Sharing risk - An economic perspective

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Abstract

Capital plays a central role for the insurance industry. First of all, it provides a cushion against adverse loss experience. By that capital mitigates the risk of ruin or insolvency and thus protects policyholder interest. Since shareholders on the other hand expect an adequate return from their investment, capital however is not for free. Keeping risk on a balance sheet thus comes at a cost for an insurer. Pooling and sharing risk in general lowers this cost because of diversification effects. The basic question then arising is how and to what extend risk should be shared from such an economic point of view.

Given this context, the paper revisits the relative retention problem originally introduced by de Finetti using concepts recently developed in risk theory and quantitative risk management. Instead of using the Variance as a risk measure we consider the Expected Shortfall (Tail-Value-at-Risk) and include capital costs and constraints on risk capital into our considerations.

Starting from a risk based capital allocation, the paper presents an optimization scheme for sharing risk in a multi risk class environment. Risk sharing takes place between two portfolios and the pricing of risk transfer includes capital costs. This allows us to shed more light on the question of how optimal risk sharing is characterized in a situation where risk transfer takes place between to parties employing similar risk and performance measures. Recent developments on the regulatory side ('risk-based supervision') pushing for common, insurance industry wide risk measures underline the importance of this question.

The paper includes a simple example illustrating optimal risk transfer in terms of retentions of common reinsurance structures.

Keywords: Sharing and pooling of risk, risk based capital, capital cost, reinsurance, optimal retentions

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1 Introduction

Actuarial and financial risk management has evolved rapidly over the last couple of years. Driving forces behind this development are manyfold. To name a few: Increasing integration of financial and actuarial risk management, a shift towards performance and risk management from an economical perspective and recent pressure on profitability in the insurance industry. Considerable theoretical progress has been achieved during the last few years, an example of this being the development of coherent risk measures. As well, increasing computer power makes it possible to address problems that have been out of reach before. In addition, current developments on the regulatory side most likely will lead to a supervisory framework in which insurers (and reinsures) will have to quantify risk based capital using prescribed advanced risk measures (see e.g. [**?**]).

Given this background, we revisit the relative retention problem originally introduced by de Finetti [8] from a more modern point of view. Instead of using Variance as a risk measure we consider the Expected Shortfall (Tail-Value-at-Risk) and include capital costs and constraints on risk based capital into our considerations. Using the Expected Shortfall as risk measure has several advantages. First, it is a coherent risk measure and second, it is in contrast to Variance an asymmetric risk measure that refers to the part of the profit or loss distribution that matters most for risk management: the tail of the distribution. Taking into account constraints on risk based capital means that capital restrictions are met. These constraints are of growing importance from a risk management and regulatory point of view. Finally, considering capital costs is of importance when it comes to deciding on how much risk should be kept from an economical perspective. The basic question to answer here is: Is it economically favorable to support risk by capital or should a transfer of risk take place, for example by buying reinsurance? And, in this respect, what are the optimal retentions of a reinsurance program?

Setting retention levels of reinsurance programs has been discussed by many authors. Starting with de Finetti [8], there has been a long history of attempts to set retention levels in such a way that given a constraint on the net profit, the overall Variance of the portfolio is minimized ([13, 2, 3, 16, 11, 12, 10]). Other approaches rely on criteria related to adjustment coefficients [4, 6] or utility theory [5]. For a comprehensive review see [7]. A limitation of most of those approaches is that they take the point of view of the primary insurer. Interests of the reinsurer are taken into account by simplified premium principles, often limited to the expected value or variance premium principle, i.e. premium principles that rely on first or second moments.

Our approach does not render de Finetti's approach superfluous since there are still situation where Variance may be the more adequate risk measure. However, by taking into account constraints on risk based capital and capital costs it adds a perspective to the problem of optimal retentions that becomes more and more important in the insurance industry: Managing risk from the balance sheet perspective.

The paper is organized as follows. In the next Section we briefly discuss some mathematical concepts and tools that will be used in the paper. The third Section derives conditions for optimal relative retentions and briefly discusses the question of setting absolute retention levels. Finally, Section four presents a simple example. The paper concludes with a summary and a discussion.

2 From risk exposure to the measurement and the price of risk

2.1 Risk measures

In insurance and finance, risk refers to the intrinsic uncertainty of the future and its impact on business goals. Mathematically, uncertainty is described by random variables. The cumulative probability distribution F_X of a random variable X is defined by $F_X(x) = \Pr(X \leq x)$. It assigns to each possible outcome x the probability F_X that the actual outcome X is smaller or equal to x. In order to quantify risk, one needs a reasonable map $\rho : F_X \mapsto \mathbb{R}$, which assigns a single risk quantifying number to F_X.

A basic example for such a risk measure is the Variance of X. It measures the variability of X by the average square deviation from the expectation $\mathrm{E}(X)$. The volatility of financial markets usually is characterized by Variance. Using Variance or standard deviation as a risk measure quantifies the risk of missing the expectation, either by falling short or surpassing it. In situations where one is not interested in the variation around the mean but in the downward variability of business result, Variance or Standard Deviation clearly do not provide all relevant information about the tail of the distribution F_X.

A common risk measure partially characterizing the tail of a distribution is the Value-at-Risk. It is defined as the α-quantile $F_X^{-1}(\alpha)$ of X and denoted by VaR_α. While VaR as risk measure is very convenient in the sense that it has a straightforward interpretation, it has considerable disadvantage. First of all, VaR just refers to one point of the probability distribution and neglects the tail beyond. It thus does not distinguish between distributions having equal α-quantiles but different tails beyond. Second, VaR is not a coherent risk measure (see [1]) since VaR is lacking sub-additivity.

A risk measure (see [1]) combining coherence and information about the full tail of a distribution is the Expected Shortfall, defined as conditional average

$$\mathrm{ES}_{\alpha,X} = E[X|X \leq F_X^{-1}(\alpha)] \tag{2.1}$$

The Expected Shortfall simply quantifies the risk related to events exceeding $\mathrm{VaR}_\alpha = F_X^{-1}(\alpha)$ as the average of all outcomes exceeding VaR_α. Note that F needs to be continuous in order for definition (2.13) to be valid. This requirement does not pose a problem in practice since every practical relevant discontinuous function F has a finite number of discontinuities and thus can easily be approximated by a continuous one.

The coherence of the Expected Shortfall means that it possesses desirable properties like

$$\rho[X+Y] \leq \rho[X] + \rho[Y] \quad \text{sub-additivity} \tag{2.2}$$
$$\rho[X] \geq \rho[Y] \quad \text{,if} \quad X \leq Y \quad \text{monotonicity} \tag{2.3}$$
$$\rho[s \cdot X] = s \cdot \rho[X] \quad \text{homogeneity} \tag{2.4}$$
$$\rho[X+a] = \rho[X] - a \quad \text{translation invariance} \tag{2.5}$$

Properties (2.3) and (2.5) will be of particular interest. Property (2.3) guarantees the risk measure ρ to scale naturally while (2.5) implies that ρ behaves naturally under translations. For details we refer the interested reader again to [1]. In summary, advantages of the Expected Shortfall are:

- refers to the tails of the distribution, i.e. the part of the distribution that matters most for risk management

- refers to the full tail of the distribution, i.e. all events beyond a threshold
- is a coherent risk measure, i.e. consistently accounts for risk diversification
- possesses natural scaling and translating properties

Based on this assessment[3], we will consider the Expected Shortfall as risk measure in the following.

2.2 Risk based capital and cost of capital

Capital plays a crucial role for an insurance company. On company level, capital provides risk mitigation and protection from insolvency or ruin due to large unexpected losses. Capital ('risk capital') thus ensures that in case of adverse loss experience liabilities still can be paid. Holding risk capital however comes at a cost since shareholders expect an adequate return from their investment. An economically meaningful analysis of insurance performance should take this cost of capital into account.

We distinguish here between risk capital and risk based capital (RBC). With the former we denote capital that is allocated to support a specific risk(s) while risk capital stands for the available amount of capital to support risk in total. In a first approximation, the cost of capital C a company is incurring for a specific risk is proportional to the amount of allocated risk capital, i.e.

$$C = \lambda \cdot \mathrm{RBC} \tag{2.6}$$

The constant λ depends on the capital structure of the company, type of business and other features. Frequently, the CAPM model is used to estimate λ.

We will now consider the simplified situation of an non-life insurer that writes one short-tail line of business over a single time period and is ceding part of the associated risk to a reinsurer. Neglecting expenses, discounting and the cost of allocated capital, the net underwriting profit distribution is

$$U = P - L - P^* + R \tag{2.7}$$

where P stands for the gross premium income of the primary insurer, P^* for paid reinsurance premiums, L for the accumulated gross loss amount and R for the accumulated recoveries due to reinsurance. We will not consider loss dependent, stochastic premiums here. Thus the basic accumulated stochastic variable is the gross loss amount L, which may include several risk classes i

$$L = \sum_i \sum_{j=1}^{N_i} l_{i,j} \tag{2.8}$$

with stochastic loss amounts $l_{i,j}$ and the stochastic number of losses N_i. Recoveries R will depend on the loss experience L, i.e. $R = R(l_{i,j}|i,j)$. Including the cost of capital, the net underwriting profit Z including capital costs becomes

$$Z = U - \lambda \cdot \mathrm{RBC} \tag{2.9}$$

where RBC stands for the capital allocated to support the *net* risk. This setup can easily be generalized to include discounting, investment return and risk, expenses and typical

[3] Sometime voiced criticism of the Expected Shortfall as risk measure include a) its complexity compared to VaR and b) the fact, that it depends on extreme tail events which are uncertain to a high degree.

features of non-life insurance like loss depended premiums or profit sharing. While in principle the situation for life insurance is similar, in practice a corresponding setup is somewhat different since life insurance typically is concerned with a multi-period time horizon and and investment risk and returns play an important role for the risk return characteristics of the business.

Insurers normally do not allocated risk capital for the full range of potential negative results but only up the probabilities corresponding to a risk tolerance level α. The determination of risk capital thus depends on the risk aversion of the insurer. More important, the determination of risk capital also depends on the risk measure under consideration and other factors. For example, the risk based capital can be defined relative to the break-even situation $Z = 0$ or relative to the expected profit $Z = \mathrm{E}[Z]$. Depending on the purpose, other possibilities exist. For a detailed summary see example [15]. Note that from a theoretical point of view there are properties that favor some of the concepts over other ones [14]. This is in particular the case for a risk capital allocation on sub-portfolio level (see Section 2.3).

Following Section 2.1 and [14], we choose the Expected Shortfall as risk measure and define risk based capital relative to the break-even situation $Z = 0$. The risk based capital for a given risk tolerance level α then is determined implicitly by

$$\begin{aligned} \mathrm{RBC}[Z] &= -\mathrm{ES}_{\alpha,Z}[Z] \\ &= -\mathrm{ES}_{\alpha,X}[U - \lambda \cdot \mathrm{RBC}[Z]] \end{aligned} \tag{2.10}$$

Using (2.5) and solving this equation for $\mathrm{RBC}[Z]$ we get

$$\mathrm{RBC}[Z] = -\frac{\mathrm{ES}_{\alpha,Z}[U]}{1-\lambda} \tag{2.11}$$

and thus

$$Z = U + \lambda \cdot \frac{\mathrm{ES}_{\alpha,Z}[U]}{1-\lambda} \tag{2.12}$$

The sign in (2.11) has been chosen in a way that the RBC is positive number under normal circumstances.

The net profit distribution Z is the basic quantity we are interested in. For the insurer, it characterizes the economical value of keeping risk on the balance sheet. Return and risk measures can be defined based on Z. The expectation $\mathrm{E}[Z]$ for example is the expected underwriting result including capital costs, i.e. the expected return after capital costs. For a given risk tolerance level α, $\mathrm{RBC}[Z]$ characterizes the monetary amount which is at risk. Other risk and performance measures can be defined easily form above expressions.

2.3 Generalization for a Portfolio of Risks

For various purposes ranging from risk management to performance evaluation, insurers typically group their business into several classes ('lines of business'). The allocation of capital usually follows this grouping which leads to a multivariate situation. While capital allocation is not straightforward, there is general agreement that a capital allocation scheme should fulfil two key properties:

- Risk capital should be allocated according to the risk contribution of the sub-portfolio to the overall risk of the portfolio.

- The sum of allocated risk capital should equal the risk capital determined for the total portfolio. A mismatch of allocated capital on sub-portfolio and total portfolio level would indicate an under- or overestimation of diversification effects, respectively.

A number of different capital allocation schemes taking into account these key properties have been proposed [15], many relying on conditional risk measures. We consider here a capital allocation scheme based on the conditional Expected Shortfall. Due to the Euler theorem, this scheme naturally accounts for the two key properties above (see e.g. [14]). Given a set of risk factors $\{X_i\}$, the conditional Expected Shortfall is defined as

$$\mathrm{CES}_{\alpha,X}[X_i] = \mathrm{E}[X_i | X \leq F_X^{-1}(\alpha)] \tag{2.13}$$

which is the mean of X_i *given that* the sum $X = \sum X_i$ is grater than or equal to the α-quantile $F_X^{-1}(\alpha)$. A straightforward property of conditional risk measures is the additivity. In case of the conditional Expected Shortfall

$$\mathrm{ES}_{\alpha,X}[X] = \mathrm{CES}_{\alpha,X}[X] = \sum_i \mathrm{CES}_{\alpha,Z}[X_i] \tag{2.14}$$

The additivity is a direct consequence of the conditional nature of $\mathrm{CES}_{\alpha,X}[X_i]$. It will play an important role in the following.

A generalization of equations (2.9) to (2.12) based on the conditional Expected Shortfall and accounting for n sub-portfolios is straightforward. The contribution of a sub-portfolio i to the total risk based capital becomes

$$\begin{aligned} \mathrm{RBC}[Z_i] &= -\mathrm{E}[Z_i | Z \leq F_Z^{-1}(\alpha)] \\ &= -\mathrm{CES}_{\alpha,Z}[Z_i] \\ &= -\frac{\mathrm{CES}_{\alpha,Z}[U_i]}{1-\lambda} \end{aligned} \tag{2.15}$$

where Z_i is the insurer's profit distribution of the ith sub-portfolio. Using the allocation scheme (2.15) for risk based capital, the profit distribution on sub-portfolio level is

$$\begin{aligned} Z_i &= U_i - \lambda \cdot \mathrm{RBC}[Z_i] \\ &= U_i + \lambda \cdot \frac{\mathrm{CES}_{\alpha,Z}[U_i]}{1-\lambda} \end{aligned} \tag{2.16}$$

The term

$$\lambda \cdot \frac{\mathrm{CES}_{\alpha,Z}[U_i]}{1-\lambda} \tag{2.17}$$

in (2.16) corresponds to capital costs for risk class i (after reinsurance). Capital costs depend on the portfolio structure of the insurer and thus takes into account diversification effects. In other terminology, (2.17) quantifies the (net) loading related to risk class i.

The total expected underwriting result after capital costs simply is

$$\mathrm{E}[Z] = \sum_i \mathrm{E}[Z_i] \tag{2.18}$$

By the additivity of the conditional Expected Shortfall we get a similar expression for the total risk of the company by summing over the contributions of the sub-portfolios

$$\mathrm{ES}_{\alpha,Z}[Z] = \sum_i \mathrm{CES}_{\alpha,Z}[Z_i] \tag{2.19}$$

The risk based capital exhibits, as a direct consequence of its definition (2.15), the similar additivity

$$\mathrm{RBC}[Z] = \sum_i \mathrm{RBC}[Z_i] \tag{2.20}$$

This demonstrates that (i) the risk capital allocation to lines of business depends on the risk characteristic on sub-portfolio level and (ii) the risk capital of the n sub-lines adds up to the total risk capital on company level. Note that $-\sum_i \mathrm{CES}_{\alpha,Z}[Z_i] \leq -\sum_i \mathrm{ES}_{\alpha,Z}[Z_i]$. The allocation scheme thus takes into account diversification on portfolio level and allocates the diversification benefit on portfolio level to the sub-portfolio level. It follows directly that capital costs on sub-portfolio level share properties (i) and (ii) as well.

3 Optimal Risk Retention

3.1 Sharing and Transferring Risk

As discussed, keeping risk on a balance sheet comes at a cost. Because of diversification effects, pooling and sharing risk will in general lower this cost. The basic question thus arising is how and to what extend risk should be retained or shared. Answering this question depends on risk characteristics and a number of other parameters. Clearly, we expect for example the type of risk measure, the risk tolerance level and capital costs as well as the price of transferring risk to have an influence on adequate risk retention levels. An equally important role play constraints on available risk capital. Constraints on risk capital are becoming more an more important because regulatory developments.

The insurance industry achieves risk sharing by risk transfer relying for example on reinsurance or by placing risk in financial markets. Reinsurance has a long history and remains the major way of risk sharing for many insurers. Highly specialized forms of reinsurance are available in the market which correspond to the need for specific forms of risk transfer. Risk transfer by reinsurance, which we will consider here, in particular distinguishes between proportional and non-proportional reinsurance. Non-proportional reinsurance intends to limit peak risks while proportional reinsurance simply scales risk exposure.

Transferring risks comes at a price. In exchange for the transfer of risk, the reinsurer receives a premium. Because of diversification benefits, the premium for risk transfer tends to be lower than the corresponding cost of keeping the risk on the balance sheet of insurer only.

The degree to which risk transfer makes sense depends on a number of parameters like capital costs and the portfolio structure of both the insurer and the reinsurer. An additional key element is the reinsurance premium principle which determines the price of risk transfer.

3.2 The price of transferring risk

3.2.1 Reinsurance premium principle

In general, reinsurance premiums are calculated from principles splitting the reinsurance premium into the expected loss and a loading. Common loadings principles are e.g. Expected Value principle or the Variance principle (see e.g. [9] CHECK). We chose here to define the loading from a more economic point of view and will consider capital costs. The general idea behind this approach is that the reinsurer, as the primary insurer, has to support the risks in his books by capital. We account for related capital costs of the reinsurer in exactly the same way as proposed in Section 2.2 for the primary insurer. This will allow to shed more light on the question of how and to what extend risk should be shared from an economic point of view.

3.2.2 Non-proportional reinsurance

Following Section 2.2, we start from the underwriting profit distribution including cost of capital. As before, quantities referring to the reinsurer will be denoted by a star (*). Z^*, the underwriting profit distribution including cost of capital C^* of the reinsurer, is

$$Z^* = P^* - R - C^* \tag{3.1}$$

where

$$C^* = \lambda^* \cdot \mathrm{RBC}^*[Z^*] \tag{3.2}$$

$\mathrm{RBC}^*[Z^*]$ denotes the reinsurer's risk based capital related to the underwriting profit distribution Z^* including cost of capital. The constant λ^* stands for the capital costs of the reinsurer. Denoting the risk tolerance level of the reinsurer by β, we get the implicit relation

$$\mathrm{RBC}^*[Z^*] = -\mathrm{ES}^*_{\beta,Z^*}[P^* - R - \lambda^* \cdot \mathrm{RBC}^*[Z^*]] \tag{3.3}$$

for the risk based capital of the reinsurer. The star in expression (3.3) indicates that there is potential difference between both the risk based capital and the risk measures of insurer and reinsurer [4]. We will come back to this issue in Section 3.3. Solving for the reinsurer's risk based capital we get

$$\mathrm{RBC}^*[Z^*] = -\frac{\mathrm{ES}^*_{\beta,Z^*}[P^* - R]}{1 - \lambda^*} \tag{3.4}$$

From equation (3.1) and expression (3.4) we define a reinsurance premium principle as follows

$$P^* = \mathrm{E}[R] + \lambda^* \cdot \mathrm{RBC}^*[Z^*] \tag{3.5}$$

The premium principle simply states that the reinsurance premium P^* equals the expected accumulated loss for the reinsurer (recovery for the primary insurer) $\mathrm{E}[R]$ plus a loading $\lambda^* \cdot \mathrm{RBC}^*[Z^*]$ reflecting the reinsurer's capital costs. Solving the implicit expression (3.5) for P^* and assuming flat reinsurance premiums (i.e. no reinstatements) yields

$$P^* = (1 - \lambda^*) \cdot \mathrm{E}[R] + \lambda^* \cdot \mathrm{ES}^*_{\beta,Z^*}[R] \tag{3.6}$$

According to (3.6) the reinsurer charges the insurer for *his* capital costs related to the transferred risk. In general, corresponding capital costs for reinsurer and primary insurer differ because $\lambda^* \neq \lambda$ and $\alpha \neq \beta$ i.e. the capital costs of insurer and reinsurer are not the same and the risk tolerance may differ. In addition, reinsurer and primary insurer allocate risk capital against different portfolios resulting in different diversification effects.

[4] Note that this notation differentiates between the risk based capital related to Z^* in the portfolio of the reinsurer $\mathrm{RBC}^*[Z^*]$ and the risk based capital related to Z^* in the portfolio of the primary insurer $\mathrm{RBC}[Z^*]$.

3.2.3 Proportional Reinsurance

In a proportional reinsurance arrangement, primary insurer and reinsurer share risk and premium on a proportional basis. The premium principle in the proportional case thus has the simple form

$$P^* = (1 - r_i) \cdot P \tag{3.7}$$

where r_i quantifies the retention of the primary insurer. In practice, a reinsurer will pay a primary insurer a commission C intended to compensate acquisition expenses related to the risk. The discussion of an adequate reinsurance premium thus becomes a question of the adequacy of the commission or brokerage. From the point of view of the primary insurer, the reinsurance premium paid for the risk transfer is

$$P^* = (1 - r_i) \cdot P - C \tag{3.8}$$

In the following, we will assume that the reinsurer charges the primary insurer for the risk transfer by adjusting the commission C. Doing so, we get formally the same expression for reinsurance premium as in the non-proportional case and the commission (or brokerage) C becomes

$$C = P^* - (1 - r_i) \cdot P \tag{3.9}$$

In the following, we will consider commissions C only implicitly and will instead focus on the 'commission adjusted' reinsurance premium P^* as defined in (3.8). In any case, recovering the explicit brokerage or commission (3.9) is straightforward.

3.2.4 Reinsurance Premium on Line of Business Level

Finally, the reinsurance premiums P_i^* per line of business needs to be considered. Following the assumption that the reinsurer charges the primary insurer for *his* capital costs (see (3.6)), reinsurance premiums on the line of business level are calculated from

$$P_i^* = \mathrm{E}[R_i] + \lambda^* \cdot \mathrm{RBC}^*[Z_i^*] \tag{3.10}$$

with

$$\mathrm{RBC}[Z_i^*] = -\mathrm{E}[Z_i^* | Z^* \leq F_{Z^*}^{-1}(\beta)] = -\mathrm{CES}^*_{\beta,Z^*}[Z_i^*] \tag{3.11}$$

This leads to

$$\mathrm{RBC}[Z_i^*] = -\frac{\mathrm{CES}^*_{\beta,Z^*}[P_i^* - R_i]}{1 - \lambda^*} \tag{3.12}$$

and

$$P_i^* = (1 - \lambda^*) \cdot \mathrm{E}[R_i] + \lambda^* \cdot \mathrm{CES}^*_{\beta,Z^*}[R_i] \tag{3.13}$$

Here the star denotes the conditional Expected Shortfall related to the reinsurer portfolio.

The reinsurance premium principle (3.13) takes into account diversification effects. Risks showing a higher degree of dependency with the portfolio of the reinsurer will require a relatively higher loading than risks with no or negative correlation.

3.3 Criteria for Optimal Risk Retention

3.3.1 Constrained Risk Retention

After having fixed notation and concepts, we come back to the initial question. What are optimal risk retention levels given portfolio structure, reinsurance premium principle, capital costs and available risk capital? Optimality in this perspective refers to the amount and type of risk that should be kept (or shared with the reinsurer, respectively) from an economical point of view. While this question itself is interesting, it does not reflect common business realities where the amount of capital supporting risk is limited and often fixed. We therefore take into account a constraint and consider the amount of available risk capital as fixed. According to de Finetti's terminology this is the relative retention problem. Optimizing the overall level of risk retention (i.e. the absolute retention problem) is briefly discussed in Section 3.3.4.

Depending on the risk characteristics of each line of business, optimal risk retentions will vary. The question to answer thus is: Given the overall risk retention in terms of the risk based capital, what are economically optimal risk retentions $r = \{r_i\}$ on line of business level? In particular, given the available risk capital, how should risk be ceded? Considering the profit distribution $Z = Z(r)$ and the risk based capital $\mathrm{RBC} = \mathrm{RBC}[Z(r)]$ as a function of retention levels $\{r_i\}$, this is a typical constrained extremum value problem

$$\mathrm{E}[Z(r)] = \max! \tag{3.14}$$

$$\mathrm{RBC}[Z(r)] = \mathrm{const} \tag{3.15}$$

with parameters $\{r_i\}$. Mathematically, constrained extremum value problems are solved by Lagrangian multiplier techniques, i.e. maximizing the expression

$$\phi = \mathrm{E}[Z(r)] + \kappa \cdot \mathrm{RBC}[Z(r)] \tag{3.16}$$

where κ represents the Lagrangian multiplier associated with the constraint (3.15) and the total net risk based capital according to (2.15) is given by

$$\mathrm{RBC}[Z] = \mathrm{ES}_{\alpha,Z}[Z] = \sum_i \frac{\mathrm{CES}_{\alpha,Z}[U_i]}{1-\lambda} \tag{3.17}$$

To keep the notation simple, we will omit in the following the explicit r-dependence of the profit distributions Z, U Z^* and U^* as well as of the recoveries R, the reinsurance premium P^* and risk based capital $RBC[Z]$ and $RBC^*[Z^*]$[5].

Independent of the reinsurance structure, the optimization problem can be formulated in terms of the reinsurance recoveries R_i and structural parameters r_i characterizing the proportional (quota share reinsurance) or non-proportional retention (excess-of-loss reinsurance) or recovery, respectively. To that extend recall that according to the Lagrange multiplier technique optima are characterized by the condition

$$\frac{\partial \phi}{\partial r_j} = 0, \quad \forall j \tag{3.18}$$

For (3.18) to be meaningful, the partial derivatives $\partial_r \mathrm{E}[Z(r)]$ and $\partial_r \mathrm{RBC}[Z(r)]$ have to exist. In [14], conditions are discussed under which the Expected Shortfall is differentiable. Note that in practice this does not pose a problem since the conditional expected shortfall is defined as an integral over the distribution F (here R_i respectively) which we assumed to be continuous or approximated by a continuous function (see Section 2.1). Thus partial derivatives do exist.

We continue by considering an explicit expression for the objective function Z, the net profit distribution. Combining explicit expressions for the insurers underwriting profit

[5] The reverse optimization problem $\mathrm{ES}_{\alpha,Z}[Z] = \min!, \mathrm{E}[Z] = \mathrm{const}$ leads to an expression equivalent to (3.16)

distribution U, the reinsurance premium P^* and the risk based capital RBC we get for Z

$$\begin{aligned} Z &= \sum_i (P_i - L_i - P_i^* + R_i) - \frac{\lambda}{1-\lambda} \cdot \mathrm{CES}_{\alpha,Z}[L_i - P_i + P_i^* - R_i] \\ &= \sum_i \frac{P_i}{1-\lambda} - \frac{1-\lambda^*}{1-\lambda} \cdot \mathrm{E}[R_i] - \frac{\lambda^*}{1-\lambda} \cdot \mathrm{CES}^*_{\beta,Z^*}[R_i] \\ &\quad - (L_i - R_i) - \frac{\lambda}{1-\lambda} \cdot \mathrm{CES}_{\alpha,Z}[L_i - R_i] \end{aligned} \tag{3.19}$$

With the corresponding expression (3.17) for the risk based capital constraint, condition (3.18) becomes

$$\begin{aligned} \frac{\partial \phi}{\partial r_j} &= \frac{\partial}{\partial r_j} \left(\mathrm{E}[Z] + \kappa \cdot \rho[Z] \right) \\ &= \frac{\partial}{\partial r_j} \Bigg[(\lambda^* - \lambda + \kappa \cdot (1 - \lambda^*)) \cdot \mathrm{E}[R_j] \\ &\quad - \sum_i \Big[\lambda^* \cdot (1-\kappa) \cdot \mathrm{CES}^*_{\beta,Z^*}[R_i] - (\lambda - \kappa) \cdot \mathrm{CES}_{\alpha,Z}[R_i] \Big] \Bigg] \overset{!}{=} 0 \quad \forall j \end{aligned} \tag{3.20}$$

Solutions $r = r_0$ of (3.14) are characterized by (3.20)[6]. They exist if three conditions are fulfilled. First, in order for λ to exist at $r = r_0$, the gradient at $r = r_0$ of Z needs to be non-zero

$$\nabla_r \rho[Z(r_0)] \neq (0, \ldots, 0) \tag{3.22}$$

In addition, for the retention vector r_0 to be solution of (3.14)

$$\mathrm{E}[Z(r)] \text{ is concave} \tag{3.23}$$

$$\lambda \cdot \mathrm{E}[Z(r)] \text{ is convex} \tag{3.24}$$

What is the role and interpretation of the Lagrange parameter κ in (3.20)? The parameter is related in our case to the overall risk retention. The overall risk retention (corresponding to the risk based capital amount RBC[Z]) implicitly determines the Lagrangian multiplier κ. Its interpretation is linked to the derivative of the objective function E[Z]. The value of the Lagrange multiplier κ at a given net risk retention is equal to the derivative of E[Z] with respect to the retentions r_i. Thus the Lagrange multiplier κ measures the marginal change of E[Z], i.e. the rate of increase (or decrease) in the maximized expected net profit E[Z] as retentions r_i are changed.

Equation (3.20) together with conditions (3.22-3.24) form the basic criteria we will consider in detail in the following. According to (3.20), optimal retentions depend on both the primary insurer's and reinsurer's portfolio, risk measure and appetite and individual capital costs. Thus, these expressions capture the essential economics behind risk transfer between primary insurer and reinsurer. Given the similar risk pricing principle, it illustrates how a 'win-win' situation between primary insurer and reinsurer can be achieved and which factors drive it.

As mentioned (3.20) condition is general. It does not depend on the type of risk transfer. For example, (3.20) characterizes optimal retention levels in non-proportional and proportional reinsurance agreements as well as combinations thereof.

[6] In vector notation (3.20) can be written as

$$\nabla_r \big[(\lambda^* - \lambda + \kappa \cdot (1 - \lambda^*)) \cdot \mathrm{E}[R] - \lambda^* \cdot (1-\kappa) \cdot \mathrm{ES}^*_{\beta,Z^*}[R] + (\lambda - \kappa) \cdot \mathrm{ES}_{\alpha,Z}[R] \big] \overset{!}{=} 0 \tag{3.21}$$

3.3.2 Equal portfolios and zero capital costs

From (3.20) it follows that the difference of the portfolios of primary and reinsurer and non-zero capital costs are crucial for the existence of a solution. To see this point consider first the case where β=α and $\text{CES}^*[R_i] = \text{CES}[R_i]$. Expression (3.20) becomes

$$\frac{\partial \phi}{\partial r_j} = (\lambda^* - \lambda + \kappa \cdot (1 - \lambda^*)) \cdot \frac{\partial}{\partial r_j} \sum_i \Big[\text{E}[R_i] - \text{CES}_{\alpha,Z}[R_i] \Big] \stackrel{!}{=} 0 \quad \forall j \tag{3.25}$$

which means that κ is independent of the retention levels r_i indicating that no solution exists. One should however keep in mind that in practice this situation will not occur since a risk transfer from the primary insurer to the reinsurer will in any case change both portfolios.

A similar situation arises if capital costs λ and λ^* are zero. This leads to

$$\frac{\partial \phi}{\partial r_j} = \kappa \cdot \frac{\partial}{\partial r_j} \sum_i \Big[\text{E}[R_i] - \text{CES}_{\alpha,Z}[R_i] \Big] \stackrel{!}{=} 0 \quad \forall j \tag{3.26}$$

Again, there is no solution since in general only with $\kappa = 0$ the condition is fulfilled. Thus both, equal portfolios and zero capital costs lead to a situation where no optimization is possible. Loosely speaking, the reason for this is that, in the first, case the loadings of reinsurer and primary insurer are proportional. In the second case the loadings itself vanish leaving $\kappa = 0$ as a degenerate solution.

3.3.3 Recovering de Finetti's Solution

It is illustrative to consider how de Finetti's solution can be recovered from (3.20) by considering simplifications and a change of the risk measure. By undoing some of the simplifications that lead to (3.20) we rewrite (3.26) to

$$\begin{aligned} \frac{\partial \phi}{\partial r_j} = \frac{\partial}{\partial r_j} \Bigg[\sum_i \Big(& \lambda \cdot (\text{CES}_{\alpha,Z}[R_i] - \text{E}[R_i]) - \lambda^* \cdot (\text{CES}^*_{\beta,Z^*}[R_i] - \text{E}[R_i]) \Big) \\ & + \kappa \cdot \Big((1 - \lambda^*) \cdot \text{E}[R_i] + \lambda^* \cdot \text{CES}^*_{\beta,Z^*}[R_i] - \text{CES}_{\alpha,Z}[R_i] \Big) \Bigg] \stackrel{!}{=} 0 \quad \forall j \end{aligned} \tag{3.27}$$

The first term in the sum (3.27)

$$\Lambda_i = \lambda \cdot (\text{CES}_{\alpha,Z}[R_i] - \text{E}[R_i]) - \lambda^* \cdot (\text{CES}^*_{\beta,Z^*}[R_i] - \text{E}[R_i]) \tag{3.28}$$

is related to loadings per risk class i and the second term corresponds to the constraint on the net risk as defined by (3.17).

Using (2.5) and changing the risk measure in (3.17) from risk based capital defined by the Expected Shortfall to de Finetti's Variance of net losses, we get

$$\frac{\partial \phi}{\partial r_j} = \frac{\partial}{\partial r_j} \sum_i \Big[\Lambda_i - \kappa \cdot \text{Var}[L_i - R_i] \Big] \stackrel{!}{=} 0, \quad \forall j \tag{3.29}$$

Finally, assume that losses L_i (and thus recoveries R_i) are independent and that differences in loadings Λ_i are not calculated according to (2.17) but from some loading principle that does not take into account portfolio and diversification effects but is solely based on the isolated profit distribution Z_i. This leads to

$$\frac{\partial \phi}{\partial r_j} = \frac{\partial}{\partial r_j}\left[\Lambda_j - \kappa \cdot \mathrm{Var}\left[L_j - R_j\right]\right] \stackrel{!}{=} 0 \quad \forall j \tag{3.30}$$

which is equivalent to the optimality condition of de Finetti's original relative retention problem.

The derivation of (3.30) illustrates the main differences between the assumptions of de Finetti's and the generalized framework. Capital costs and risk measure are only one part, different portfolios and the independence of losses L_i and loadings Λ_i are the other. The independence of the L_i is crucial for the derivation of optimal relative retentions in de Finetti's setting since it guarantees the additivity of the Variance as risk measure. If the L_i's are not independent, Covariance terms enter the calculation which complicates the situation. For an approach taking into account dependencies by Covariance, see [12]. In our case, dependencies between L_i's are naturally covered by considering a risk measure conditional to the overall portfolio result.

3.3.4 Unconstrained risk retention

If there is no constraint on the overall net risk retention of the primary insurer, optimal retentions can be obtained by simply abandoning the constraint on risk based capital in (3.14), i.e.

$$\mathrm{E}[Z] = \max! \tag{3.31}$$

This leads to a similar criteria as (3.20) with $\kappa = 0$ in ϕ

$$\begin{aligned} \frac{\partial \phi}{\partial r_j} &= \frac{\partial}{\partial r_j}\left[(\lambda^* - \lambda)\cdot \mathrm{E}[R_j] - \sum_i \left[\lambda^* \cdot \mathrm{CES}^*_{\beta,Z^*}[R_i] - \lambda \cdot \mathrm{CES}_{\alpha,Z}[R_i]\right]\right] \\ &= \frac{\partial}{\partial r_j}\sum_i \left[\lambda \cdot (\mathrm{CES}_{\alpha,Z}[R_i] - \mathrm{E}[R_i]) - \lambda^* \cdot (\mathrm{CES}^*_{\beta,Z^*}[R_i] - \mathrm{E}[R_i])\right] \stackrel{!}{=} 0 \quad \forall j \end{aligned} \tag{3.32}$$

The implicit assumption in this case is that the primary insurer does not care about the net risk retention level. In our context this means that there is no constraint related to the available risk capital or, respectively, the risk capital to be deployed. In the case of this unconstrained maximization problem, the condition to be fulfilled is

$$\mathrm{E}[Z(r)] \text{ is concave} \tag{3.33}$$

If existing, the solution are retention levels which will maximize the overall profit after capital costs on portfolio level.

3.4 Reinsurance Structures

Based on the optimality criterion (3.20), this section presents a brief discussion of optimal retentions for common reinsurance structures and discusses implementation questions. The Section also relates to Section 4, where a numerical example is considered.

We consider here the simplified situation where both proportional and non-proportional reinsurance are characterized by one parameter only. For a non-proportional reinsurance structure on a single claim basis (e.g. excess-of-loss), the parameter is the attachment point above which risk is transferred, or equivalently the deductible. No limit is considered. For a proportional reinsurance agreement, the parameter is the relative amount of risk that is retained.

3.4.1 Proportional Reinsurance

Proportional reinsurance leads to recoveries

$$R_i = (1 - r_i) \cdot L_i = \sum_i (1 - r_i) \cdot \sum_{j=1}^{N_i} S_{j,i} \tag{3.34}$$

where r_i is the proportional risk retention by the primary insurer (per line of business or risk category i). Inserting this expression in (3.20) leads to the following expression

$$\begin{aligned} \frac{\partial \phi}{\partial r_j} &= (\lambda^* - \lambda + \kappa \cdot (1 - \lambda^*)) \cdot \mathrm{E}[L_j] \\ &- \frac{\partial}{\partial r_j} \left[\sum_i r_i \cdot \left[\lambda^* \cdot (1 - \kappa) \cdot \mathrm{CES}^*_{\beta, Z^*}[L_i] - (\lambda - \kappa) \cdot \mathrm{CES}_{\alpha, Z}[L_i] \right] \right] \stackrel{!}{=} 0 \quad \forall j \end{aligned} \tag{3.35}$$

On first sight, this expression may appear to be independent of r_i. However, this is not the case since the risk measures $\mathrm{CES}_{\alpha,Z}$ and $\mathrm{CES}^*_{\beta,Z^*}$ by $Z = Z[r]$ and $Z^* = Z^*[r]$ implicitly depend on r. Loosely speaking, the linearity of underwriting result and capital costs under a proportional reinsurance agreement thus is lost once a risk based commission as defined by (3.9) is taken into account.

While proportional reinsurance leads to a simplified optimality criterion (3.35), the implicit dependence of risk measures $\mathrm{CES}^*_{\beta,Z^*}[R_i]$ and $\mathrm{CES}_{\alpha,Z}[R_i]$ on r_i remains and a further simplification is not straightforward. In general, a numerical evaluation of (3.35) is necessary.

3.4.2 Non-proportional Reinsurance

Non-proportional reinsurance on an excess-of-loss basis applies to single losses. In general, the recovery from a reinsurance structure is limited to loss amounts between retentions r_i and a maximum claim amounts m_i. As discussed, we consider here only the case where $m_i = \infty$ (unlimited layer). The total recovery thus is

$$R_i = \sum_{j=1}^{N_j} \max[l_{i,j} - r_i, 0] \tag{3.36}$$

While finding analytical expressions in terms of the severity and frequency distributions S_i and N_i for $\mathrm{E}[R_i]$ is straightforward (see e.g. [9]), expressions for $\mathrm{CES}^*_{\beta,Z^*}[R_i]$ and $\mathrm{CES}_{\alpha,Z}[R_i]$ and corresponding derivatives with respect to retention levels r_i are hard to workout. We thus do not attempt to write (3.20) in terms of severity and frequency distributions for S_i and N_i. Instead, we shall consider a numerical evaluation.

3.4.3 Stop Loss Reinsurance

Stop loss reinsurance applies to the aggregated loss amounts. The recoveries for a reinsurance structure with aggregate retentions r_i and limits $m_i = \infty$ are

$$R_i = \max[L_i - r_i, 0] \tag{3.37}$$

As is the case with excess-of-loss reinsurance, risk measures $\mathrm{CES}^*_{\beta,Z^*}[R_i]$ and $\mathrm{CES}_{\alpha,Z}[R_i]$ implicitly depend on r_i. Finding analytical expressions for derivatives of $\mathrm{CES}^*_{\beta,Z^*}[R_i]$ and $\mathrm{CES}_{\alpha,Z}[R_i]$ with respect to r_i is thus not straightforward. Therefore we shall consider numerical evaluations.

3.5 Some Remarks

3.5.1 Technicalities

In a real-world situation, there is no way around using numerical techniques when evaluating partial derivatives in expressions like (3.20). While for simplified situations de Finetti's approach yields analytical expressions, this is in general not possible in our case. On one hand, analytical solutions certainly are nice to have. On the other hand, from the practical point of view, analytical solutions are often of limited use since they do not reflect business reality in an adequate way. For example, de Finetti's analytical solution explicitly assume that risk classes are independent. Fortunately, available computer power and efficient software makes it now possible to address problems numerically that have been out of reach in the past. The framework for the optimization of retentions outlined above is an example for this.

A straightforward way of addressing the risk retention problem numerically are Monte Carlo methods that nowadays are frequently used for Asset Liability Management (ALM) and Dynamic Financial Analysis (DFA). In such a scenario based setting, the Expected Shortfall and its conditional counterpart, the conditional Expected Shortfall, transform simply into averages over subsets of scenarios. Computing these conditional averages can be done easily and efficiently. From the practical point of view, another important property of risk measures used here is stability. Compared to Value-at-Risk, the Expected Shortfall is a much more stable quantity since it is defined as an average over quantiles instead of a single quantile. This is of special importance when it comes to computing partial derivatives e.g. in (3.20). Numerically, the main problem to be tackled in the Monte Carlo setting are discontinuities due to finite resolution in both the objective functions Z_i and the constraint $\mathrm{RBC}(Z)$. One option is to continuously approximate or smooth the corresponding functions. Besides, numerical optimization schemes available in advanced mathematical software such as MATLAB provide a wide range of methods and parameters that makes it possible to resolve numerical problems related to discontinuities.

3.5.2 Practical Feasibility and Relevance

In practice, risk transfer does take place without a detailed knowledge of the portfolio structures on the other side of the transaction. In general, portfolio information is passed to the market only indirectly by quoted rates. The concept of 'leading and following reinsurers' implies that rates do not take into account the portfolio structure of a major part of market participants. The current (re-)insurance market thus is not well enough developed in order to fully allow for risk transfer guided by the scheme outlined here. Nevertheless, approximations based, for example, on simplified portfolio models and considering optimal retentions as first indications to be adjusted manually later on prove the feasibility of the scheme in practice.

The situation is somewhat different when it comes to intercompany risk transfer. Intercompany risk transfer plays an important role for the capital management of multinational (re-)insurance groups. Instead of capital, risk is often transfer between balance sheets. Under the assumption that multinational groups have a risk management process and system in place that allows for a quantitative assessment of risk on portfolio and sub-portfolio level, risk transfer guided by principles discussed above will allow the group to use its capital in an efficient way despite the fact that capital might be tied up on individual balance sheets.

An important force that currently is shaping the (re-)insurance industry is the regulatory shift towards risk-based supervision. In Europe, Solvency II for instance will impose a risk-based regulatory framework based on prescribe risk measures and risk tolerance levels. Thus this framework introduces comparability into the market and paves the way towards a market place that eventually may allow for a more economical and transparent risk sharing.

4 An illustrative Examples

In this section we present illustrative results of a simple model. The model demonstrates risk transfer from a portfolio consisting of two risk classes to a portfolio that is similar but more diversified. Losses are modelled by frequency and severity distributions. Even this simple example shows how diverse effects of risk transfer are and that sharing the 'right' part of the risk has a considerable impact on the economic bottom line.

The example is based on 50'000 Monte Carlo scenarios. Characteristics of risk classes are listed in Table 1 and risk tolerance levels and cost of capital in Table 2. We chose the risk tolerance for both portfolios to be the same but consider different capital costs. The first portfolio could be identified to a primary insurer, the second portfolio to a reinsurer. Different capital costs are considered in order to illustrate the dependency on this parameter. Note that for the first, risk ceding portfolio capital costs are lower than the capital costs of the second, risk receiving portfolio.

In Figures 1 to 4 we plot optimal retentions, diversification and expected net loss and results as well as the loading of the reinsurance premium, all as a function of the net risk based capital. A proportional and a non-proportional excess-of-loss reinsurance agreement is considered. For illustration, we also plot a number of other relations, e.g. between retentions and reinsurance premium loading. Finally we also present surface plots illustrating the behavior of the net result over the whole retention range. Note that this type of figure is based on a 'brut-force' calculation taking into account a large number of possible combinations of retention levels r_i. For the determination of retention levels these calculations are not used. All Figures are based on the assumption that the primary insurer receives adequate premiums that compensate the expected loss and capital costs.

The quantitative behavior observed in Figures 1 to 4 is understood in terms of factors driving the expected net underwriting result of the primary insurer Z and the factors driving the reinsurance premium R. Main drivers are diversification and capital costs. Note for instance that the overall net result is increasing as retentions are lowering. Behind this feature is the fact the the diversification effects tend to increase as a bigger part of the

	Severity			Frequency
	Typ	μ	σ	λ
Risk Class 1a	Normal	2.0	1.0	2
Risk Class 1b	LogNormal	2.0	1.5	4
Risk Class 2a	Normal	2.0	1.0	4
Risk Class 2b	LogNormal	2.0	1.5	8

Table 1: Risk characteristics of the risk ceding (risk classes 1a and 1b) and the risk receiving portfolio (risk classes 2a and 2b). Risk classes are independent. Note that the risk receiving portfolio is more diversified

risk is shared. As well, sharing both types of risk turns out to be better than sharing just on type of risk. As can be seen from Figures 1 and 3 as well as Figures 2 and 4, optimal retentions are characterized by ceding similar parts of the risk. This again has to do with diversification effects. Finally note that in both the proportional and non-proportional case there is an absolute maximum of the net result. First of all, we assumed that capital costs for the portfolio taking over risk is higher than capital costs of the risk ceding portfolio. The more risk is ceded, the more the higher capital costs of the risk receiving portfolio impact, through the reinsurance premium, the net result. An interesting result of its own is the fact that from an absolute point of view the proportional reinsurance structure is favored over the non-proportional one. This can be traced back to diversification effects which tend to be bigger for higher net risk retention levels in the case of the proportional reinsurance structure.

	Risk Tolerance	Cost of Capital
Portfolio 1	10%	9%
Portfolio 2	10%	14%

Table 2: Risk tolerance levels and cost of capital for risk transferring portfolio (1) and risk receiving portfolio (2).

Looking at Figures 3, another important feature becomes apparent. There are situations, where optimal retentions are hard to isolate because another combination of retention levels is nearly as optimal. In this case numerical instabilities introduce noisy fluctuations in retention levels. However note that these fluctuations do not affect the maximization of the overall net result. In this sense, the noisy fluctuations observed especially in the lower left pane of Figure 3 is an allocation problem.

When interpreting these results, it should be kept in mind that the discussed behavior is specific for the simple model we have chosen. If for example dependency structures are considered either in the risk ceding, the risk receiving or between risk ceding and receiving portfolio, optimal retentions are clearly different. Again, this is due to different diversification effects. A positive dependency between risks in the transferring and receiving portfolio e.g. leads to a corresponding higher reinsurance premium and will thus favor transferring other, less dependent risks. Clearly, changing the risk characteristics (e.g. severity distribution) will have also an impact on optimal retentions.

5 Summary and Conclusions

The question of optimal risk transfer has a long history. In terms of minimizing the Variance of the overall net result, the setting of optimal retention levels of reinsurance programs dates back to de Finetti's 1940 paper [8]. A lot of work on this problem has been done since then, ranging from approaches relying on criteria related to adjustment coefficients to utility theory. The advent of new actuarial and financial risk management methods like e.g. coherent risk measures and Monte Carlo techniques used for ALM or DFA makes it possible to address the problem of optimal risk transfer from yet another point of view.

In this paper we consider the impact of risk transfer on the economic bottom line of an insurer by taking into account capital costs. Capital costs relate to risk based capital which we define by means of the Expected Shortfall. On sub-portfolio level, capital is allocated conditional to the overall portfolio thus consistently taking into account diversification effects. We assume that for both, the risk ceding and risk receiving portfolio similar capital allocation and risk pricing procedure are used. Starting from these assumptions, we derive a criterion for optimal risk retention levels on sub-portfolio level. Optimal retentions maximize the net result of including capital costs given a constraint on available risk based capital.

A simple numerical example illustrates the optimization scheme. While specific, the example illustrates generic features of the economics of risk transfer: To be most advantageous, not only risk but also risk diversification should be shared.

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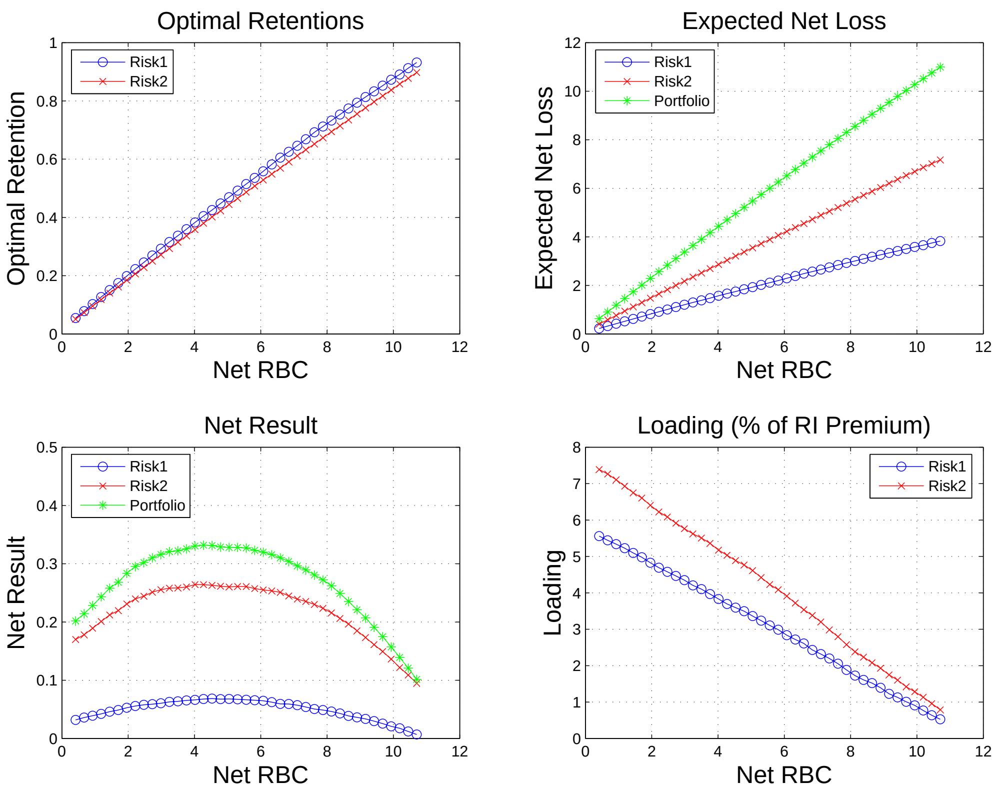

Figure 1: Proportional reinsurance. Optimal retentions and net result as a function of the total net risk based capital (left panes). Right panes show the expected net loss (top) and the loading of the reinsurance premium, again as a function of the net risk based capital.

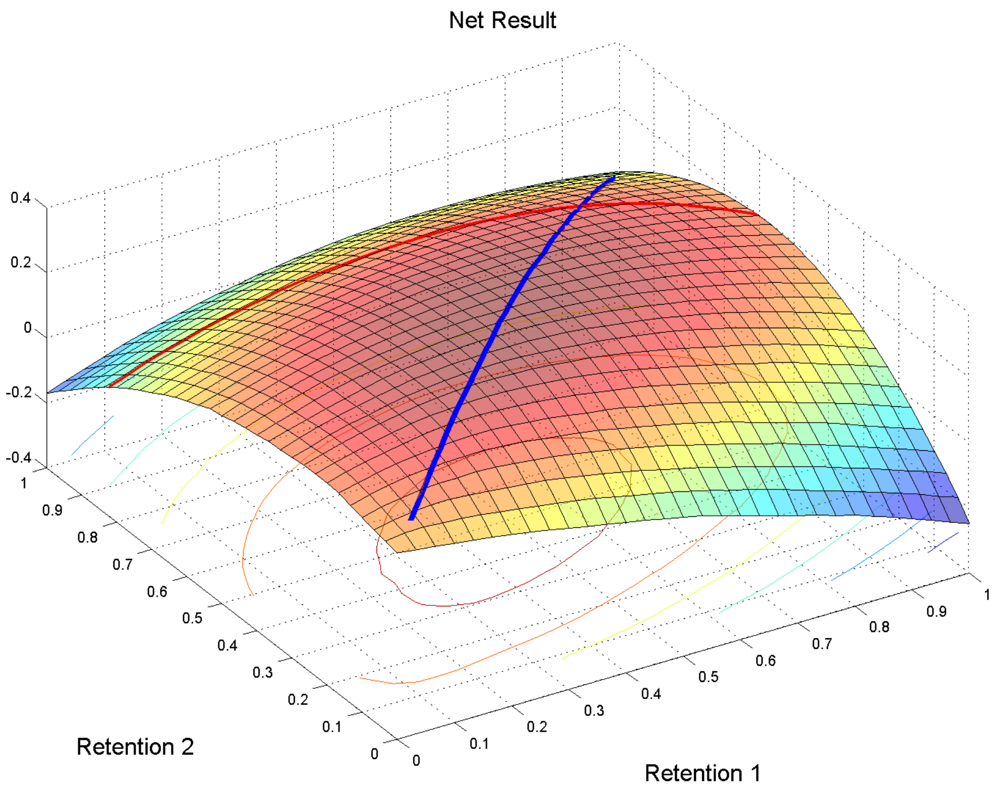

Figure 2: Proportional reinsurance. Net result as a function of retention levels. The blue (dark) line indicates optimal retentions as calculated from condition (3.20). The red (light) line represents the constraint net result at RBC=8.5. It illustrates the range of achievable net results with fixed RBC amount.

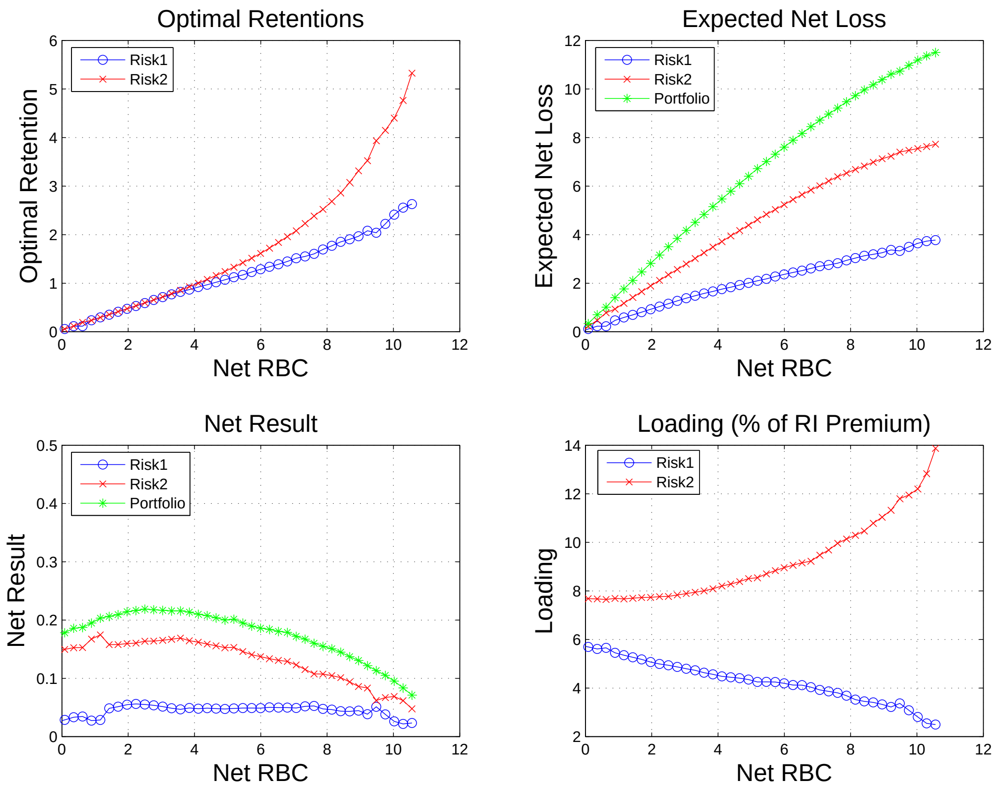

Figure 3: Non-proportional excess-of-loss reinsurance. Optimal retentions (or deductibles) and net result as a function of the total net risk based capital (left panes). Right panes show the expected net loss (top) and the loading of the reinsurance premium, again as a function of the net risk based capital. Note the noisy behavior of optimal retentions around RBC=9 and RBC=1.While the optimal net result on sub-portfolio level is affected (see lower left pane), the optimal overall net result is not. For a discussion see Section 4.

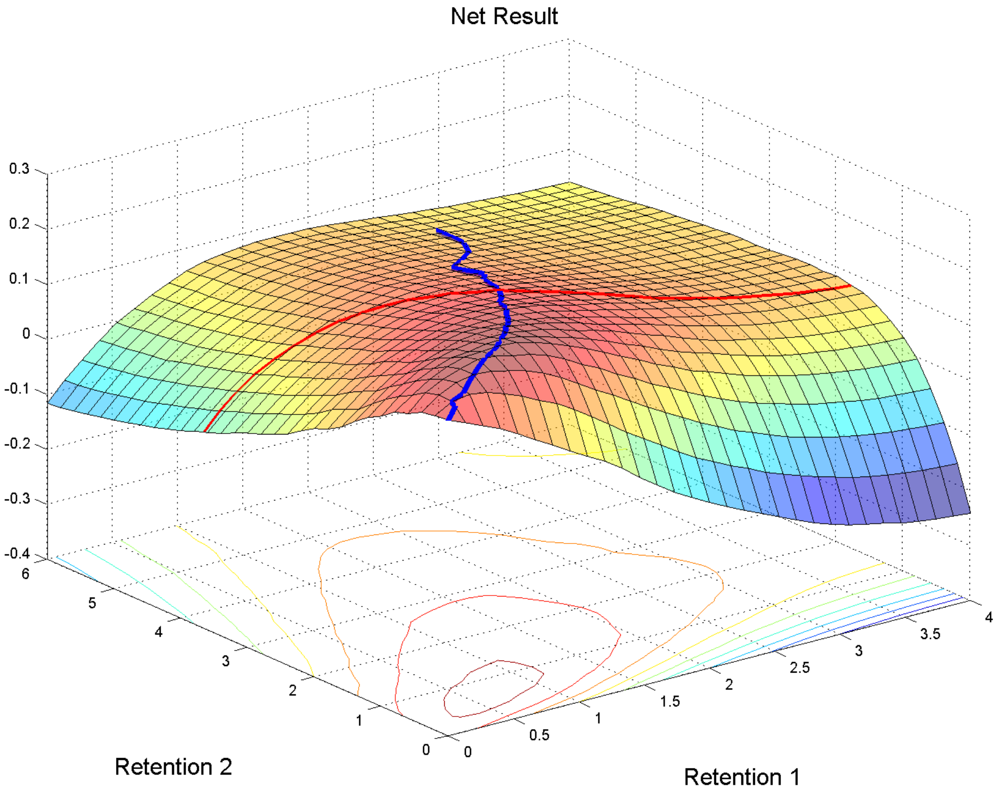

Figure 4: Non-proportional excess-of-loss reinsurance. Net result as a function of retention levels (or deductibles). The blue (dark) line indicates optimal retentions as calculated from condition (3.20). The red (light) line represents the constraint net result at RBC=8.5. It illustrates the range of achievable net results with fixed RBC amount.