The correlated chain-ladder method for reserving in case of multiple excess layers*

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Abstract

A handy forecasting method for calculating loss reserves in case of multiple excess layers is given. It is in some sense a further development of the classical chain ladder method to the context of correlated claims developments in multiple layers.

*The Paper is submitted to the international ASTIN-Colloquium at Zurich in 2005.

1 Introduction

During the last decades a lot was written on how to calculate adequately loss reserves in nonlife insurance. Certain survey books were published (see e.g. Taylor (1986), (2000), Institute of Actuaries (1998), Radtke & Schmidt (2004)). During the last ten years many attempts were made to refine and extend previous methods (see e.g. Kremer (1997), (1999a), (1999b), Doray (1996a), Taylor (2003)). New aspects were considered and totally new approaches presented (see e.g. Doray (1996b), (1997) and Herbst (1999).

That all was done for the situation where one has only one run-off-triangle. Recently at ASTIN Washington (2001) there was considered a more complicated situation. More concretely that of calculating loss reserves for the situation of K correlated loss triangles. Such situations appear when one looks at the loss development of an excess-of-loss cover with K layers. Two papers on that topic were given, one is Taylor (1996), the other Zehnwirth (2001). It is clear that the naive way to apply the classical chain-ladder method separately to each triangle is not very suitable, since then the correlations between different triangles are not incorporated into the forecasting procedure. One likes to have a further development of the chain-ladder technique that fits nicely to the new situation. Such a further development is given in the sequel.

2 Problem

Denote with the random variable $X_{ij}^{(k)}$ on $(\Omega, \mathcal{A}, P)$ the total claims amount or the number of claims of a (collective of) risk(s) in development year no. j with respect to its accident year no. i and this for the layer no. k of an excess-of-loss cover with K layers. Let n denote the number of periods until claims settlement for all layers no. $k = 1, \ldots, K$. Altogether one has K **run-off triangles**

$$X_{\triangle}^{(k)} = \left(X_{ij}^{(k)},\ j = 1, \ldots, n - i + 1,\ i = 1, \ldots, n \right), \quad k = 1, \ldots, K \ ,$$

each $X_{\triangle}^{(k)}$ for layer no. k. The problem consists in finding adequate, quite handy forecasts $\widehat{X}_{ij}^{(k)}$ for the $X_{ij}^{(k)}$ in the unknown right-lower triangles, meaning that one has

$$\begin{aligned} i &= 2, \ldots, n \\ j &= n - i + 2, \ldots, n \\ k &= 1, \ldots, K \ . \end{aligned}$$

3 Basic assumptions

For giving an adequate forecasting procedure suppose that the model

$$\boxed{X_{ij}^{(k)} = \alpha_j^{(k)} \cdot X_{i,j-1}^{(k)} + e_{ij}^{(k)}} \tag{3.1}$$
$$\text{with} \quad i = 1, \ldots, n \qquad j = 1, \ldots, n$$

holds for each triangle $k = 1, \ldots, K$, where $\alpha_j^{(k)}$ is an unknown growth factor and the $e_{ij}^{(k)}$ are error terms with

$$E(e_{ij}^{(k)} | \cdot) = 0 \tag{3.2}$$

where the conditioning "$|\cdot)$" is with respect to all $X_{i,j-1}^{(\ell)}, \ i = 1, \ldots, n, \ \ell = 1, \ldots, K$.

Suppose that one has:

(A) for fixed k the vectors $\left(X_{i1}^{(k)}, e_{i2}^{(k)}, \ldots, e_{in}^{(k)} \right)$ with $i = 1, \ldots, n$ are pairwise independent ($k = 1, \ldots, K$).

(B) for fixed k_1, k_2 with $k_1 \neq k_2$ the vectors

$$\left(X_{i_1 1}^{(k_1)}, e_{i_1 2}^{(k_1)}, \ldots, e_{i_1 n}^{(k_1)} \right), \ \left(X_{i_2 1}^{(k_2)}, e_{i_2 2}^{(k_2)}, \ldots, e_{i_2 n}^{(k_2)} \right)$$

are for $i_1 \neq i_2$ independent (for all k_i, i_j).

The vectors

$$\left(X_{i1}^{(k_1)}, e_{i2}^{(k_1)}, \ldots, e_{in}^{(k_1)} \right), \ \left(X_{i1}^{(k_2)}, e_{i2}^{(k_2)}, \ldots, e_{in}^{(k_2)} \right)$$

usually are dependent for $k_1 \neq k_2$. Assume for them

(C)
$$\begin{aligned} E\left(e_{ij}^{(k_1)} \cdot e_{ij}^{(k_2)} | \cdot \right) &= c_j^{(k_1,k_2)} \cdot \sqrt{X_{i,j-1}^{(k_1)} \cdot X_{i,j-1}^{(k_2)}} \\ &=: C_{ij}^{(k_1,k_2)} \end{aligned}$$

for $i = 1, \ldots, n \quad j = 2, \ldots, n$
$k_i = 1, \ldots, K (i = 1, 2)$ with $k_1 \neq k_2$,

where the conditioning "$|\cdot)$" is with respect to all $X_{i,j-1}^{(k)}, i = 1, \ldots, n,\ k = 1, \ldots, K$ (j fixed). The $c_j^{(k_1,k_2)}$ are unknown for $j = 2, \ldots, n, \quad k_i = 1, \ldots, K$.

Something like (C) has also to be assumed for $k_1 = k_2$, meaning

$$\text{(D)} \quad \begin{aligned} Var\left(e_{ij}^{(k)} | \cdot\right) &= v_j^{(k)} \cdot X_{i,j-1}^{(k)} \\ &=: V_{ij}^{(k)} \end{aligned}$$

for $i = 1, \ldots, n \quad j = 1, \ldots, n$
$k = 1, \ldots, K$,

where the conditioning "$|\cdot)$" is with respect to all $X_{i,j-1}^{(\ell)}, i = 1, \ldots, n,\ \ell = 1, \ldots, K$ (j fixed). The $v_j^{(k)}$ are unknown for all $j = 2, \ldots, n, \quad k = 1, \ldots, K$.

4 IBNR method

According to (3.1) and (3.2) it is nearlying to calculate the adequate forecasts $\widehat{X}_{ij}^{(k)}$ of $X_{ij}^{(k)}$ according to the recursions

$$\begin{aligned} \widehat{X}_{ij}^{(k)} &= \alpha_j^{(k)} \cdot \widehat{X}_{i,j-1}^{(k)} , \quad \text{for } j \geq n-i+3 \\ \widehat{X}_{i,n-i+2}^{(k)} &= \alpha_{n-i+2}^{(k)} \cdot X_{i,n-i+1}^{(k)} . \end{aligned} \tag{4.1}$$

But like told, the factors $\alpha_j^{(k)}$ are unknown model parameters. They have to be estimated adequately from the data of **all** K triangles. For fixed j the $\alpha_j^{(k)},\ k = 1, \ldots, K$ can be estimated with methods of regression analysis, since for fixed j one has the **linear model**

$$\boxed{X_j = D_j \cdot \alpha_j + e_j}$$

with vectors

$$\begin{aligned} \alpha_j &= \left(\alpha_j^{(1)}, \ldots, \alpha_j^{(K)}\right)^T \\ X_j &= \left(X_{1j}^{(1)}, \ldots, X_{n-j+1,j}^{(1)} | \ldots | X_{1j}^{(K)}, \ldots, X_{n-j+1,j}^{(K)}\right)^T \\ e_j &= \left(e_{1j}^{(1)}, \ldots, e_{n-j+1,j}^{(1)} | \ldots | e_{1j}^{(K)}, \ldots, e_{n-j+1,j}^{(K)}\right)^T \end{aligned}$$

and design-matrix

$$D_j = \left(\begin{array}{c|c|c|c}
\begin{matrix} X^{(1)}_{1,j-1} \\ \vdots \\ X^{(1)}_{n-j+1,j-1} \end{matrix} & 0 & \cdots & 0 \\
\hline
0 & \begin{matrix} X^{(2)}_{1,j-1} \\ \vdots \\ X^{(2)}_{n-j+1,j-1} \end{matrix} & \cdots & 0 \\
\hline
\vdots & \vdots & \ddots & \vdots \\
\hline
0 & 0 & \cdots & \begin{matrix} X^{(K)}_{1,j-1} \\ \vdots \\ X^{(K)}_{n-j+1,j-1} \end{matrix}
\end{array}\right)$$

of type $(K \cdot (n-j+1)) \times K$.

As estimator for α_j one is willing now, to take the well-known **Aitken-estimator**

$$\boxed{\widehat{\alpha}_j = \left(D_j^T W_j^{-1} D_j\right)^{-1} D_j^T W_j^{-1} X_j} \tag{4.2}$$

with the covariance matrix

$$W_j = Cov(e_j | \cdot)$$

where conditioning "$|\cdot)$" is with respect to the given $X^{(k)}_{i,j-1}$ (j fixed). Accordiing to assumptions (A) - (C) one gets for W_j the matrix shown on the next page.

Having the $\widehat{\alpha}_j = \left(\widehat{\alpha}^{(1)}_j, \ldots, \widehat{\alpha}^{(K)}_j\right)^T$ one inserts the components $\widehat{\alpha}^{(k)}_j$ for the $\alpha^{(k)}_j$ in (4.1) and then makes with recursions (4.1) the desired forecasts of the unknown

$$X^{(k)}_{ij}, \quad j \geq n-i+2, \ i \geq 2, \ \text{all } k.$$

$$
W_j = \left(\begin{array}{cc|cc|c|cc}
V_{1j}^{(1)} & 0 & C_{1j}^{(1,2)} & 0 & & C_{1j}^{(1,K)} & 0 \\
\ddots & & \ddots & & \cdots & \ddots & \\
0 & V_{n-j+1,j}^{(1)} & 0 & C_{n-j+1,j}^{(1,2)} & & 0 & C_{n-j+1,j}^{(1,K)} \\
\hline
C_{1j}^{(2,1)} & 0 & V_{1j}^{(2)} & 0 & & & \\
\ddots & & \ddots & & \cdots & \vdots & \\
0 & C_{n-j+1,j}^{(2,1)} & 0 & V_{n-j+1,j}^{(2)} & & & \\
\hline
\vdots & & \vdots & & \ddots & \vdots & \\
\hline
C_{1j}^{(K,1)} & 0 & & & & V_{1j}^{(K)} & 0 \\
\ddots & & \cdots & & \cdots & \ddots & \\
0 & C_{n-j+1,j}^{(K,1)} & & & & 0 & V_{n-j+1,j}^{(K)}
\end{array}\right) \tag{4.3}
$$

Obviously the thing is not such easily done, like written before. In W_j the parameters $v_j^{(k)}$, $c_j^{(k_1,k_2)}$ are unknown for all k, k_1, k_2! Also they have to be estimated!

Having the estimators $\widehat{\alpha}_j^{(k)}$ for the $\alpha_j^{(k)}$, the nearlying estimators for $v_j^{(k)}$, $c_j^{(k_1,k_2)}$ named by $\widehat{v}_j^{(k)}$, $\widehat{c}_j^{(k_1,k_2)}$ are

$$
\widehat{v}_j^{(k)} = \frac{\sum\limits_{i=1}^{n-j+1} \left(X_{ij}^{(k)} - \widehat{\alpha}_j^{(k)} \cdot X_{i,j-1}^{(k)} \right)^2}{\sum\limits_{i=1}^{n-j+1} X_{i,j-1}^{(k)}} \tag{4.4}
$$

$$
\widehat{c}_j^{(k_1,k_2)} = \frac{\sum\limits_{i=1}^{n-j+1} \left(X_{ij}^{(k_1)} - \widehat{\alpha}_j^{(k_1)} \cdot X_{i,j-1}^{(k_1)} \right) \left(X_{ij}^{(k_2)} - \widehat{\alpha}_j^{(k_2)} \cdot X_{i,j-1}^{(k_2)} \right)}{\sum\limits_{i=1}^{n-j+1} \sqrt{X_{i,j-1}^{(k_1)}} \sqrt{X_{i,j-1}^{(k_2)}}} \tag{4.5}
$$

Now, how to combine all? Nearlying is the following iterative estimation procedure:

One starts with the very special classical situation that

$$\boxed{c_j^{(k_1,k_2)} = 0 \quad \forall j, k_2, k_2} \; .$$

Then from (4.2) one gets the **classical chain-ladder estimators**

$$\begin{aligned}
\widehat{\alpha}_j^{(k)} &= \frac{\sum_{i=1}^{n-j+1} X_{ij}^{(k)}}{\sum_{i=1}^{n-j+1} X_{i,j-1}^{(k)}} \\
&=: \widehat{\alpha}_j^{(k)}(0) \; .
\end{aligned}$$

These one inserts into (4.4), (4.5), what gives first estimators $\widehat{v}_k^{(k)}(0), \widehat{c}_j^{(k_1,k_2)}(0)$ for the $v_j^{(k)}, c_j^{(k_1,k_2)}$. These one inserts into (4.3) and computes new estimates $\widehat{\alpha}_j^k(1)$ for the $\alpha_j^{(k)}$ with formula (4.2). Again one inserts these into (4.4), (4.5), leading to new estimates $\widehat{v}_j^{(k)}(1), \widehat{c}_j^{(k_1,k_2)}(1)$ for the $v_j^{(k)}, c_j^{(k_1,k_2)}$.

One goes on with this iteratively until the estimates $\widehat{\alpha}_j^k(m)$ have converged. The results one uses as final estimates $\widehat{\alpha}_j^{(k)}$ for the unknown $\alpha_j^{(k)}$.

Note that the above estimation procedure does not make sense for $j = n$. In that situation one clearly takes

$$\widehat{X}_{in}^{(k)} = \frac{X_{1n}^{(k)}}{X_{1,n-1}^{(k)}} \cdot X_{i,n-1}^{(k)}$$

for $i \geq 2$, the classical chain ladder advice.

5 Stricter case

One can be willing to assume that

$$\begin{aligned}
v_j^{(k)} &= v_j \quad \text{for all } k = 1, \ldots, K \\
c_j^{(k_1,k_2)} &= c_j \quad \text{for all } k_i = 1, \ldots, K, \; k_1 \neq k_2 \; .
\end{aligned} \tag{5.1}$$

Then one has less to estimate. From (4.4), (4.5) one gets also senseful estimators $\widehat{v}_j, \widehat{c}_j$ for

v_j and c_j

$$\begin{aligned} \widehat{v}_j &= \frac{1}{K} \cdot \sum_{k=1}^{K} \widehat{v}_j^{(k)} \\ \widehat{c}_j &= \frac{2}{K \cdot (K-1)} \cdot \sum_{k_1=1}^{K} \sum_{k_2=1}^{k_1-1} \widehat{c}_j^{(k_1,k_2)} \end{aligned}$$

These one puts into (4.3) (with (5.1)) and applies the same iterative estimation procedure like in the previous section.

But note the assumptions (5.1) are quite restrictive and perhaps too unrealistic.

6 Final remark

Note that the layer no. 1 is the ground-up layer which may not be reinsured, but for which is good experience available. Layers no. 2,3,... are the layers taken by reinsurers.

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