Decision principles derived from risk measures

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Back to the future

- "WORST CASE RISK MEASUREMENT"
  (1982) F. De Vylder
  $v(c) = \sup_{\mu \in M}(\int f d\mu | \int g_i d\mu = c_i, i = 1, \ldots, n)$
- "COHERENT RISK MEASURES"
  (1981) P. Huber (Robust Statistics)
  $X \leq Y \Rightarrow \rho(X) < \rho(Y)$
  $\rho(aX + b) = a\rho(X) + b$
  $\rho(X + Y) \leq \rho(X) + \rho(Y)$
- "CONVEX RISK MEASURES"
  (1985) H. Gerber and O. Deprez
  (Convex principles of premium calculation)
- "DISTORTION RISK MEASURES"
  (1987) Yaari, The Dual theory of choice under risk (Econometrica)

Risk Measures $\Leftrightarrow$ Decision Principles

• *Risk Measure*: functional that assigns a real number to a random variable, by means of a set of axioms (top level)

Axioms are fixed by economic actors or agents

• *Decision Principle*: idem but at a lower level

! A "DERIVED" FUNCTIONAL !

• "Markowitz": In one context one basic set of axioms is appropriate while in another context a different set is appropriate.

• E.g. Premium Principles, Capital Allocation, Solvency Capital Principle (regulatory, economic or rating capital)

Premium Principle

- Insurer $E[u(\omega + \rho(X) - X)] \geq u(\omega)$
- $\tilde{u}(\tilde{\omega} - \rho(X)) \geq E[\tilde{u}(\tilde{\omega} - X)]$
- $\tilde{\rho}(X) < \rho(X) \leq \rho^{+}(X)$
- Hierarchy between measures and principles:
  - Expected Utility = Risk Measure (units: arbitrary) (axioms)
  - Decision Principle = derived functional ($\neq$ properties) (units: money) (properties, eventually derived axioms)

Solvency capital principle

- Risk: $(X - \rho(X))_+ + i\rho(X)$
- Risk Measure: Π AXIOMS?
- Total Risk: $\Pi[(X - \rho(X))_+ + i\rho(X)]$
- Example: $\Pi(.) = E(.)$

Total Risk: $E[(X - \rho(X))]_+ + i\rho(X)$

Derived Measure: $\rho(X) = F_X^{-1}(1 - i)$

Reinsurance Principle

- Insurer $(X-d)_+ \rightarrow$ Reinsurer and he retains $X-(X-d)_+$
- If reinsurance price = $\rho(X,d) \rightarrow \inf_d \Pi[X-(X-d)_+ + \rho(X,d)]$
- For a utility risk measure:

$E[u(r(X)-X)] \leq E[u((X-d)_+ - X)]$ and $E[r(X)] = E[(X-d)_+]$

But: $E[\tilde{u}(\omega+\rho-r(X))] \geq E[\tilde{u}(\omega+\rho-(X-d)_+)]$

Bridge actuarial-financial pricing

- Axiom: expected utility $u(x) = -\alpha \exp(-\alpha x)\ \alpha > 0$
- Derived pricing principle

$$\sup_{\phi} -\alpha E[\exp(-\alpha E(\phi(X)X) - X)]$$

Solution: $\phi(X) = \frac{e^{\alpha X}}{E(e^{\alpha X})}$

Remark: $\alpha = \frac{|\ln(\epsilon)|}{u}$.

Incomplete information

- $V(\rho, i, m) = \sup_{F \in G} \left(\int_a^b (x-\rho)_+ dF(x) + i\rho \right)$

whereby $\int_a^b x dF(x) = c$ and $\int_a^b dF(x) = 1$,

$= \left(c - \frac{1}{2}(a+m)\right) \frac{(b-\rho)^2}{(b-m)(b-a)} + i\rho$

then $\rho^\star = b - i\frac{(b-m)(b-a)}{2c-(a+m)}$.

Financial pricing

- Ordered Esscher-Girsanov transforms implies ordered prices. If the price measure is applied to normally distributed random variables, this axiom is equivalent to "respect for second-order stochastic dominance".
- The price measure is appropriately normalized such that the price of c non-random units is equal to c non-random units.
- Additivity for sums of Esscher-Girsanov transforms. If the price measure is applied to normally distributed random variables, the axiom is equivalent to "superadditivity and comonotonic additivity of the price measure".
- Topological conditions, which are necessary to establish the mathematical proofs.

Esscher transform

- $dF_X^{(h)}(x) = \frac{e^{hx} dF_X(x)}{E[e^{hX}]}, \quad h \in \mathbb{R}$.
- $\psi_X(h) = \int_{-\infty}^{+\infty} x dF_X^{(h)}(x) = \frac{E[Xe^{hX}]}{E[e^{hX}]}$.
- A1. If $\frac{E[Xe^{hX}]}{E[e^{hX}]} \leq \frac{E[Ye^{hY}]}{E[e^{hY}]}$ for all $h \leq 0$, then $\pi[X] \leq \pi[Y]$;
- A2. $\pi[c] = c$, for all c;
- A3. $\pi[X+Y] = \pi[X] + \pi[Y]$ when X and Y are independent;
- A4. If X_n converges weakly to X, with $\min[X_n] \to \min[X]$, then $\lim_{n\to+\infty} \pi[X_n] = \pi[X]$.

Esscher transform

- A price measure $\pi[.]$ satisfies the set of axioms A1-A4 if and only if there exists some non-decreasing function

$$H : [-\infty, 0] \to [0, 1]$$

such that

$$\pi[X] = \int_{[-\infty,0]} \psi_X(h) dH(h).$$

The Esscher-Girsanov theorem

- $\phi_X(x) = \Phi^{-1}(F_X(x))$
- For the cdf $F_X(.)$ with differential $dF_X(.)$ corresponding to a given r.v. X, and a given real number v, we define by

$$dF_X^{(h,v)}(x) = \frac{e^{hv\phi_X(x)}}{E[e^{hv\phi_X(x)}]} dF_X(x) = e^{hv\phi_X(x) - \frac{1}{2}h^2v^2} dF_X(x), \ h \in \mathbb{R},$$

  its Esscher-Girsanov transform with parameters h and v (absolute risk aversion and penalty parameter, respectively).
- $\psi_X^v(h) = \int_{-\infty}^{+\infty} x dF_X^{(h,v)}(x) = E[X e^{hv\phi_X(x) - \frac{1}{2}h^2v^2}], \quad h \in \mathbb{R}.$

Esscher-Girsanov price

- $\pi^v[X] = \rho^v[\psi_X^v]$.
- B1. If $\psi_X^v(h) \leq \psi_Y^v(h)$ for all $h \leq 0$, then $\rho^v[\psi_X^v] \leq \rho^v[\psi_Y^v]$;
- B2. $\rho^v[c] = c$, for all c;
- B3. $\rho^v[\psi_X^v + \psi_Y^v] = \rho^v[\psi_X^v] + \rho^v[\psi_Y^v]$;
- B4. If $\psi_{X_n}^v(h)$ converges to $\psi_X^v(h)$ for all $h \in [-\infty, 0]$, then $\lim_{n\to+\infty} \rho^v[\psi_{X_n}^v] = \rho^v[\psi_X^v]$.
- If X and Y are two normally distributed r.v.'s with linear correlation coefficient ρ_{XY}, then

$$\psi_{X+Y}^v(h) = \mu_X + \mu_Y + hv\sqrt{\sigma_X^2 + 2\rho_{XY}\sigma_X\sigma_Y + \sigma_Y^2}.$$

Esscher-Girsanov price

- A functional $\rho^v[.]$ satisfies the set of axioms B1-B4 if and only if there exists some non-decreasing function $H : [-\infty, 0] \to [0, 1]$ such that
$$\rho^v[\psi_X^v] = \int_{[\infty,0]} \psi^v(X)(h)dH(h).$$
- For the cdf $F_{X_n}(.)$ with differential $dF_{X_n}(.)$ corresponding to a given r.v. X_n and a given real-valued function $v(.)$, we define by

$$\begin{aligned} dF^{(h,v(.))}_{X_n|X_0}(x_n|x_0) &= \int_{x_{n-1}} \ldots \int_{x_1} e^{h \sum_{j=0}^{n-1} v(x_j)\phi_{X_{j+1}|X_j}(x_{j+1}|x_j) - \frac{1}{2}h^2 v(x_j)^2} \\ &\times \quad dF_{X_n|X_{n-1}}(x_n|x_{n-1}) \ldots dF_{X_1|X_0}(x_1|x_0) \end{aligned}$$

its discrete Esscher-Girsanov transform with parameter h and penalty function $v(.)$.

Financial Derivative Pricing by Esscher-Girsanov Transforms

- $S_0 = s_0, \quad dS_t = \mu(S_t)dt + \sigma(S_t)dB_t$
- $Z_t^T(S,h,v(.)) = h\int_t^T v(S_\tau)dB_\tau - \frac{1}{2}h^2\int_t^T v(S_\tau)^2 d\tau,$
- $E_t[e^{Z_t^T(S,h,v(.))}] = 1,$
- $\pi_t^{v(.)}[g(S_T)] =$ $\int_{[-\infty,0]} E_t[e^{-\int_t^T r(S_\tau)d\tau}\phi^{(h)}(S_T,T)e^{Z_t^T(S,h,v(.))}]dH(h)$ (1).
- The function $\phi^{(h)}(.,.)$ will be chosen such that the calculation of the Feynman-Kac path integral on the right-hand side of (1) becomes feasible. Whatever function $\phi^{(h)}(.,.)$ introduced, the right-hand side of (1) only depends on the terminal values $\phi^{(h)}(S_T,T) = g(S_T)$.

Financial Derivative Pricing by Esscher-Girsanov Transforms

- Let $\phi^{(h)}(S_T, T) = g(S_T)$. Then

$$\begin{aligned} \pi_t^{v(.)}[g(S_T)] &= \int_{[-\infty,0]} E_t[e^{-\int_t^T r(S_\tau)d\tau} \phi^{(h)}(S_T, T) e^{Z_t^T(S,h,v(.))}] dH(h) \\ &= \int_{[-\infty,0]} \phi^{(h)}(S_t, t) dH(h), \end{aligned}$$

whenever $\phi^{(h)}(.,.)$ is the solution of the PDE

$$\begin{aligned} &= \frac{\partial \phi^{(h)}(x,\tau)}{\partial \tau} + (\mu(x) + hv(x)\sigma(x)) \frac{\partial \phi^{(h)}(x,\tau)}{\partial x} \\ &= \frac{1}{2}\sigma(x)^2 \frac{\partial^2 \phi^{(h)}(x,\tau)}{\partial x^2} = r(x)\phi^{(h)}(x,\tau), \quad \tau \in [t, T]. \end{aligned}$$

Financial Derivative Pricing by Esscher-Girsanov Transforms

- Suppose that S is a tradable asset. If $v(x) = \frac{\mu(x)-r(x)x}{\sigma(x)}$ and

$$H(h) = \begin{cases} 1, & h \geq -1 \\ 0, & \text{otherwise}, \end{cases}$$

then $\pi_t^{v(.)}[g(S_T)]$ coincides with the approximate arbitrage-free price of the financial derivative $g(S_T)$.