_{ij}^{Re} \frac{ED_{ij} X_{ij}}{VarD_{ij} X_{ij}}$$

where b is a constant that is determined by the constraint on the expected gain.

Clearly this result is not useful as it will not be possible from an administrative point of view to apply a different line to each policy in the portfolio. We then move on to the more interesting case of the table of lines.

b. Surplus treaty with a table of lines

$$Z(\mathbf{R}) = \sum_{j=1}^{4} \sum_{i=1}^{n_j} ((1+\xi_{ij}) ES_{ij} - (1+\xi_{ij}^{Re}) \tau_{ij} ES_{ij} - (1-\tau_{ij}) S_{i})$$

with

$$\tau_{ij} = \max\left(0, 1 - \frac{R_{j}}{SI_{ij}}\right).$$

Glineur and Walhin (2004) have used convex optimization to prove that the optimal lines are

$$R_j = b \frac{\sum_{i=1}^{n_j} \xi_{ij}^{Re} E[D_{ij} L_{ij}] SI_{ij}}{\sum_{i=1}^{n_j} Var[D_{ij} L_{ij}]} \quad , \quad j = 1, 2, 3, 4$$

where b is a constant that is determined by the constraint.

On the reasonable assumption that the X_{ij} and D_{ij} are identically distributed within the class j and that the reinsurance loading is the same for each risk within the class j, the formula is reduced to

$$R_j = b\xi_j^{Re}\frac{ED_jX_j}{VarD_jX_j}$$

where b is a constant that is determined by the constraint on the expected gain.

We obtain

Case	R_1	R_2	R_3	R_4	E	σ
10	5844858	7628597	10842395	16701348	5000000	24511489

Table 5.5 Optimal surplus treaty with a table of line with expected gain = 5000000

5.2 RORAC's Type Results

Now let us compute the RORAC (Return On Risk Adjusted Capital) for different reinsurance structures.

Let us assume that the required solvency level, RSL, is given by the Tail Value at Risk at the level $\varepsilon = 99\%$.

Using our shifted gamma approximation, we have

$$\begin{aligned} RSL &= E[S \mid S > VaR_S(\varepsilon)] \\ &= E[Z \mid Z > VaR_Z(\varepsilon)] + x_0 \\ &= \frac{\alpha}{\beta}\frac{1}{1-\varepsilon}\left(1 - F(\alpha+1, \beta, VaR_Z(\varepsilon))\right) + x_0 \end{aligned}$$

where $VaR_Z(\varepsilon) = F^{-1}(\alpha, \beta, \varepsilon)$.

The retained premium is equal to

$$P^R = (1+\xi)ES - (1+\xi^{Re})ES^{Re}.$$

The risk adjusted capital is obtained by deducting the retained premium from RSL. In other words, the risk adjusted capital is the required solvency level minus the premium that is charged to the policyholders plus the premium that is charged by the reinsurers:

$$RAC = RSL - P^R$$

and RORAC is defined as

$$RORAC = \frac{P^R - ES^R}{RAC}$$

For the direct (i.e. before any reinsurance) portfolio, we obtain the following:

$$\begin{aligned} ES = ES^R &= 293\,751\,934 \\ CV &= 0.20 \\ \gamma &= 0.62 \\ VaR &= 452\,547\,891 \\ TVaR &= 483\,141\,978 \\ P = P^R &= 308\,439\,531 \\ RAC &= 174\,702\,447 \\ RORAC &= 8.41\%. \end{aligned}$$

Now let us compare the RORAC for the reinsurance structures that have been analysed in the previous section:

Case	*CV*	γ	*TVaR*	*RORAC*
1	0.20	0.62	255 521 124	5.25%
2	0.19	0.51	248 418 187	5.68%
3	0.16	0.24	227 868 224	7.41%
4	0.16	0.25	228 686 935	7.32%
5	0.16	0.30	228 769 300	7.31%
6	0.16	0.28	228 915 033	7.29%
7	0.17	0.26	230 640 846	7.11%
8	0.16	0.25	227 430 799	7.45%
9	0.16	0.29	228 247 460	7.36%
10	0.16	0.25	226 965 212	7.51%

Table 5.6. RORAC for our 10 alternatives

We can make the following observations:

1. as for the de Finetti's criterion, the differences are not that large
2. the ranking is not exactly the same as the de Finetti's one. In particular the second best alternative under de Finetti's criterion (case 9) is now

outperformed by the classical surplus (case 3). This is due to the fact that the de Finetti's criterion does not account for the skewness. Case 9 is penalized in the RORAC criterion due to its larger skewness.

3. we also observe that the RORAC in this reinsurance structure is less than in the case of no reinsurance. Obviously buying reinsurance at that level destroys value. This is due to the fact that the reduction in risk is not counter-balanced by the cost of reinsurance ($\xi^{Re} = 7\% > 5\% = \xi$).

Now let us analyse other situations:

Case	*Line*	*CV*	γ	*RORAC*	$\frac{ES^{Re}}{ES}$
1	5000000	0.16	0.24	4.16%	61.31%
2	7500000	0.16	0.24	7.58%	46.03%
3	10000000	0.16	0.25	9.05%	33.91%
4	12500000	0.17	0.26	9.71%	24.82%
5	15000000	0.17	0.28	9.98%	18.12%
6	17500000	0.17	0.29	10.06%	13.20%
7	20000000	0.18	0.30	10.06%	9.59%
8	22500000	0.18	0.31	10.00%	6.91%

Table 5.7 RORAC as a function of the line of a surplus treaty

We observe that the optimal line is about $20\,000\,000$ providing a $RORAC = 10.06\%$, instead of 8.41% without reinsurance.

Now we consider the RORAC for surplus treaties with a table of lines. We choose the method of the inverse rate and we choose the lines so as to get the same global cession as in the previous table.

Case	R_1	R_2	R_3	R_4	*CV*	γ	*RORAC*	$\frac{ES^{Re}}{ES}$
1	2792144	5430844	11891468	25987731	0.16	0.29	4.06%	61.31%
2	4373473	8506598	18626192	40705865	0.16	0.28	7.47%	46.03%
3	6066679	11799959	25837392	56465292	0.16	0.28	8.93%	33.91%
4	7857669	15283513	33465040	73134831	0.17	0.29	9.58%	24.82%
5	9739358	18943481	41478968	90648548	0.17	0.30	9.86%	18.12%
6	11697749	22752639	49819564	108876170	0.17	0.31	9.96%	13.20%
7	13736088	26717298	58500649	127847900	0.18	0.32	9.96%	9.59%
8	15858279	30845054	67538854	147600082	0.18	0.33	9.92%	6.91%

Table 5.8. RORAC in function of the table of lines (inverse rate method)

We observe that the RORAC is smaller in case of a table of lines than it is in case of a classical surplus with one fixed line.

Now we consider the RORAC for surplus treaties with a table of lines that is built using the de Finetti optimal table of lines:

Case	R_1	R_2	R_3	R_4	*CV*	γ	*RORAC*	$\frac{ES^{Re}}{ES}$
1	3949974	5155430	7327325	11286824	0.15	0.24	4.22%	61.31%
2	6007752	7841203	11144567	17166807	0.16	0.25	7.68%	46.03%
3	8113889	10590093	15051518	23184974	0.16	0.26	9.15%	33.91%
4	10247187	13374433	19008852	29280752	0.17	0.27	9.79%	24.82%
5	12397936	16181549	22998558	35426392	0.17	0.28	10.05%	18.12%
6	14573268	19020751	27033868	41642281	0.17	0.29	10.13%	13.20%
7	16743363	21853117	31059460	47843201	0.18	0.30	10.11%	9.59%
8	18964227	24751746	35179233	54189193	0.18	0.31	10.05%	6.91%

Table 5.9. RORAC in function of the table of lines (de Finetti's optimal table)

Previous results are confirmed: this method of building up a table of lines is more efficient than the two methods of practitioners. Note that in our numerical example, it becomes more efficient than the surplus with a single line.

6. CONCLUSION

We have analysed the optimality properties of an insurance portfolio covered by a proportional reinsurance. The numerical application has confirmed that quota share reinsurance is suboptimal when compared to all other types of proportional reinsurance. In fact, quota share reinsurance will only be of interest to the ceding company when the loading of the reinsurer is smaller than the loading of the insurer. This is possible if one refers to the diversification possibilities that are offered to the reinsurer. So one may argue that less capital needs to be remunerated from the reinsurer's point of view. On the other hand, one may argue that the reinsurer's shareholders may require a higher cost of capital due to the agency costs (see Hancock et al. (2001) for details) that apply when underwriting a business that is less well understood by the reinsurer than the primary insurer. This means that ceding companies should provide as much information as possible to reinsurers in order to reduce these agency costs.

We have also observed that surplus reinsurance with a table of lines based on the inverse frequency method, or inverse rate method, is not, in our numerical example, optimal when compared to surplus reinsurance with one single line. This goes against the traditional belief of practitioners. Obviously we have not proved that it is always true but we have simply shown that a table of lines is not always optimal.

On the other hand we have derived the optimal table of lines using the de Finetti's criterion. This table of lines is more efficient, in our numerical example, than the other proportional reinsurance programmes.

Eventually, one should note that the reinsurer's loading would most probably not remain constant in case of surplus treaties with increasing retentions. Indeed, when increasing the retentions, the reinsured business becomes less well balanced, implying a larger volatility for the reinsurer. Clearly the reinsurer will apply higher capital charges in these cases. Moreover the fixed management expenses of the reinsurer will be more important in those treaties where the cession is small. One therefore has to be cautious with the previous conclusions and always ask quotes from the reinsurer when analysing the optimality of a reinsurance programme.

7. REFERENCES

[1] Bernegger, S. (1997), The Swiss Re Exposure Curves and The MBBEFD Distributions Class, Astin Bulletin, 27, 99–111.
[2] de Finetti, B. (1940), Il problema dei pieni, Giorn. Inst. Ital. Attuari, 11, 1–88.
[3] Denuit, M. and Vermandele, C. (1998), Optimal Reinsurance and Stop-Loss Order, Insurance : Mathematics and Economics, 22, 229–233.
[4] Dhaene, J. and Vandebroek, M. (1995), Recursions for the Individual Model, Insurance : Mathematics and Economics, 16, 31–38.
[5] Dufresne, F. and Niederhauser, E. (1997), Some Analytical Approximations of Stop-Loss Premiums , Bulletin of the Swiss Association of Actuaries, 25–47.
[6] Glineur, F. and Walhin, J.F. (2004), de Finetti's Retention Problem for Proportional Reinsurance Revisited , Unpublished manuscript.
[7] Hancock, J., Huber, P. and Koch, P. (2001), Value Creation in the Insurance Industry, Risk Management and Insurance Review, 4, 1-9.
[8] Klugman, S.A., Panjer, H.H., Willmot, G.E. (1998), Loss Models, Wiley.
[9] Panjer, H.H. (1981), Recursive Evaluation Of A Family Of Compound Distributions, Astin Bulletin, 12, 22–26.