CREDIBLE LOSS RATIO CLAIMS RESERVES: THE BENKTANDER, NEUHAUS AND MACK METHODS REVISITED

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Abstract

The Benktander(1976) and Neuhaus(1992) credibility loss reserving methods are reconsidered in the framework of a credible loss ratio reserving method. The presented new method follows closely the review by Mack(2000) and leads to similar but different results. Two advantages over the Mack method are worthwhile to be mentioned. First, a pragmatic estimation of the required parameters leads to a straightforward calculation of the optimal credibility weights and mean squared errors of the credible reserves. An advantage of the collective loss ratio IBNR claims reserve over the Bornhuetter/Ferguson reserve in Mack(2000) is that different actuaries come always to the same results provided they use the same actuarial premiums.

Key words

IBNR, loss ratio method, credibility model, Benktander reserve, Neuhaus reserve

1. Introduction.

The design of the considered IBNR claims reserving method is inspired from and similar but different from the Benktander method reviewed in Mack(2000), Section 2.

Usually, the estimation of IBNR claims reserves is based on a full loss triangle of paid claims statistics, and some methods require additionally the knowledge of a measure of exposure for each underwriting period. The considered credible loss reserving method requires slightly less information. We suppose that there are n underwriting periods, for which one knows besides actuarial premiums V_i, $i = 1,...,n$, used as a measure of exposure, loss ratios m_k, $k = 1,...,n$, which represent the amount of claims per unit of actuarial premium required in the reporting period k. Additionally, we require the knowledge of the accumulated paid claims C_{in-i+1} for each underwriting period $i = 1,...,n$, which are reported in the latest period of development n.

Since the sum $\sum_{k=1}^{n} m_k$ represents the loss ratio over all reporting periods, the quantity $U_i^{BC} = V_i \cdot \sum_{k=1}^{n} m_k$ is nothing else than the loss ratio estimate or burning cost of the total ultimate claims required for the underwriting period i. This estimate is similar to the Bornhuetter/Ferguson prior estimate U_0 of the total ultimate claims in Mack(2000). By definition of the loss ratios, the quantities $V_i \cdot \sum_{k=1}^{n-i+1} m_k$, $i = 1,...,n$, represent the loss ratio estimate or burning cost of the paid claims for the underwriting period i, which are required in the current or latest period of development n. The *loss ratio payout* defined by

$$p_i = \frac{V_i \cdot \sum_{k=1}^{n-i+1} m_k}{U_i^{BC}} = \frac{\sum_{k=1}^{n-i+1} m_k}{\sum_{k=1}^{n} m_k}, \quad i = 1,...,n, \tag{1.1}$$

is a loss ratio estimate of the proportion of the total ultimate claims, which is expected to be paid for the underwriting period i. From this an estimate of the total ultimate claims is obtained by grossing up the latest accumulated paid claims amount. Since it is based solely on the individual latest claims experience of an underwriting year, it is called *individual total ultimate claims* amount and is given by

$$U_i^{ind} = \frac{C_{in-i+1}}{p_i}, \quad i = 1,...,n. \tag{1.2}$$

This estimate is similar to the chain-ladder estimate in Mack(2000). A corresponding estimate of the IBNR claims reserve, called *individual loss ratio IBNR claims reserve*, is defined by

$$R_i^{ind} = q_i \cdot U_i^{ind}, \quad i = 1,...,n, \tag{1.3}$$

where $q_i = 1 - p_i$ represents the proportion of the total ultimate claims, which is expected to be paid in the future for the underwriting period i. On the other side, the above burning cost estimate of the total ultimate claims leads to the alternative IBNR claims reserve

$$R_i^{coll} = q_i \cdot U_i^{BC}, \quad i = 1,...,n. \tag{1.4}$$

It is called *collective loss ratio IBNR claims reserve* because it depends solely on the portfolio claims experience of all underwriting periods.

Like the Bornhuetter/Ferguson and the chain-ladder estimates in Mack(2000), the considered collective and individual loss ratio IBNR claims reserve estimates represent extreme positions. Indeed, the individual claims reserve considers the latest accumulated claims amount to be fully credible predictive for future claims and ignores the prior burning cost estimate of the total ultimate claims, while the collective claims reserve ignores the current accumulated paid claims and relies fully on this prior estimate. Therefore it is natural to apply the credibility mixture to those reserves and use the *credible loss ratio IBNR claims reserve* estimate

$$R_i^c = Z_i \cdot R_i^{ind} + (1 - Z_i) \cdot R_i^{coll}, \quad i = 1,...,n, \tag{1.5}$$

where Z_i is the credibility weight associated to the individual loss ratio reserve. It interesting to reconsider two popular choices of the credibility weights proposed in the literature. As the credibility weight should increase similarly as the accumulated claims C_{in-i+1} develop, Gunnar Benktander(1976) proposed the credibility weight $Z_i^{GB} = p_i, \quad i = 1,...,n$. This leads to the *Benktander loss ratio IBNR claims reserve*

$$R_i^{GB} = p_i \cdot R_i^{ind} + q_i \cdot R_i^{coll}, \quad i = 1,...,n. \tag{1.6}$$

According to Mack(1997), p.242, the choice made by Walter Neuhaus(1992) corresponds to the credibility weight $Z_i^{WN} = \sum_{k=1}^{n-i+1} m_k$. It leads to the *Neuhaus loss ratio IBNR claims reserve*

$$R_i^c = Z_i^{WN} \cdot R_i^{ind} + (1 - Z_i^{WN}) \cdot R_i^{coll}, \quad i = 1,...,n. \tag{1.7}$$

It is remarkable that in numerical examples these simple choices are both quite close to an *optimal credible loss ratio IBNR claims reserve*, whose credibility weights are derived in Section 3.

The organisation of the paper is as follows. In Section 2, the mentioned collective and individual loss ratio IBNR claims reserves are introduced. Section 3 defines the credible loss ratio IBNR claims reserves. Formulas for the optimal credibility weights and the mean squared error of the credible loss ratio reserve are derived in Section 4. The practical evaluation of these quantities is based on a pragmatic estimation method, which is motivated in Section 5. In Section 6, the remaining unspecified parameters of the pragmatic estimation method are chosen such that they minimize the variance of the optimal credible claims reserve. This is a desirable property because from a statistical point of view estimates with lower variances are usually preferred. Fortunately, the minimum variance optimal credible claims reserve is parameter-free and comparable in simplicity to the Neuhaus and Benktander loss ratio reserves. Finally, Section 7 presents numerical examples. Based on our pragmatic estimation method, we observe that the Neuhaus and Benktander loss ratio reserves are quite close to the optimal credible reserve. In our examples, the Neuhaus reserve is closer to the optimal one than the Benktander reserve for all underwriting years. Through application of a credible loss ratio reserving method, the reduction in mean squared error is substantial. In absence of sufficient information to estimate the optimal credibility weights, the three simple credible methods are highly recommended for actuarial practice.

2. The collective and individual loss ratio IBNR claims reserves.

The *total ultimate claims* of the claims incurred in a given year, known in the future when all claims have been closed and paid out, is defined as follows:

total ultimate claims = paid claims + outstanding claims reserve

+ IBNR claims reserve (**I**ncurred **B**ut **N**ot **R**eported claims reserve)

Let n be the number of periods, here one-year periods, for which historical data on paid claims is available. Let S_{ik}, $1 \le i,k \le n$, be the paid claims in period i and reported in period $i+k-1$. The restriction to the claims of a calendar year yields the condition $i+k-1 \le n$. Under the assumption that after n periods all claims occurred in a period are known and closed, the amount $\sum_{k=1}^{n} S_{ik}$ is the total amount of claims occurred in period i. However, at the end of the calendar year only the amount $\sum_{k=1}^{n-i+1} S_{ik}$ is known. The required amount for the incurred but not reported claims of period i, called *i-th period IBNR claims reserve*, is equal to $R_i = \sum_{k=n-i+2}^{n} S_{ik}$, $i = 2,...,n$. An estimate for the total required amount of incurred but not reported claims over all periods $R = \sum_{i=2}^{n} R_i$ is called *total IBNR reserve*. In the following, the sums $C_{ik} = \sum_{j=1}^{k} S_{ij}$, $1 \le i,k \le n$, denote the accumulated paid claims incurred in calendar year i and reported after $i+k-1$ years of development.

Usually, the estimation of IBNR claims reserves is based on a full loss triangle of paid claims statistics S_{ik}, $1 \le i,k \le n$, subject to the restriction $i+k-1$. Some methods require additionally the knowledge of a measure of exposure V_i for each underwriting period $i = 1,...,n$. To fix ideas, we suppose that V_i are actuarial premiums. Then the quotients $\frac{S_{ik}}{V_i}$ represent the claims loss ratio in the underwriting period i and reporting period $i+k-1$. The considered credible loss reserving method requires slightly less information. Besides the actuarial premiums V_i, we suppose that the loss ratios

$$m_k = \frac{\sum_{i=1}^{n-k+1} S_{ik}}{\sum_{i=1}^{n-k+1} V_i}, \quad k = 1,\ldots,n, \tag{2.1}$$

which represent the amount of claims per unit of actuarial premium required in the reporting period k, are given known quantities. Additionally, we require the knowledge of the accumulated paid claims C_{in-i+1} for each underwriting period $i = 1,\ldots,n$, which are reported in the latest period of development n.

Remark 2.1.

It is possible to use other exposure measures. For example, suppose that instead of actuarial premiums the number N_i of accumulated claims for the underwriting period i at the latest period of development n is known. Then one proceeds as follows to estimate an exposure measure V_i similar to an actuarial premium measure. Consider the total burning costs of the latest period of development $BC = \sum_{i=1}^{n} C_{in-i+1}$ and let $N = \sum_{i=1}^{n} N_i$ be the total number of claims for the same period of development. Then the ratio BC/N represents the burning cost per claims and $V_i = (N_i/N) \cdot BC$ is an appropriate exposure measure, which can be used instead of actuarial premiums.

The design of the considered IBNR claims reserving method is inspired from and similar but different from the Benktander method reviewed in Mack(2000), Section 2. Since the sum $\sum_{k=1}^{n} m_k$ represents the loss ratio over all reporting periods, the quantity

$$U_i^{BC} = V_i \cdot \sum_{k=1}^{n} m_k \tag{2.2}$$

is nothing else than the loss ratio estimate or burning cost of the total ultimate claims required for the underwriting period i. This estimate is similar to the Bornhuetter/Ferguson prior estimate U_0 of the total ultimate claims in Mack(2000). There is a close connection between (2.2) and the IBNR claims reserve set according to the loss ratio reserving method as defined in Mack(1997), Section 3.2.2, p. 230-234. The usual *loss ratio IBNR claims reserve* for the underwriting period i is defined by

$$R_i^{LR} = V_i \cdot \sum_{k=n-i+2}^{n} m_k, \quad i = 2,\ldots,n, \quad R_1^{LR} = 0. \tag{2.3}$$

The corresponding amount of estimated total ultimate claims is given by

$$U_i^{LR} = R_i^{LR} + C_{in-i+1}, \quad i = 1,\ldots,n. \tag{2.4}$$

It is remarkable that the total ultimate claims over all underwriting periods are the same for the two different estimates (2.2) and (2.4).

Theorem 2.1. The burning cost estimate of the total ultimate claims over all underwriting periods coincides with the estimate obtained from the loss ratio reserving method, that is

$$\sum_{i=1}^{n} U_i^{LR} = \sum_{i=1}^{n} U_i^{BC} . \tag{2.5}$$

Proof. Changing the order of summation several times and using the definition of the loss ratio m_k, one has

$$\sum_{i=1}^{n} C_{in-i+1} = \sum_{i=1}^{n} \sum_{k=1}^{n-i+1} S_{ik} = \sum_{k=1}^{n} \sum_{i=1}^{n-k+1} S_{ik} = \sum_{k=1}^{n} m_k \cdot \sum_{i=1}^{n-k+1} V_i = \sum_{i=1}^{n} V_i \cdot \sum_{k=1}^{n-i+1} m_k . \tag{2.6}$$

It follows that

$$\sum_{i=1}^{n} U_i^{LR} = \sum_{i=1}^{n} C_{in-i+1} + \sum_{i=1}^{n} V_i \sum_{k=n-i+2}^{n} m_k = \sum_{i=1}^{n} V_i \sum_{k=1}^{n} m_k = \sum_{i=1}^{n} U_i^{BC} . \; \Diamond \tag{2.7}$$

Starting from the loss ratio estimates and the latest accumulated paid claims, there is another way to define an estimate of the total ultimate claims. By definition of the loss ratios, the quantities $V_i \cdot \sum_{k=1}^{n-i+1} m_k, \quad i = 1,...,n,$ represent the loss ratio estimate or burning cost of the paid claims for the underwriting period i, which are required in the current or latest period of development n. The *loss ratio payout* defined by

$$p_i = \frac{V_i \cdot \sum_{k=1}^{n-i+1} m_k}{U_i^{BC}} = \frac{\sum_{k=1}^{n-i+1} m_k}{\sum_{k=1}^{n} m_k}, \quad i = 1,...,n, \tag{2.8}$$

is a loss ratio estimate of the proportion of the total ultimate claims, which is expected to be paid for the underwriting period i. From this a second estimate of the total ultimate claims is obtained by grossing up the latest accumulated paid claims amount. Since it is based solely on the individual latest claims experience of an underwriting year, it is called *individual total ultimate claims* amount and is given by

$$U_i^{ind} = \frac{C_{in-i+1}}{p_i}, \quad i = 1,...,n . \tag{2.9}$$

This estimate is similar to the chain-ladder estimate in Mack(2000). The corresponding estimate of the IBNR claims reserve is called *individual loss ratio IBNR claims reserve* and satisfies the relations

$$R_i^{ind} = U_i^{ind} - C_{in-i+1} = q_i \cdot U_i^{ind} = \frac{q_i}{p_i} \cdot C_{in-i+1}, \quad i = 1,...,n, \tag{2.10}$$

where one sets $q_i = 1 - p_i, \quad i = 1,...,n$. Similarly, the first loss ratio estimate (2.2) leads to the alternative IBNR claims reserve

$$R_i^{coll} = q_i \cdot U_i^{BC}, \quad i = 1,...,n. \tag{2.11}$$

It is called *collective loss ratio IBNR claims reserve* because it depends solely on the portfolio claims experience of all underwriting periods. It immediately seen that the latter estimate coincides with the usual loss ratio IBNR claims reserve (2.3). It follows that the associated *collective total ultimate claims* amount coincides with the loss ratio estimate (2.4), that is

$$U_i^{coll} = R_i^{LR} + C_{in-i+1} = U_i^{LR}, \quad i = 1,...,n. \tag{2.12}$$

This estimate is similar to the Bornhuetter/Ferguson posterior estimate of the total ultimate claims in Mack(2000).

3. Credible loss ratio IBNR claims reserves.

Like the Bornhuetter/Ferguson and the chain-ladder estimates in Mack(2000), the considered collective and individual loss ratio IBNR claims reserve estimates represent extreme positions. Indeed, the individual claims reserve $R_i^{ind} = \frac{q_i}{p_i} \cdot C_{in-i+1}$ considers the latest accumulated claims amount to be fully credible predictive for future claims and ignores the prior estimate U_i^{BC} of the total ultimate claims, while the collective claims reserve $R_i^{coll} = q_i \cdot U_i^{BC}$ ignores the current accumulated paid claims and relies fully on this prior estimate. An advantage of the collective loss ratio IBNR claims reserve over the Bornhuetter/Ferguson reserve in Mack(2000) is that different actuaries come always to the same results provided they use the same actuarial premiums. This is also true for the individual claims reserve without restriction.

In view of these extreme positions, it is natural to apply the credibility mixture to those reserves and use the *credible loss ratio IBNR claims reserve* estimate

$$R_i^c = Z_i \cdot R_i^{ind} + (1 - Z_i) \cdot R_i^{coll}, \quad i = 1,...,n, \tag{3.1}$$

where Z_i is the credibility weight associated to the individual loss ratio reserve. It interesting to reconsider two popular choices of the credibility weights proposed in the literature. As the credibility weight should increase similarly as the accumulated claims C_{in-i+1} develop, Gunnar Benktander(1976) proposed the credibility weight $Z_i^{GB} = p_i, \quad i = 1,...,n$. This leads to the *Benktander loss ratio IBNR claims reserve*

$$R_i^{GB} = p_i \cdot R_i^{ind} + q_i \cdot R_i^{coll}, \quad i = 1,...,n. \tag{3.2}$$

According to Mack(1997), p.242, the choice made by Walter Neuhaus(1992) corresponds to the credibility weight $Z_i^{WN} = \sum_{k=1}^{n-i+1} m_k$. It leads to the *Neuhaus loss ratio IBNR claims reserve*

$$R_i^{WN} = Z_i^{WN} \cdot R_i^{ind} + \left(1 - Z_i^{WN}\right) \cdot R_i^{coll}, \quad i = 1,...,n. \tag{3.3}$$

It is remarkable that in numerical examples (see Section 6) these simple choices are both quite close to an *optimal credible loss ratio IBNR claims reserve*, whose credibility weights are derived in Section 3.

On the other side, remarks similar to those made by Mack(2000) at the end of Section 2 can be made. The functions $R_i(U_i) = q_i U_i$ and $U_i(R_i) = R_i + C_{in-i+1}$ are not inverse to each other except for $U_i = U_i^{ind}$. Similarly to the "iterated Bornhuetter/Ferguson method", there is an "iterated collective loss ratio reserving method". The successive iteration of the collective and Benktander loss ratio reserving methods for an arbitrary start point U_i^0 leads in the infinite limit to the individual loss ratio reserving method. It is worthwhile to state this result, which paraphrases Theorem 1 in Mack(2000).

Theorem 3.1. For an arbitrary starting point $U_i^{(0)} = U_i^0$, the iteration rule

$$R_i^{(m)} = q_i \cdot U_i^{(m)}, \quad U_i^{(m+1)} = C_{in-i+1} + R_i^{(m)}, \quad m = 0,1,2,..., \tag{3.4}$$

gives credibility mixtures

$$\begin{aligned} U_i^{(m)} &= \left(1 - q_i^m\right) \cdot U_i^{ind} + q_i^m \cdot U_i^0, \\ R_i^{(m)} &= \left(1 - q_i^m\right) \cdot R_i^{ind} + q_i^m \cdot R_i^0 \end{aligned} \tag{3.5}$$

between the collective and individual loss ratio reserving methods, which starts at the collective method and lead via the Benktander method finally to the individual method for $m = \infty$.

4. The optimal credibility weights and the mean squared error.

In the following, we suppose that the burning cost estimate U_i^{BC} of the total ultimate claims is an estimation function which is independent from C_{in-i+1}, R_i and U_i, and has expectation $E\left[U_i^{BC}\right] = E\left[U_i\right]$ and variance $Var\left[U_i^{BC}\right]$.

Theorem 4.1. The optimal credibility weights Z_i^* which minimize the mean squared error $mse\left(R_i^c\right) = E\left[\left(R_i^c - R_i\right)^2\right]$ is given by

$$Z_i^* = \frac{p_i}{q_i} \cdot \frac{Cov\left[C_{in-i+1}, R_i\right] + p_i q_i \cdot Var\left[U_i^{BC}\right]}{Var\left[C_{in-i+1}\right] + p_i^2 \cdot Var\left[U_i^{BC}\right]}. \tag{4.1}$$

Proof. By definition (3.1) of the credible loss reserve one has

$$\begin{aligned} &E\left[\left(R_i^c - R_i\right)^2\right] = E\left[\left(Z_i \cdot \left(R_i^{ind} - R_i^{coll}\right) + R_i^{coll} - R_i\right)^2\right] \\ &= Z_i^2 \cdot E\left[\left(R_i^{ind} - R_i^{coll}\right)^2\right] - 2Z_i \cdot E\left[\left(R_i^{ind} - R_i^{coll}\right)\left(R_i - R_i^{coll}\right)\right] + E\left[\left(R_i^{coll} - R_i\right)^2\right] \end{aligned}$$

From the first order condition

$$\frac{\partial}{\partial Z_i} E\left[\left(R_i^c - R_i\right)^2\right] = 2Z_i \cdot E\left[\left(R_i^{ind} - R_i^{coll}\right)^2\right] - 2 \cdot E\left[\left(R_i^{ind} - R_i^{coll}\right)\left(R_i - R_i^{coll}\right)\right] = 0,$$

one obtains that

$$Z_i^* = \frac{E\left[\left(R_i^{ind} - R_i^{coll}\right)\left(R_i - R_i^{coll}\right)\right]}{E\left[\left(R_i^{ind} - R_i^{coll}\right)^2\right]}, \quad i = 1,\dots,n. \tag{4.2}$$

Inserting the expressions $R_i^{ind} = q_i \cdot U_i^{ind} = \frac{q_i}{p_i} \cdot C_{in-i+1}$ and $R_i^{coll} = q_i \cdot U_i^{BC}$ one gets

$$Z_i^* = \frac{p_i}{q_i} \cdot \frac{Cov\left[C_{in-i+1} - p_i U_i^{BC}, R_i - q_i U_i^{BC}\right]}{Var\left[C_{in-i+1} - p_i U_i^{BC}\right]}. \tag{4.3}$$

By definition of the loss ratio payout one has $E\left[C_{in-i+1}\right] = p_i E\left[U_i\right] = p_i E\left[U_i^{BC}\right]$, hence $E\left[R_i\right] = E\left[U_i - C_{in-i+1}\right] = (1 - p_i) \cdot E\left[U_i^{BC}\right] = q_i \cdot E\left[R_i^{BC}\right]$. Using this and the assumption that U_i^{BC} is independent from C_{in-i+1}, R_i and U_i, one gets the desired formula (4.1). ◊

To estimate the optimal credibility weights, one needs a model for $Var\left[C_{in-i+1}\right]$ and $Cov\left[C_{in-i+1}, R_i\right]$. Consider the following model for the loss ratio payout:

$$E\left[\frac{C_{in-i+1}}{U_i} \middle| U_i\right] = p_i, \quad Var\left[\frac{C_{in-i+1}}{U_i} \middle| U_i\right] = p_i q_i \beta_i^2(U_i), \quad i = 1,\dots,n. \tag{4.4}$$

The factor q_i ensures that $Var\left[\frac{C_{in-i+1}}{U_i} \middle| U_i\right] = 0$ when $i = 1$ and that $Var\left[\frac{C_{in-i+1}}{U_i} \middle| U_i\right] \to 0$ in case of very small values p_i. In the following the notation $\alpha_i^2(U_i) = U_i^2 \cdot \beta_i^2(U_i)$ is used.

<u>Theorem 4.2</u>. Under the assumption of model (4.4), the optimal credibility weights Z_i^* which minimize the mean squared error $mse\left(R_i^c\right) = E\left[\left(R_i^c - R_i\right)^2\right]$ is given by

$$Z_i^* = \frac{p_i}{p_i + t_i^*}, \text{ with} \tag{4.5}$$

$$t_i = \frac{E\left[\alpha_i^2(U_i)\right]}{Var\left[U_i^{BC}\right] + Var\left[U_i\right] - E\left[\alpha_i^2(U_i)\right]}, \quad i = 1,\dots,n. \tag{4.6}$$

<u>Proof</u>. From (4.4) one obtains

$$E\left[C_{in-i+1} \middle| U_i\right] = p_i U_i, \quad Var\left[C_{in-i+1} \middle| U_i\right] = p_i q_i \alpha_i^2(U_i), \quad i = 1,\dots,n. \tag{4.7}$$

It follows that

$$\begin{aligned}Var[C_{in-i+1}] &= E\left[Var[C_{in-i+1}|U_i]\right]+Var\left[E[C_{in-i+1}|U_i]\right]\\ &= p_iq_iE\left[\alpha_i^2(U_i)\right]+p_i^2Var[U_i]\\ &= p_iE\left[\alpha_i^2(U_i)\right]+p_i^2\cdot\left(Var[U_i]-E\left[\alpha_i^2(U_i)\right]\right),\end{aligned} \tag{4.8}$$

and

$$Cov[C_{in-i+1},U_i]=E\left[Cov[C_{in-i+1},U_i|U_i]\right]+Cov\left[E[C_{in-i+1}|U_i],U_i\right]=p_iVar[U_i]. \tag{4.9}$$

From (4.8) and (4.9) one obtains further

$$Cov[C_{in-i+1},R_i]=Cov[C_{in-i+1},U_i]-Var[C_{in-i+1}]=p_iq_i\cdot\left(Var[U_i]-E\left[\alpha_i^2(U_i)\right]\right). \tag{4.10}$$

Inserting (4.8) and (4.10) into (4.1) the desired formula follows. ◊

Simple formulas for the mean squared errors are derived in a similar way.

Theorem 4.3. Under the assumption of model (4.4), the following formulas for the mean squared error hold:

$$\begin{aligned}mse\left(R_i^{coll}\right) &= E\left[\alpha_i^2(U_i)\right]\cdot q_i\left(1+\frac{q_i}{t}\right),\\ mse\left(R_i^{ind}\right) &= E\left[\alpha_i^2(U_i)\right]\cdot\frac{q_i}{p_i},\\ mse\left(R_i^{c}\right) &= E\left[\alpha_i^2(U_i)\right]\cdot\left(\frac{Z_i^2}{p_i}+\frac{1}{q_i}+\frac{(1-Z_i)^2}{t_i}\right)\cdot q_i^2.\end{aligned} \tag{4.11}$$

Proof. Using (4.8) and (4.9) one obtains

$$\begin{aligned}Var[R_i] &= Var[U_i-C_{in-i+1}]=Var[U_i]-2Cov[C_{in-i+1},U_i]+Var[C_{in-i+1}]\\ &= Var[U_i]\cdot\left(1-2p_i+p_i^2\right)+p_iq_iE\left[\alpha_i^2(U_i)\right]=q_i^2\cdot Var[U_i]+p_iq_iE\left[\alpha_i^2(U_i)\right]\\ &= q_iE\left[\alpha_i^2(U_i)\right]+q_i^2\cdot\left(Var[U_i]-E\left[\alpha_i^2(U_i)\right]\right).\end{aligned} \tag{4.12}$$

Since $E\left[R_i^{coll}\right]=q_iE\left[U_i^{BC}\right]=q_iE[U_i]=E[U_i-C_{in-i+1}]=E[R_i]$ and by assumption $Cov\left[R_i^{coll},R_i\right]=q_iCov\left[U_i^{BC},R_i\right]=0$, one has

$$\begin{aligned}mse\left(R_i^{coll}\right) &= E\left[\left(R_i^{coll}-R_i\right)^2\right]=Var\left[R_i^{coll}-R_i\right]=Var\left[R_i^{coll}\right]+Var[R_i]\\ &= q_i^2\cdot Var\left[U_i^{BC}\right]+q_i^2\cdot\left(Var[U_i]-E\left[\alpha_i^2(U_i)\right]\right)+q_i\cdot E\left[\alpha_i^2(U_i)\right]=E\left[\alpha_i^2(U_i)\right]\cdot\left(q_i+\frac{q_i^2}{t}\right),\end{aligned} \tag{4.13}$$

where the last equality follows from (4.6) of Theorem 4.2. Similarly, one has $E\left[R_i^{ind}\right]=E[R_i]$ and it follows that

$$mse\left(R_i^{ind}\right) = E\left[\left(R_i^{ind} - R_i\right)^2\right] = Var\left[R_i^{ind} - R_i\right] = Var\left[R_i^{coll}\right] - 2Cov\left[R_i^{ind}, R_i\right] + Var\left[R_i\right]$$
$$= \left(\frac{q_i}{p_i}\right)^2 \cdot Var\left[C_{in-i+1}\right] - 2\frac{q_i}{p_i} Cov\left[C_{in-i+1}, R_i\right] + Var\left[R_i\right]. \tag{4.14}$$

Inserting (4.8), (4.10) and (4.12) one gets without difficulty the desired formula (4.11). The third formula follows from

$$mse\left(R_i^c\right) = E\left[\left(Z_i(R_i^{ind} - R_i) + (1-Z_i)(R_i^{coll} - R_i)\right)^2\right]$$
$$= Z_i^2 mse\left(R_i^{ind}\right) + 2Z_i(1-Z_i)E\left[\left(R_i^{ind} - R_i\right)\left(R_i^{coll} - R_i\right)\right] + (1-Z_i)^2 mse\left(R_i^{coll}\right), \tag{4.15}$$

and

$$E\left[\left(R_i^{ind} - R_i\right)\left(R_i^{coll} - R_i\right)\right] = Cov\left[R_i^{ind} - R_i, R_i^{coll} - R_i\right] = Var\left[R_i\right] - Cov\left[R_i^{ind}, R_i\right]$$
$$= Var\left[R_i\right] - \frac{q_i}{p_i} Cov\left[C_{in-i+1}, R_i\right] = q_i E\left[\alpha_i^2(U_i)\right] \tag{4.16}$$

using the formulas for $mse\left(R_i^{ind}\right)$ and $mse\left(R_i^{coll}\right)$. ◊

5. A pragmatic estimation method.

To evaluate the optimal credibility weights and the mean squared errors, it is necessary to estimate the quantities $Var\left[U_i^{BC}\right]$, $Var\left[U_i\right]$ and $E\left[\alpha_i^2(U_i)\right]$. The proposed estimators are based on a full loss triangle of paid claims statistics S_{ik}, $1 \le i,k \le n$, subject to the restriction $i+k-1$, and the knowledge of exposures V_i for each underwriting period $i = 1,...,n$. For $Var\left[U_i^{BC}\right]$ we use the following standard estimate, which follows from the analysis by Mack(1997), p. 231-233:

$$Var\left[\hat{U}_i^{BC}\right] = V_i^2 \cdot \left(\sum_{k=1}^{n} \frac{s_k^2}{w_k}\right), \text{ with} \tag{5.1}$$

$$w_k = \sum_{i=1}^{n-k+1} V_i, \quad s_k^2 = \frac{1}{n-k} \sum_{i=1}^{n-k+1} V_i \left(\frac{S_{ik}}{V_i} - m_k\right)^2, \quad k = 1,...,n-1,$$
$$s_n^2 = \min\left\{s_k^2 \middle| k = 1,...,n-1\right\} \tag{5.2}$$

It is intuitively appealing that U_i should be at least as volatile than the burning cost estimate U_i^{BC} (similar to the fact that $Var[U]$ should be larger than $Var[U_0]$ in Mack(2000)). As pragmatic estimates we assume that

$$Var\left[\hat{U}_i\right] = f \cdot Var\left[\hat{U}_i^{BC}\right], \quad E\left[\hat{U}_i\right] = U_i^{BC}, \tag{5.3}$$

with some factor $f \geq 1$, and that an estimate of $\beta_i^2(U_i)$ in (4.4) is a constant β_i^2 for all underwriting periods $i = 1,...,n$. The quantities f and β_i can be statistically justified as follows. Arguing that U_i^{BC} is a good unbiased point estimate of the total ultimate claims, it is reasonable to assume that U_i belongs to the confidence interval $\left[U_i^{BC} - c \cdot \sqrt{\widehat{Var}\left[U_i^{BC}\right]}, U_i^{BC} + c \cdot \sqrt{\widehat{Var}\left[U_i^{BC}\right]}\right]$ for some constant $c > 1$. On the other side U_i should belong to the confidence interval $\left[\hat{E}[U_i] - d \cdot \sqrt{\widehat{Var}[U_i]}, \hat{E}[U_i] + d \cdot \sqrt{\widehat{Var}[U_i]}\right]$ for some constant $d \leq c$ (U_i at least as volatile than U_i^{BC}). If $\hat{E}[U_i] = U_i^{BC}$, one must have $\widehat{Var}[U_i] = f \cdot \widehat{Var}\left[U_i^{BC}\right]$ for some constant $f \geq 1$. On the other side, if the ratio C_{in-i+1}/U_i given U_i has a $\text{Beta}(a_i p_i, a_i q_i)$ distribution with some constant $a_i > 0$, then one has necessarily $\beta_i^2(U_i) = (a_i + 1)^{-1}$, which is a constant independent of U_i. Recalling that $\alpha_i^2(U_i) = U_i^2 \cdot \beta_i^2(U_i)$, one obtains from (5.3) the following estimate

$$\hat{E}\left[\alpha_i^2(U_i)\right] = \beta_i^2 \cdot \left\{ f \cdot \widehat{Var}\left[U_i^{BC}\right] + \left(U_i^{BC}\right)^2 \right\}. \tag{5.4}$$

The above estimates are inserted in the formulas of Theorems 4.2 and 4.3 to get estimates of the optimal credibility weights and the mean squared errors. In particular, an optimal credibility estimate is obtained from

$$\hat{t}_i = \frac{\beta_i^2 \cdot (f + \hat{u})}{1 + f - \beta_i^2 \cdot (f + \hat{u})}, \tag{5.5}$$

where $\hat{u}_i = \dfrac{\left(U_i^{BC}\right)^2}{\widehat{Var}\left[U_i^{BC}\right]} = \dfrac{\left(\sum_{k=1}^{n} m_k\right)^2}{\sum_{k=1}^{n} \frac{s_k^2}{w_k}} = \hat{u}$ is an estimate of the inverse of the coefficient of variation of the burning cost estimate U_i^{BC} of the total ultimate claims, which is independent of the underwriting period.

6. <u>The optimal credible claims reserve with minimum variance</u>.

The pragmatic estimation method of Section 5 depends on the unknown parameter $(f, \beta_1, ..., \beta_n)$. In the present Section, we compare the variances of the individual, collective and optimal credible claims reserves in order to determine a set of parameters $(f, \beta_1, ..., \beta_n)$, which minimizes the variance of the optimal credible claims reserve. This is a desirable property because from a statistical point of view estimates with lower variances are usually preferred. Fortunately, the minimum variance optimal credible claims reserve is parameter-free and attained at the parameter value

$$t_i^* = \tfrac{1}{2}\left(q_i + \sqrt{q_i^2 + 4}\right), \quad i = 1,\dots,n, \tag{6.1}$$

in Theorem 4.2. It compares with the Benktander estimate

$$t_i^{GB} = q_i, \quad i = 1,\dots,n, \tag{6.2}$$

and the Neuhaus estimate

$$t_i^{WN} = q_i + \frac{1 - \sum_{k=1}^{n} m_k}{\sum_{k=1}^{n} m_k}, \quad i = 1,\dots,n. \tag{6.3}$$

All three methods yield monotone decreasing credibility weights in the underwriting periods. Since $t_1^* = 1$ the optimal credibility weights satisfy the inequality

$$Z_i^* \le \tfrac{1}{2}, \quad i = 1,\dots,n, \tag{6.4}$$

where the equality is attained for the first underwriting period. Note that usually the Benktander and Neuhaus methods lead to higher credibility weights. The proposed new method is the special case $f = 1$ of the following more general result.

Theorem 6.1. Under the assumption of Sections 4 and 5, the optimal credibility weights Z_i^* which minimize the mean squared error $mse(R_i^c) = E\left[(R_i^c - R_i)^2\right]$ and the variance $Var[R_i^c]$ are given by

$$Z_i^* = \frac{p_i}{p_i + t_i^*}, \text{ with} \tag{6.5}$$

$$t_i^* = \frac{f - 1 + q_i + \sqrt{(f+1)^2 + q_i^2}}{2}, \quad i = 1,\dots,n. \tag{6.6}$$

Proof. Recall that $R_i^c = Z_i \cdot R_i^{ind} + (1 - Z_i) \cdot R_i^{coll}$, $i = 1,\dots,n$, with $R_i^{ind} = \frac{q_i}{p_i} \cdot C_{in-i+1}$, $R_i^{coll} = q_i \cdot U_i^{BC}$. By the assumption at the beginning of Section 4, the covariance between the individual and the collective reserve vanishes, that is $Cov\left[R_i^{ind}, R_i^{coll}\right] = \frac{q_i^2}{p_i} \cdot Cov\left[C_{in-i+1}, U_i^{BC}\right] = 0$. It follows that

$$Var[R_i^c] = Z_i^2 \cdot Var[R_i^{ind}] + (1 - Z_i)^2 \cdot Var[R_i^{coll}], \text{ with}$$

$$Var[R_i^{ind}] = \left(\frac{q_i}{p_i}\right)^2 \cdot Var[C_{in-i+1}], \quad Var[R_i^{coll}] = q_i^2 \cdot Var[U_i^{BC}].$$

Using (4.8) and the estimates (5.3), (5.4) and the definition of $\hat{u}$ after (5.5), one gets the estimate

$$Var\left[\hat{C}_{in-i+1}\right]=\left(f\cdot p_i\cdot\left[(1-p_i)\beta_i^2+p_i\right]+p_i(1-p_i)\beta_i^2\hat{u}\right)\cdot Var\left[U_i^{BC}\right].$$

Inserted into the above formulas, one obtains the estimate

$$Var\left[\hat{R}_i^c\right]=\left(Z_i^2\cdot\left[\beta^2\cdot(f+\hat{u})\cdot\frac{1-p_i}{p_i}+f\right]+(1-Z_i)^2\right)\cdot Var\left[\hat{R}_i^{coll}\right].$$

A calculation shows that the minimum variance is attained at (6.6). ◊

7. Numerical examples.

It is instructive to illustrate the obtained results at some practical examples. Let us start with the published full loss triangle of paid claims and exposures in Mack(1997), Table 3.1.5.1:

Table 7.1: loss triangle of paid claims

underwriting period	development year 1	2	3	4	5	6
1	4'370	1'923	3'999	2'168	1'200	647
2	2'701	2'590	1'871	1'783	393	-
3	4'483	2'246	3'345	1'068	-	-
4	3'254	2'550	2'547	-	-	-
5	8'010	4'108	-	-	-	-
6	5'582	-	-	-	-	-

The used exposures and relevant parameters, which are obtained through application of the described method, are summarized in Table 7.2.

Table 7.2: parameters of credible loss ratio method

underwriting period	parameters V	m	p	q	t	Z*
1	13'085	0.29667	1	0	1.00000	0.50000
2	14'258	0.17770	0.94496	0.05504	1.02790	0.47898
3	16'114	0.20072	0.88010	0.11990	1.06175	0.45323
4	15'142	0.11549	0.75153	0.24847	1.13192	0.39902
5	16'905	0.05826	0.52808	0.47192	1.26342	0.29477
6	20'224	0.04945	0.33026	0.66974	1.38945	0.19205

The next two Tables compare the loss ratio IBNR claims reserves and the total ultimate claims obtained through application of the collective, individual, Neuhaus, Benktander and optimal methods.

Table 7.3: credible loss ratio IBNR reserves

underwriting period	IBNR method collective	individual	Neuhaus	Benktander	optimal
all periods	25'154	26'972	25'913	25'999	25'648
2	705	544	568	553	628
3	1'736	1'518	1'564	1'544	1'637
4	3'380	2'761	2'962	2'915	3'133
5	7'166	10'829	8'904	9'101	8'246
6	12'167	11'320	11'916	11'887	12'004

Table 7.4: credible loss ratio ultimate claims

underwriting period	IBNR method collective	individual	Neuhaus	Benktander	optimal
all periods	85'992	87'810	86'751	86'837	86'486
1	14'307	14'307	14'307	14'307	14'307
2	10'043	9'882	9'906	9'891	9'966
3	12'878	12'660	12'706	12'686	12'779
4	11'731	11'112	11'313	11'266	11'484
5	19'284	22'947	21'022	21'219	20'364
6	17'749	16'902	17'498	17'469	17'586

The Table 7.5 displays mean squared errors of the different methods expressed as ratios to the minimal mean squared error of the optimal credible IBNR reserve. For this the minimum variance estimator of Section 6 is applied with $f = 1$ and $t_i = \frac{1}{2}\left(q_i + \sqrt{q_i^2 + 4}\right)$.

Table 7.5: mean squared standard errors (ratio to minimal error)

underwriting period	IBNR method collective	individual	Neuhaus	Benktander	optimal
2	1.024953	1.029526	1.014880	1.023617	1
3	1.048206	1.070158	1.026708	1.042762	1
4	1.077381	1.175539	1.037043	1.060395	1
5	1.087147	1.498822	1.032353	1.054596	1
6	1.066623	2.179197	1.019775	1.034510	1

The Neuhaus and Benktander loss ratio reserves are quite close to the optimal credible reserve. In the present situation, the Neuhaus reserve is closer to the optimal one than the

Benktander reserve for all underwriting years. Through application of a credible loss ratio reserving method, the reduction in mean squared error is substantial. In absence of sufficient information to estimate the optimal credibility weights, the three simple credible methods are highly recommended for actuarial practice.

The next practical example stems from a slightly modified real life project. The same conclusions as before are made. Again, the minimum variance estimator of Section 6 is applied with $f = 1$ and $t_i = \frac{1}{2}\left(q_i + \sqrt{q_i^2 + 4}\right)$.

Table 7.6: loss triangle of paid claims

underwriting period	development year 1	2	3	4	5	6
1	3'789'045	2'860'826	506'651	151'996	65'141	24'203
2	3'582'774	2'687'080	1'250'163	535'784	880'143	-
3	4'221'853	3'166'390	2'249'388	207'853	-	-
4	4'074'429	2'949'557	1'162'885	-	-	-
5	1'227'618	3'906'617	-	-	-	-
6	6'839'930	-	-	-	-	-

Table 7.7: parameters of credible loss ratio method

underwriting period	parameters V	m	p	q	t	Z*
1	8'000'000	0.40230	1	0	1.00000	0.50000
2	9'000'000	0.33129	0.99687	0.00313	1.00157	0.49882
3	10'000'000	0.13971	0.93925	0.06075	1.03084	0.47676
4	10'000'000	0.03317	0.90488	0.09512	1.04869	0.46319
5	10'000'000	0.05560	0.76012	0.23988	1.12711	0.40277
6	12'000'000	0.00303	0.41685	0.58315	1.33322	0.23819

Table 7.8: credible loss ratio IBNR reserves

underwriting period	IBNR method collective	individual	Neuhaus	Benktander	optimal
all periods	10'600'143	12'714'477	11'219'478	11'340'434	11'340'434
2	27'228	28'101	28'067	27'663	27'882
3	586'303	636'809	632'085	610'382	622'334
4	918'019	860'619	867'892	891'432	873'689
5	2'315'070	1'620'276	1'805'379	2'035'227	1'745'863
6	6'753'523	9'568'672	7'886'055	7'424'066	8'070'666

Table 7.9: credible loss ratio ultimate claims

underwriting period	IBNR method collective	individual	Neuhaus	Benktander	optimal
all periods	56'940'469	59'054'803	57'559'804	57'329'097	57'680'760
1	7'397'862	7'397'862	7'397'862	7'397'862	7'397'862
2	8'963'172	8'964'045	8'964'011	8'963'607	8'963'826
3	10'431'787	10'482'293	10'477'569	10'455'866	10'467'818
4	9'104'890	9'047'490	9'054'763	9'078'303	9'060'560
5	7'449'305	6'754'511	6'939'614	7'169'462	6'880'098
6	13'593'453	16'408'602	14'725'985	14'263'996	14'910'596

Table 7.10: mean squared standard errors (ratio to minimal error)

underwriting period	IBNR method collective	individual	Neuhaus	Benktander	optimal
2	1.001559	1.001574	1.001344	1.001554	1
3	1.027256	1.032831	1.022142	1.025650	1
4	1.040063	1.053810	1.031405	1.036429	1
5	1.076054	1.167220	1.051307	1.059868	1
6	1.078146	1.799366	1.037095	1.043965	1

As a third example, let us analyze whether the published and practically used A.M. Best loss development factors are "best" in the sense of the proposed credible loss ratio method. We compare the inverse of the loss ratio payout factors obtained from the ratio $\frac{U_i}{C_{in-i+1}}$ for the various methods. As a single illustration, we just look at the A.M. Best Table of paid claims for General Liability claims made policies, but not that similar results hold for other insurance categories. Table 7.11 lists the used triangle of paid claims and Table 7.12 displays the calculated factors. The optimal credibility weights are calculated using the minimum variance estimator of Section 6 with $f = 1$ and $t_i = \frac{1}{2}\left(q_i + \sqrt{q_i^2 + 4}\right)$.

Table 7.11: loss triangle of paid claims for General Liability claims made policies

underwriting period	development year 1	2	3	4	5	6	7	8	9	10
1994	394'645	991'758	1'601'166	2'048'315	2'286'923	2'464'913	2'578'083	2'632'786	2'678'889	2'691'701
1995	353'192	969'542	1'592'009	1'974'155	2'272'665	2'438'559	2'566'985	2'671'750	2'741'844	
1996	388'218	1'078'557	1'646'205	2'171'862	2'443'991	2'709'179	2'827'232	2'934'048		
1997	389'758	1'082'588	1'874'868	2'427'603	2'867'146	3'125'564	3'277'475			
1998	449'549	1'312'383	2'190'150	2'989'522	3'671'202	4'103'170				
1999	391'147	1'398'570	2'506'347	3'448'315	4'134'317					
2000	683'853	1'720'774	3'136'015	4'112'141						
2001	586'225	1'962'547	3'283'757							
2002	810'359	2'149'450								
2003	648'230									

Table 7.12: inverse of loss ratio payout factors

	A.M. Best	optimal	Benktander	Neuhaus	collective	individual
1994	1.000	1.000	1.000	1.000	1.000	1.000
1995	1.010	1.005	1.005	1.005	1.005	1.005
1996	1.030	1.027	1.027	1.027	1.027	1.027
1997	1.066	1.062	1.062	1.062	1.061	1.062
1998	1.114	1.112	1.113	1.113	1.112	1.113
1999	1.226	1.220	1.221	1.221	1.219	1.222
2000	1.471	1.440	1.439	1.439	1.441	1.438
2001	1.986	1.920	1.914	1.915	1.927	1.903
2002	3.475	3.344	3.328	3.331	3.366	3.245
2003	9.903	9.669	9.652	9.655	9.696	9.285

One notes that the A.M. Best factors slightly but systematically overestimate the optimal and nearly optimal Benktander and Neuhaus factors.

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