Decision Principles Derived from Risk Measures

Marc J. Goovaerts[†,‡], **Rob Kaas**[†] **and Roger J.A. Laeven**[†,*]

[†]University of Amsterdam, Dept. of Quantitative Economics, Roetersstraat 11,
1018 WB Amsterdam, The Netherlands

[‡]Catholic University of Leuven, Dept. of Applied Economics, Naamsestraat 69,
B-3000 Leuven, Belgium

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Abstract

In this paper, we argue that there exists a distinction between risk measures and decision principles. Though both can be regarded as functionals assigning a real number to a random variable, we think that there is a hierarchy between the two concepts. Risk measures operate on the first "level", quantifying the risk in the situation under consideration, while decision principles operate on the second "level", being derived from the risk measure. We will briefly illustrate this distinction with several examples of economic situations encountered in the insurance and financial industry.

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[*]Corresponding author. E-mail: R.J.A.Laeven@uva.nl, Phone: +31 20 525 7317, Fax: +31 20 525 4349.

1 Introduction

We introduce a distinction between *risk measures* and *decision principles*, such as premium principles, capital allocation principles and solvency capital principles. The difference between risk measures and decision principles comes from the different "levels" at which they operate. That is, there is a hierarchy between the two concepts.

A *risk measure* is a functional that assigns a real number to a random variable (the risk). Justifications of risk measures are generally based on axiomatic characterizations, imposing a set of axioms, which we denote henceforth by $\mathcal{S}$. A risk measure is appropriate if and only if its characterizing axioms are. Axiomatic characterizations can be used to justify a risk measure, but also to criticize it.

The particular set of axioms must reflect the risk perception of the economic *actors* or *agents* involved in the situation under consideration. The economic relevance of the axioms thus depends on the actors involved as well as on the specific situation under study. The axioms should be formalized such as to be representative for all the actors in the evaluation of any "feasible" risk.

A useful meaning of the numerical value of a risk measure is given by the *certainty equivalent*. For a given risk measure and a given random variable, the certainty equivalent is the real number that is equally risky (equivalent).

A *decision principle* is a "derived" functional assigning a real number to a random variable. The derivation is generally based on an optimization procedure, e.g., by minimizing the total risk as measured by a risk measure. Notice that both risk measures and decision principles are functionals mapping random variables to the real line. Hence, mathematically they are similar concepts. Though, justifications and derivations differ fundamentally.

The explicit distinction between risk measures and decision principles has important consequences for the management of risk in the insurance and financial industry.

By using different sets of axioms, reflecting different perceptions of risk, e.g., for different actors (management, regulator), or for different economic situations, appropriate decision principles can be derived.

Let us quote Markowitz (1959, Chapter 10) in this respect: "We might decide that in one context one basic set of principles is appropriate, while in another context a different set of principles should be used. We might find that some patterns of preferences are consistent with a set of preferences while other patterns are not." Here, "principles" mean "axioms".

We think that a "two level" procedure in which risk measures determine the risk of an economic transaction, and decision principles are derived in a second stage, is a proper approach and a valuable tool to demonstrate what are the essential parameters or concepts to be determined. For example, the assumption of expected utility is justified by the belief that for the economic situations considered, monotonicity and the so-called *sure-thing principle* (which amounts to the independence axiom when probabilities are known and given) are reasonable assumptions. These axioms, jointly with continuity, axiomatize expected utility and clarify the validity and limitations of it. Then, in a second stage, the particular functional form of e.g., the exponential premium principle and the role of the absolute coefficient of risk aversion are derived; see Section 2 for further details.

In this approach the "selection" of appropriate preference axioms takes place on the first "level" of measuring the total risk of an economic operation, while properties of decision principles follow as consequences.

In actuarial science, for many years, risk measures have been important objects of study; see e.g., Bühlmann (1970), Gerber (1979) and Goovaerts, De Vylder & Haezendonck (1984) for early accounts.

Recent developments aiming for international convergence of solvency capital principles, both in insurance and finance, have further increased the importance of the topic. In this context, we refer the interested reader e.g., to the Basel Capital Accords.

In what follows, we will elucidate the concepts of risk measures and decision principles in different economic situations encountered in practice. To fix our framework, we state the following definitions and remarks:

Definition 1.1 *We fix a measurable space $(\Omega, \mathcal{F})$. A risk is a random variable defined on $(\Omega, \mathcal{F})$ and is denoted by X. It represents the final net loss of a position (contingency) currently held. When $X > 0$, we call it a loss, whereas if $X \leq 0$, we call it a gain. The class of all random variables on $(\Omega, \mathcal{F})$ is denoted by Φ.*

Definition 1.2 *A risk measure π is a functional assigning a real number to any random variable defined on $(\Omega, \mathcal{F})$. Thus, π is a mapping from Φ to $\mathbb{R}$.*

Remark 1.1 *In general, no integrability conditions need to be imposed on the elements of Φ. This may cause the problem that a functional π, though finite on finite measurable spaces, becomes infinite on more general measurable spaces. Therefore, some authors prefer to extend the range in the definition of a risk measure to $\mathbb{R} \cup \{+\infty\}$. Then, in case $\pi[X] = +\infty$, we say that the risk is unacceptable or non-insurable. Though, for the purposes of this paper, we can restrict to finite measurable spaces and leave such extension out of consideration.*

Remark 1.2 *In the classical sense, it is assumed that risks with identical distribution functions lead to the same value of the risk measure, that is, if for two risks X, Y we have that $F_X(x) = F_Y(x)$ for all real x, then $\pi[X] = \pi[Y]$. In general this assumption need not be imposed.*

Remark 1.3 *Though the units of the elements of Φ (the risks) are considered to be monetary units (e.g., Dollars, Euros), the units of π are not necessarily monetary.*

Remark 1.4 *A risk measure establishes a complete ordering within the class of random variables Φ. In this paper, risk measures are exclusively used to compare (order) the "riskiness" of elements of Φ.*

Definition 1.3 *A risk measure is $\mathcal{S}$-consistent if it is characterized by imposing the set $\mathcal{S}$ of axioms on the elements or on a subclass of the elements of Φ.*

Remark 1.5 *It should be noted that some decision principles derived in this paper can also be axiomatized directly. A direct axiomatic characterization need not be inferior as long as the axioms are well-chosen and undeniable. Though, in this paper, the term "decision principle" is exclusively used for a "derived" functional.*

1.1 A note on the literature

The academic research on risk measures has recently experienced a revival. As is well-known, the study of risk measures has a long history in Actuarial Science and Probability Theory. Furthermore, functionals representing preferences have been the object of study of Economic Theory, in particular the realm of Decision under Uncertainty, for almost a century.

It is inevitable that a revival leads to the restatement (or reinvention) of known results, perhaps in a slightly different framework. Examples of such restatements are listed below (without being exhaustive). We hope that this short list will enhance careful citation in future research.

- *Distortion risk measures* were developed in the economics literature; see Schmeidler (1989), Quiggin (1982) and Yaari (1987). In fact, an axiomatic characterization of the Choquet expectation (of which a distortion risk measure is a particular example) was already in Greco (1982) (see Denneberg (1994) for a translation of the main results into English); see also Theorem 3 of Anger (1977).

- An axiomatic characterization of the upper (lower) expectation (also known as *coherent risk measure*) was established by Huber (1981). Related results in an economic environment are in Gilboa & Schmeidler (1989).
- *Convex risk measures* were studied already in Deprez & Gerber (1985).
- A particular example of a *spectral risk measure* was characterized axiomatically by Gerber & Goovaerts (1981); see also Section 6 of this paper for further remarks in this direction.
- Worst case risk measurement was studied already by De Vylder (1982, and subsequent papers); see also Laeven, Goovaerts & Kaas (2005) and Section 7 of this paper for further details.

2 Premium principles

A main example of a decision principle in an insurance context, is an insurance premium principle. When deriving a premium principle, two viewpoints can be taken, the one of the insurer and the one of the insured. In the former case, the premium principle reflects the minimum premium for which the insurer is willing to sell the insurance, while in the latter case the premium principle reflects the maximum premium that a individual is willing to pay to cede the risk.

From the viewpoint of the insurer, a premium principle can e.g., be derived such that the probability of ruin is sufficiently small (see e.g., Gerber (1974), Bühlmann (1985) or Kaas *et al.* (2001), Section 5.2, for further details in this direction).

Another well-known approach following Von Neumann & Morgenstern (1944), is to specify a utility function $u : \mathbb{R} \to \mathbb{R}$ and consider the expected utility $\mathbb{E}[u(w+\rho[X]-X)]$, where w denotes the initial wealth of the insurer and ρ denotes the premium principle, assigning a real number to any random variable (risk) in a particular class Φ. In the Von Neumann-Morgenstern framework, the utility function u is subjective, whereas the probability measure is assumed to be objective, known and given. In the more general framework of Savage (1954), also the probability measure can be subjective. In the latter framework, the probability measure need not be based on objective statistical information, but may be based on subjective judgements of the decision situation under consideration, just as u is.

From the expected utility expression, an *equivalent utility principle* (also known under the misnomer *zero utility principle*) can be established as follows: for a given random variable X in some class Φ and a given real number w, the equivalent utility premium $\rho^-[X]$ of the insurer is derived by solving

$$\mathbb{E}[u(w + \rho^-[X] - X)] = u(w). \tag{1}$$

Under the assumption that u is continuous, a solution exists. If the insurer is risk-averse, one easily proves that $\rho^-[X] > \mathbb{E}[X]$. Obviously, the insurer will sell the insurance if and only if he can charge a premium $\rho[X]$ that satisfies $\rho[X] \geq \rho^-[X]$.

Next, we consider the viewpoint of the insured. An insurance treaty that for a given random variable X in some class Φ leaves the insured with final wealth $\bar{w} - \rho[X]$, will be preferred to full self insurance, which leaves the insured with final wealth $\bar{w} - X$, if and only if $\bar{u}(\bar{w} - \rho[X]) \geq \mathbb{E}[\bar{u}(\bar{w} - X)]$, where $\bar{u}$ denotes the utility function of the insured. The equivalent utility premium $\rho^+[X]$ is derived by solving for the maximal premium in the above inequality. Clearly, the insured will buy the insurance if and only if $\rho[X] \leq \rho^+[X]$. One can easily verify that a risk-averse insured is willing to pay more than the pure net premium $\mathbb{E}[X]$. An insurance treaty can be signed both by the insurer and by the insured only if the premium satisfies $\rho^-[X] \leq \rho[X] \leq \rho^+[X]$.

Notice that the properties of an equivalent utility premium are determined by the properties of the utility function. That is, the properties that the decision principle satisfies are determined by the risk measure that it is derived from.

It is well-known that if the utility function is non-decreasing, which is implied by the monotonicity axiom that is imposed to axiomatize expected utility, then $X \leq_{\text{fosd}} Y$ implies both $\rho^+[X] \leq \rho^+[Y]$ and $\rho^-[X] \leq \rho^-[Y]$. Here and in the following "$\leq_{\text{f(s)osd}}$" denotes "is smaller in first (second) order stochastic dominance than". If moreover the expected utility maximizer is risk-averse, which is equivalent to concavity of the utility function, then $-Y \leq_{\text{sosd}} -X$ implies both $\rho^+[X] \leq \rho^+[Y]$ and $\rho^-[X] \leq \rho^-[Y]$.

An interesting problem, which is related to the problem described above, is the problem of how to distribute between individual policies the aggregate premium income of an insurer. Taking into account the "specific nature" of insurance pricing, the mechanism that distributes the aggregate premium income can to some extent be rather arbitrary, because the premium of an individual policy reflects two types of randomness: on the one hand, it reflects the randomness of the single random variable (the risk) under consideration and on the other hand it reflects the randomness of the aggregate insurance portfolio, as well its heterogeneity and the subsidisation that takes place within the portfolio.

For the case of mutually independent policies, Bühlmann (1985) proposes to distribute to policy j, $j = 1, \ldots, n$ with n denoting the total number of policies, the premium

$$\rho[X_j] = \frac{1}{\alpha} \log \mathbb{E}[\exp(\alpha X_j)], \tag{2}$$

which can be regarded as the equivalent utility premium of an insurer with a negative exponential utility function with coefficient of absolute risk aversion equal to α. It is well-known that (2) is additive for independent random variables. In Bühlmann (1985), the allocation (2) arises not as an equivalent utility premium but rather in a ruin theoretic framework when the probability of ruin is bounded from above by ε and the insurer has a negative exponential utility function. In this case, it turns out that α takes the value of $\frac{\log \varepsilon}{w}$, with w denoting the initial wealth of the insurer.

In a recent paper, Goovaerts *et al.* (2004) present a new axiomatic characterization of risk measures that are additive for independent random variables. The risk measure characterized there is of the same form as the premium in expression (2), though mixed with respect to the parameter α.

3 Solvency capital principles

In the insurance and financial industry, solvency capital serves as a buffer against the contingency that assets turn out to be insufficient to cover future liabilities. Several types of solvency capital need to be distinguished, namely regulatory capital, economic (or management) capital, rating capital and book capital. The discussion below focuses on economic capital; for further details we refer to Laeven & Goovaerts (2004) and Dhaene *et al.* (2005).

We argue that the optimal amount of economic capital should be derived in a tradeoff between risk exposure on the one hand and the cost of economic capital on the other hand. A rather similar tradeoff is encountered in mathematical statistics when determining the appropriate confidence level for the testing of hypotheses. Hypothesis testing is not a one dimensional problem (with one criterion) but it is a problem where two possible errors are to be considered: the so-called *type I error* of rejecting a true null hypothesis and the *type II error* of failing to reject a false null hypothesis.

Suppose that we axiomatize a risk measure π, imposing a set of axioms that is representative for all the economic agents in the specific situation under study. Furthermore, let ρ denote an economic capital principle. Clearly,

$$X = \min(X, \rho[X]) + (X - \rho[X])_+,$$

with $(X - \rho[X])_+ = \max(X - \rho[X], 0)$. In the insurance and financial industry, the "lower layer" $\min(X, \rho[X])$ can be regarded as the risk borne by the shareholders. When $X \leq 0$ (hopefully the usual case), a profit is made (and typically a dividend is paid out). When $X > 0$, the shareholders experience a loss. Because the loss experienced by the shareholders can never exceed the total amount of solvency capital invested, the loss is capped at $\rho[X]$. The risk $\min(X,\rho[X])$ will be evaluated by the shareholders and an opportunity cost of capital will be charged.

Let i denote the opportunity cost of capital (in excess of the risk-free rate of interest). Then we consider

$$\pi[X1_{\{X<0\}} + i\rho[X] + (X-\rho[X])_+], \tag{3}$$

where, as usual, 1_A denotes the indicator function of event A. One easily verifies that if π is an expectation operator, the optimal solvency capital principle that minimizes (3) is given by

$$\rho[X] = F_X^{-1}(1-i).$$

Here, as usual, we denote by F_X^{-1} the left-continuous generalized inverse distribution function of the random variable X. That is, the optimal solvency capital principle is the Value-at-Risk at level $1-i$.

4 Reinsurance principles

A particular example of risk transfer is reinsurance. The reinsurance market is a market with a restricted number of players and a large though limited financial capacity. Therefore, the mechanisms in this market follow their own paradigms.

We consider an insurer that wants to cede "the tail" of its risk and performs an analysis of the optimal retention in a stop-loss reinsurance contract. In such a contract, the insurer transfers the risk $(X-d)_+$ to the reinsurer, while retaining the risk $X-(X-d)_+$, with d the retention (level) of the contract. Let the price of a stop-loss reinsurance contract for a given risk X and a retention d be denoted by $\rho(X;d)$. Furthermore, let π denote the risk measure of the insurer. Then, the insurer faces the following optimization problem:

$$\inf_d \pi\left[X-(X-d)_+ + \rho(X;d)\right]. \tag{4}$$

Given π and ρ, the optimal retention can readily be derived.

It is well-known that stop-loss reinsurance is optimal from the viewpoint of the insurer. That is, if $r:\mathbb{R}\to\mathbb{R}$ denotes the payoff function of a reinsurance contract, assuming $0\le r(x)\le x$ since gains on insurance are generally forbidden, then for any r the equality $\mathbb{E}[r(X)] = \mathbb{E}[(X-d)_+]$ implies

$$\mathbb{E}[u(r(X)-X)] \le \mathbb{E}[u((X-d)_+ - X)],$$

with u non-decreasing and concave. This says that stop-loss reinsurance is preferred to any other form of reinsurance by all risk averse expected utility maximizers. Notice however that this conclusion crucially depends on the silent assumption that if $\mathbb{E}[r(X)] = \mathbb{E}[(X - d)_+]$, then the price charged for contract r is the same as the price $\rho(X; d)$ charged for the stop-loss contract. Clearly, this will generally not be the case in practice. To verify this, reconsider (1). Since $-(X - d)_+ \leq_{\text{sosd}} -r(X)$ whenever $\mathbb{E}[r(X)] = \mathbb{E}[(X - d)_+]$, we find that for all risk averse expected utility maximizers

$$\mathbb{E}[u(w + \rho - r(X))] \geq \mathbb{E}[u(w + \rho - (X - d)_+)].$$

Hence, the equivalent utility premium for $r(X)$ is smaller than the equivalent utility premium for $(X - d)_+$.

In the framework of the previous section, the insurer might also consider a more general problem. Let the risk X be decomposed as follows:

$$X = X_1 + X_2 + X_3 + X_4,$$

with

$$\begin{aligned}
X_1 &= X 1_{\{X \leq 0\}} : \text{ the profit layer;} \\
X_2 &= \min(X 1_{\{X > 0\}}, c) : \text{ the reinsurance layer with retention 0 and cap } c; \\
X_3 &= \min((X 1_{\{X > 0\}} - c)_+, \rho[X]) : \text{ the economic capital layer;} \\
X_4 &= (X - \rho[X])_+ : \text{ the residual risk layer.}
\end{aligned}$$

We note that the random variables X_1, X_2, X_3, X_4 are *comonotonic* (which is an abbreviation of *common monotonic*); we defer until Section 6 a formal definition of this strong dependence notion. By specifying the cost of economic capital, the stop-loss reinsurance pricing principle and the risk measure of the insurer, an optimal retention policy can be derived. We leave this to the reader. We note that the different layers of the risk X are measured or priced by different agents and thus by different principles.

5 A bridge between actuarial and financial pricing principles

A versatile tool to unify actuarial and financial pricing principles is the time-honoured *Esscher transform*; the interested reader is referred to Bühlmann (1980), Goovaerts, DeVylder & Haezendonck (1984), Gerber & Shiu (1996), Bühlmann *et al.* (1996) and Goovaerts & Laeven (2005) for various contributions in this direction.

The Esscher transform was originally introduced in Esscher (1932), who suggested to use this transform instead of the original distribution function, to apply the well-known Edgeworth approximation to.

We will show that the Esscher transform also appears in an optimal premium problem when the risk measure is exponential-like. We consider the expected utility framework, assuming the utility function $u : \mathbb{R} \to \mathbb{R}$ to be negative exponential, i.e.,

$$u(x) = -\alpha \exp(-\alpha x), \qquad \alpha > 0. \tag{5}$$

It is well-known (see e.g., Kaas *et al.* (2001), Chapter 1) that the corresponding equivalent utility premium principle is the exponential premium principle. Gerber (1974) proved that the equivalent utility principle is the exponential premium principle if and only if it is additive for independent random variables; see also Goovaerts *et al.* (2004) in this respect.

Now suppose that, rather than adopting the equivalent utility principle, we use expected utility maximization with a negative exponential utility function in an optimal premium problem that, for any random variable X in some class Φ, only allows premiums of the form $\mathbb{E}[\varphi(X)X]$, with $\varphi(\cdot)$ a real-valued, continuous and strictly increasing function satisfying $\mathbb{E}[\varphi(X)] = 1$. Premiums of this form can be expressed as expectations under a transformed probability measure and hence can be regarded as financial pricing principles, consistent with a no arbitrage setup. The random gain of an insurer charging a premium $\mathbb{E}[\varphi(X)X]$ is given by $\mathbb{E}[\varphi(X)X] - X$. Let Φ be the class of all random variables with support $[a, b]$, $a < b$. Henceforth, we denote by Ψ the class of all functions φ that satisfy the aforementioned conditions. Then, we state the following problem:

$$\max_{\Psi} \; -\alpha \mathbb{E}\left[\exp\left(-\alpha(\mathbb{E}[\varphi(X)X] - X)\right)\right], \qquad X \in \Phi. \tag{6}$$

Problem (6) has been solved by Goovaerts, De Vylder & Haezendonck (1984), though their proof contains some inaccuracies. Below we solve it again.

For notational convenience, we write $\varphi(X) = Y$, hence $X = \varphi^{-1}(Y)$, and $f(y) = \varphi^{-1}(y)$ for all real y, hence $X = f(Y)$. Our proof, as is the one in Goovaerts, De Vylder & Haezendonck (1984), is based on variational calculus. For *any* real-valued, continuous and strictly increasing function f (we do not restrict ourselves here to functions f that satisfy $\mathbb{E}[f^{-1}(X)] = 1$ for all $X \in \Phi$) we write

$$f(Y) = f_0(Y) + \varepsilon f_1(Y) + \varepsilon^2 f_2(Y) + \ldots,$$

with f_i, $i = 0, \ldots, n$, real-valued, continuous functions. Necessary and sufficient conditions for f_0 to give rise to an optimum are given by

$$\frac{d}{d\varepsilon}\left(-\alpha \mathbb{E}\left[\exp\left(-\alpha(\mathbb{E}[f_0(Y)Y] + \varepsilon \mathbb{E}[f_1(Y)Y] + \ldots - f_0(Y) - \varepsilon f_1(Y) - \ldots)\right)\right]\right)_{|\varepsilon=0} = 0,$$

and

$$\frac{d^2}{d\varepsilon^2}\left(-\alpha\mathbb{E}\left[\exp\left(-\alpha(\mathbb{E}[f_0(Y)Y]+\varepsilon\mathbb{E}[f_1(Y)Y]+\ldots-f_0(Y)-\varepsilon f_1(Y)-\ldots)\right)\right]\right)_{|\varepsilon=0}\leq 0,$$

which have to hold for any Y. The first order condition can be cast into the form

$$\mathbb{E}\left[e^{\alpha f_0(Y)}\left(f_1(Y)-\mathbb{E}[Yf_1(Y)]\right)\right]=0,$$

or equivalently

$$\mathbb{E}\left[\left(e^{\alpha f_0(Y)}-\mathbb{E}[e^{\alpha f_0(Y)}]Y\right)f_1(Y)\right]=0.$$

Because of the arbitrariness of f_1, this reduces to

$$e^{\alpha f_0(y)}=\mathbb{E}[e^{\alpha f_0(Y)}]y, \tag{7}$$

for any real y, and hence

$$e^{\alpha x}=\mathbb{E}[e^{\alpha X}]\varphi(x),$$

for any $x\in[a,b]$. Notice that (7) implies that f_0 must satisfy $\mathbb{E}[f_0^{-1}(X)]=1$ for any $X\in\Phi$, which guarantees that $\mathbb{E}[\varphi(X)]=1$ for any $X\in\Phi$.

To verify that for this particular function φ a maximum is obtained in problem (6), one may substitute the right-hand side of (7) into the second order condition. This proves that

$$\mathbb{E}[\varphi(X)X]=\frac{\mathbb{E}[Xe^{\alpha X}]}{\mathbb{E}[e^{\alpha X}]},$$

is the optimal premium that solves (6).

6 Spectral risk measures

It was noticed already in the introduction of this paper that an important tool to justify (or criticize) a risk measure is an axiomatic characterization. In this section we will outline a general method to axiomatize mixtures (i.e., probability weighted averages) of distribution characterizing functions (that have a one-to-one correspondence to a distribution function).

An example of such a mixture is the *spectral risk measure*. Spectral risk measures were introduced by Acerbi (2002). They are defined as probability weighted averages of Tail-Value-at-Risks. Recall that the Tail-Value-at-Risk is defined as follows:

$$\mathrm{TVaR}_h[X]=\frac{1}{1-h}\int_h^1 F_X^{-1}(h),\quad h\in(0,1). \tag{8}$$

It is not difficult to verify that the random variables X and Y have the same distribution if and only if $\mathrm{TVaR}_h[X] = \mathrm{TVaR}_h[Y]$ for all $h \in (0, 1)$. Hence, the Tail-Value-at-Risk is what we call a distribution characterizing function. The interested reader is referred to Kusuoka (2001) for an axiomatic characterization of the spectral risk measure. Using the general method introduced below, we present a new and different axiomatic characterization of spectral risk measures.

As a matter of fact, we argue that the term "spectral risk measure" should not be restricted to probability weighted averages of Tail-Value-at-Risks, but should rather be used for any mixture of distribution characterizing functions. Examples of spectral risk measures (in the general sense) include the mixture of Esscher premiums and the mixture of exponential premiums (both of which in the context of this paper may be called decision principles rather than risk measures); see Gerber & Goovaerts (1981) and Goovaerts *et al.* (2004).

Before we state the main results, we first introduce the notion of *comonotonicity.* We state the following (equivalent) definitions for a pair of random variables to be *comonotonic*; we follow Denneberg (1994), Proposition 4.5.

Definition 6.1 *We fix a measurable space* $(\Omega, \mathcal{F})$. *A pair of random variables* $X, Y : \Omega \to \mathbb{R}$ *is said to be comonotonic if*

(i) there is no pair $\omega_1, \omega_2 \in \Omega$ *such that* $X(\omega_1) < X(\omega_2)$ *while* $Y(\omega_1) > Y(\omega_2)$*;*

(ii) there exists a function $Z : \Omega \to \mathbb{R}$ *and non-decreasing functions* f, g *such that*

$$X(\omega) = f(Z(\omega)), \qquad Y(\omega) = g(Z(\omega)), \qquad \text{for all } \omega \in \Omega; \tag{9}$$

Comonotonicity is a very strong positive dependence notion. Definition (ii) points out that if random variables are comonotonic, all multivariate problems are reduced to univariate ones. The interested reader is referred to Dhaene *et al.* (2002) for an elaborate study of comonotonicity and its applications in insurance and finance.

In the remainder of this section, we restrict ourselves to bounded random variables, unless otherwise stated. For a given random variable X we consider a real-valued distribution characterizing function $\varphi_X : (a, b) \to \mathbb{R}$. We assume that φ_X is non-decreasing and satisfies $\varphi_c(h) = c$ for all real c and $a < h < b$.

We introduce a continuous random variable H_0 with a strictly increasing distribution function F_{H_0} supported in $[a, b]$ with $a < b$ and having positive jumps at both a and b. Here, $[a, b]$ is taken such that it coincides with the (closed) domain of φ_X. In the case where $a = -\infty$ and $b = +\infty$, the random variable H_0 is defective.

We consider the random variable $\varphi_X(H_0)$. Since φ_X depends on the distribution of X rather than on the random variable X itself, we can assume without loss of generality that H_0 is independent of the indices used.

Then we introduce a functional ξ that assigns a real number to any function φ_X. We state the following set of axioms that ξ should satisfy:

A1. If $\varphi_X(H_0) \leq \varphi_Y(H_0)$ almost surely then $\xi[\varphi_X(H_0)] \leq \xi[\varphi_Y(H_0)]$;

A2. $\xi[c] = c$, for all real c;

A3. $\xi[\varphi_X(H_0) + \varphi_Y(H_0)] = \xi[\varphi_X(H_0)] + \xi[\varphi_Y(H_0)]$;

Then we state the following theorem:

Theorem 6.1 *The functional ξ satisfies the set of axioms A1-A3 if and only if there exists some non-decreasing function $H : [a, b] \to [0, 1]$ such that*

$$\xi[\varphi_X(H_0)] = H(a)\varphi_X(a) + \int_{(a,b)} \varphi_X(h)dH(h) + (1 - H(b))\varphi_X(b). \tag{10}$$

Proof: The proof of this theorem can be established along the same lines as the proof of Theorem 3 of Goovaerts *et al.* (2004). □

As an example, we provide a new axiomatic characterization of the spectral risk measures. We state the following set of axioms that a risk measure π should satisfy:

B1. If $\text{TVaR}_h[X] \leq \text{TVaR}_h[Y]$ for all $h \in (0, 1)$, then $\pi[X] \leq \pi[Y]$;

B2. $\pi[c] = c$, for all real c;

B3. $\pi[X + Y] = \pi[X] + \pi[Y]$, when X and Y are comonotonic;

First, notice that axiom B1 implies law invariance, that is, if X and Y have the same distribution, they must be assigned the same value by the risk measure. Notice furthermore that if X is smaller than Y in first-order stochastic dominance sense, then $\text{TVaR}_h[X] \leq \text{TVaR}_h[Y]$ for all $0 < h < 1$. Thus, axiom B1 guarantees monotonicity of the risk measure π.

Comonotonic additivity as an axiom was first imposed by Greco (1982), Schmeidler (1989) and Yaari (1986); see also Theorem 3 of Anger (1977).

Then we state the following corollary:

Corollary 6.1 *The risk measure π satisfies the set of axioms B1-B3 if and only if there exists some non-decreasing function $H : [0,1] \to [0,1]$ such that*

$$\pi[X] = H(0)\min[X] + \int_{(0,1)} \mathrm{TVaR}_h[X]dH(h) + (1 - H(1))\max[X]. \tag{11}$$

Proof: The proof of this corollary follows from the result of Theorem 6.1 by defining $\pi[X] := \xi[\varphi_X(H_0)]$ and taking $\varphi_X(h) = \mathrm{TVaR}_h[X]$. □

7 Incomplete Information

A drawback of the setup considered so far, is that it presumes the distribution function of the random variable under consideration (the risk) to be known completely. Clearly, this is not always the case in practical situations. When only partial (i.e., incomplete) information on the random variable is available, e.g., mean, variance, one may restrict oneself to the set of "admissible" distribution functions, satisfying the constraints implied by the information available, and then maximize the risk measure over this set. Doing so, one obtains what is often called a "worst case risk measurement", which can be regarded as a prudent assessment of the risk.

To illustrate this approach, we consider again the solvency capital problem, now under incomplete information. In particular, we assume that the mean of the random variable is available and equal to c and, in addition, we assume that the distribution is unimodal with given mode m.

For a fixed and given value of ρ, we then solve

$$V(\rho; i, m) = \sup_{F \in \mathcal{G}_{\mathrm{u(m)}}} \left(\int_a^b (x-\rho)_+ dF(x) + i\rho \mid \int_a^b x dF(x) = c, \int_a^b dF(x) = 1 \right), \tag{12}$$

in which $\mathcal{G}_{\mathrm{u(m)}}$ denotes the set of all unimodal distributions with fixed mode m, a and b are the infimum and the supremum of the distribution, respectively (which can be set arbitrarily small or large), and i denotes the cost of solvency capital. The value of this problem at its solution can be derived analytically (see Laeven *et al.* (2005)) and is given by

$$V(\rho; i, m) = \left(c - \frac{1}{2}(a+m) \right) \frac{(b-\rho)^2}{(b-m)(b-a)} + i\rho. \tag{13}$$

To derive the optimal amount of solvency capital, which is denoted by ρ^*, $V(\rho; m)$ must be minimized with respect to ρ. It is not difficult to verify from (13) that ρ^* is given by

$$\rho^* = b - i\frac{(b-m)(b-a)}{2c-(a+m)}. \tag{14}$$

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